Group Decision-Making Model With Incomplete Fuzzy Preference Relations Based on Additive Consistency

Enrique Herrera-Viedma, Francisco Chiclana, Francisco Herrera, and Sergio Alonso

***Abstract*—In decision-making problems there may be cases in which experts do not have an in-depth knowledge of the problem to be solved. In such cases, experts may not put their opinion forward about certain aspects of the problem, and as a result they may present incomplete preferences, i.e., some preference values may not be given or may be missing. In this paper, we present a new model for group decision making in which experts' preferences can be expressed as incomplete fuzzy preference relations. As part of this decision model, we propose an iterative procedure to estimate the missing information in an expert's incomplete fuzzy preference relation. This procedure is guided by the additive-consistency (AC) property and only uses the preference values the expert provides. The AC property is also used to measure the level of consistency of the information provided by the experts and also to propose a new induced ordered weighted averaging (IOWA) operator, the AC-IOWA operator, which permits the aggregation of the experts' preferences in such a way that more importance is given to the most consistent ones. Finally, the selection of the solution set of alternatives according to the fuzzy majority of the experts is based on two quantifier-guided choice degrees: the dominance and the nondominance degree.**

***Index Terms*—Additive consistency (AC), aggregation, choice degree, group decision making (GDM), incomplete preference relations, induced ordered weighted averaging (IOWA) operator.**

I. Introduction

GROUP decision making (GDM) consists of multiple individuals interacting to reach a decision. Each decision maker (expert) may have unique motivations or goals and may approach the decision process from a different angle, but have a common interest in reaching eventual agreement on selecting the "best" option(s) [8], [25]. To do this, experts have to express their preferences by means of a set of evaluations over a set of alternatives. It has been a common practice in research to model GDM problems in which all the experts express their preferences using the same preference representation format. However, in real practice, this is not always possible because each expert has their unique characteristics with regard to knowledge, skills, experience, and personality, which implies that different experts may express their evaluations by means of different preference representation formats. In fact, this is an issue that recently has attracted the attention of many researchers in the area of GDM, and as a result, different approaches to integrating different preference representation formats have been proposed [1]–[3], [9], [15], [16], [44], [45]. In these research papers, many reasons are provided for fuzzy preference relations to be chosen as the base element of that integration. Among these reasons, it is worth noting that they are a useful tool in the aggregation of experts' preferences into group preferences (see also [5], [10], [17], [20], [31], and [32]).

As aforementioned, each expert has his/her own experience concerning the problem being studied, which also may imply a major drawback, that of an expert not having a perfect knowledge of the problem to be solved [21]–[23], [28], [33]. Indeed, there may be cases where an expert would not be able to efficiently express any kind of preference degree between two or more of the available options. This may be due to an expert not possessing a precise or sufficient level of knowledge of part of the problem, or because that expert is unable to discriminate the degree to which some options are better than others. Therefore, it would be of great importance to provide the experts with tools that allow them to express this lack of knowledge in their opinions. In this paper, we present and use the concept of an incomplete fuzzy preference relation, i.e., a fuzzy preference relation with some of its values missing or unknown, as the tool for modeling situations with incomplete information.

Another important issue to bear in mind when information is provided by experts is that of "consistency" [3], [4], [17]. Due to the complexity of most decision-making problems, experts' preferences may not satisfy formal properties that fuzzy preference relations are required to verify. Consistency is one of them, and it is associated with the transitivity property.

Many properties have been suggested to model transitivity of fuzzy preference relations and, consequently, consistency may be measured according to which of these different properties is required to be satisfied. One of these properties is the "additive transitivity," which, as shown in [17], can be seen as the parallel concept of Saaty's consistency property in the case of the multiplicative preference relation.

It is obvious that consistent information, i.e., information which does not imply any kind of contradiction, is more

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relevant or important than the information containing some contradictions. The general procedure for the inclusion of importance degrees in GDM problems involves the transformation of the preference values under the importance degrees to generate new values [11], [38]. This activity is carried out by means of a transformation function. Examples of such a function used in these cases include the minimum operator [11], [38], the exponential function [35], or generally a t-norm operator [46]. An alternative way of implementing these importance degrees in the resolution process of a GDM problem is by using them to induce the ordering of the preference values prior to their aggregation, which is in the definition of an induced ordered weighted averaging (IOWA) operator [40], [42]. In the case of quantifier-guided IOWA operators, the importance degrees can also be used to calculate their corresponding weighting vector [6], [12].

The aim of this paper is to present a new decision model to deal with GDM problems with the incomplete fuzzy preference relations based on the additive-consistency (AC) property. This new model is composed of two steps: the estimation of missing preference values and the selection of alternatives. To do this, we define a new AC measure for fuzzy preference relations, which is based on the additive transitivity property [31]. We use this AC measure to propose a new IOWA operator, which we call the AC-IOWA operator. For the first step, we propose an iterative procedure based on AC to estimate the missing values of the incomplete fuzzy preference relation. We will show that under certain conditions the incomplete fuzzy preference relation can be completed, i.e., all its missing values can be successfully estimated. The main difference between the approach we propose and those already proposed [22], [23] is that the completion of a particular expert's incomplete fuzzy preference relation is carried out using only the information he/she provides and no other expert's information is needed. It is important to point out that because no information from external sources is used, the estimated information is consistent with the original expert's opinions. Finally, following the choice scheme proposed in [10], i.e., aggregation followed by exploitation, we will design a selection process for GDM problems with the incomplete fuzzy preference relations based on the concept of fuzzy majority and the AC-IOWA operator. The aggregation step of a GDM problem consists in combining the experts' individual preferences into a group collective one in such a way that it summarizes or reflects the properties contained in all the individual preferences. This aggregation is carried out by applying the proposed AC-IOWA operator. The exploitation phase transforms the global information about the alternatives into a global ranking of them. This can be done in different ways, the most common one being the use of a ranking method to obtain a score function. To do this, two quantifier-guided choice degrees of alternatives are used: the dominance and the nondominance degree.

The rest of the paper is set out as follows. In Section II, we present the GDM problem and its corresponding resolution process when working with the incomplete fuzzy preference relations. In Section III, AC measures for both complete and the incomplete fuzzy preference relations are defined. Section IV presents the iterative procedure to estimate the missing values of the incomplete fuzzy preference relations and the sufficient conditions to successfully estimate all the missing values. The AC-IOWA operator and a detailed description of the selection process used to solve GDM problems with the incomplete fuzzy preference relations are presented in Section V. Finally, an example as to how to apply the new decision-making model presented in this paper is given in Section VI and our concluding remarks will be pointed out in Section VII. The Appendix presents in greater detail the concept of fuzzy quantifiers to model the concept of fuzzy majority in the decision process.

II. GDM With Incomplete Fuzzy Preference Relations

The problem we deal with is that of choosing the best alternative(s) among a finite set, $X = \{x_1, \ldots, x_n\}$, $(n \geq 2)$. The alternatives will be classified from best to worst, using the information known according to a set of experts, i.e., $E = \{e_1, \ldots, e_m\}$, $(m \geq 2)$. Each expert $e_k \in E$, will provide his/her preferences by means of a fuzzy preference relation:

Definition 1 [20], [29]: A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, i.e., it is characterized by a membership function $\mu_P : X \times X \longrightarrow [0, 1]$.

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ij})$, being $p_{ij} = \mu_P(x_i, x_j) (\forall i, j \in \{1, \ldots, n\})$ interpreted as the preference degree or intensity of the alternative x_i over x_j : $p_{ij} = 1/2$ indicates indifference between x_i and $x_j (x_i \sim x_j)$, $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, and $p_{ij} > 1/2$ indicates that x_i is preferred to $x_j (x_i \succ x_j)$. Based on this interpretation, we have that $p_{ii} = 1/2$ $\forall i \in \{1, \ldots, n\}$ $(x_i \sim x_i)$.

As we have already mentioned, missing information is a problem that we have to deal with because usual decision-making procedures assume that experts are able to provide preference degrees between any pair of possible alternatives, which is not always possible. We note that a missing value in a fuzzy preference relation is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another, in which case he/she may decide not to "guess" to maintain the consistency of the values already provided. It must be clear then that when an expert is not able to express the particular value p_{ij}, because he/she does not have a clear idea of how better alternative x_i is over alternative x_j, this does not automatically mean that he/she prefers both options with the same intensity.

In order to model these situations, in the following definitions, we express the concept of the incomplete fuzzy preference relation:

Definition 2: A function $f : X \longrightarrow Y$ is partial when not every element in the set X necessarily maps onto an element in the set Y. When every element from the set X maps onto one element of the set Y, then we have a total function.

Definition 3: The incomplete fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$ that is characterized by a partial membership function.

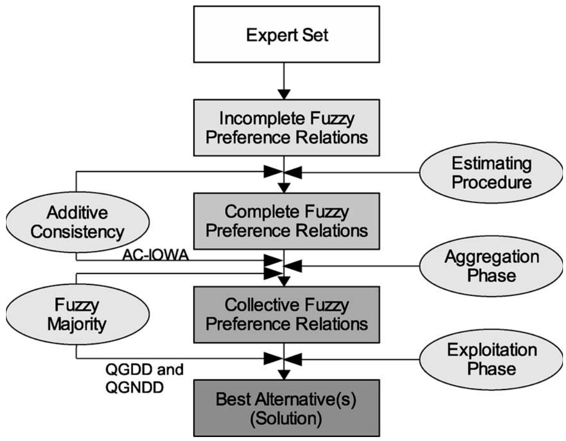

Fig. 1. Resolution process of a GDM with the incomplete fuzzy preference relations.

As per this definition, we call a fuzzy preference relation complete when its membership function is a total one. Clearly, Definition 1 includes both definitions of complete and the incomplete fuzzy preference relations. However, as there is no risk of confusion between a complete and the incomplete fuzzy preference relation, in this paper, we refer to the first type as simply fuzzy preference relation.

In this context, to obtain a set of solution alternatives $X_{\mathrm{sol}} \subset X$, the first step of a resolution process of GDM problems with the incomplete fuzzy preference relations might be the application of some kind of mechanism to infer or estimate the missing values. Therefore, the resolution process presents the scheme given in Fig. 1. Once the experts provide their (incomplete) preference relations, two main steps are applied.

1) Estimation of missing information. In this step, the incomplete fuzzy preference relations are completed. To do this, an iterative procedure to estimate the missing values of the incomplete fuzzy preference relation is presented in Section IV.
2) Application of a selection process, which is carried out in two sequential phases.
   a) *Aggregation phase.* A collective fuzzy preference relation is obtained by aggregating all completed individual fuzzy preference relations. This aggregation is carried out by applying a special type of IOWA operator [40]–[42], the AC-IOWA operator (Section V-A2), which is based on the concept of AC [31] (Section III) and that is guided by a linguistic quantifier representing the concept of fuzzy majority (of experts) desired to be implemented in the resolution process.
   b) *Exploitation phase.* Using again the concept of fuzzy majority (of alternatives), two choice degrees of alternatives are used: the quantifier-guided dominance degree (QGDD) and the quantifier-guided nondominance degree (QGNDD) [1]. These choice degrees will act over the collective preference relation resulting in a global ranking of the alternatives, from which the set of solution alternatives will be obtained.

The next section presents a new consistency measure for fuzzy preference relations based on the concept of AC. An extended measure for incomplete fuzzy preference is also given.

III. AC Measure

The previous Definition 1 of a fuzzy preference relation does not imply any kind of consistency property. In fact, preference values of a fuzzy preference relation can be contradictory. To make a rational choice, properties to be satisfied by such fuzzy preference relations have been suggested, among which we can cite [17]: triangle condition, weak transitivity, max–min transitivity, max–max transitivity, restricted max–min transitivity, restricted max–max transitivity, and additive transitivity.

As shown in [17], additive transitivity for fuzzy preference relations can be seen as the parallel concept of Saaty's consistency property for multiplicative preference relations [30]. The mathematical formulation of the additive transitivity was given by Tanino in [31]

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5), \qquad \forall i, j, k \in \{1, \ldots, n\}. \quad (1)$$

This kind of transitivity has the following interpretation: suppose we want to establish a ranking between three alternatives x_i, x_j, and x_k, and that the information available about these alternatives suggests that we are in an indifference situation, i.e., $x_i \sim x_j \sim x_k$. When giving preferences, this situation would be represented by $p_{ij} = p_{jk} = p_{ik} = 0.5$. Suppose now that we have a piece of information that says $x_i \prec x_j$, i.e., $p_{ij} < 0.5$. This means that p_{jk} or p_{ik} have to change; otherwise there would be a contradiction, because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$, then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must then conclude that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ij} = p_{ik}$, and so $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ij} - 0.5) = (p_{ik} - 0.5)$. We have the same conclusion if $p_{ik} = 0.5$. In the case of $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ik} has to be equal to or lower than p_{ij}, being equal only in the case of $p_{jk} = 0.5$, as we have already shown. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_i over x_k should be equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is, $p_{ik} - 0.5 = (p_{ij} - 0.5) + (p_{jk} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$.

Additive transitivity implies additive reciprocity. Indeed, because $p_{ii} = 0.5\ \forall i$, if we make $k = i$ in (1), then we have: $p_{ij} + p_{ji} = 1\ \forall i, j \in \{1, \ldots, n\}$. Therefore, the additive transitivity is the only property that we will assume throughout this paper.

Expression (1) can be rewritten as

$$p_{ik} = p_{ij} + p_{jk} - 0.5, \qquad \forall i, j, k \in \{1, \ldots, n\}. \quad (2)$$

We will consider a fuzzy preference relation to be "additive consistent" when for every three options in the problem $x_i, x_j, x_k \in X$ their associated preference degrees p_{ij}, p_{jk}, p_{ik}

fulfil (2). An additive consistent fuzzy preference relation will be referred as consistent throughout the paper, as this is the only transitivity property we are considering.

Expression (2) can be used to calculate the value of a preference degree using other preference degrees in a fuzzy preference relation. Indeed, the preference value $p_{ik} (i \neq k)$ can be estimated using an intermediate alternative x_j in three different ways.

1) From $p_{ik} = p_{ij} + p_{jk} - 0.5$, we obtain the estimate

$$cp_{ik}^{j1} = p_{ij} + p_{jk} - 0.5. \quad (3)$$

2) From $p_{jk} = p_{ji} + p_{ik} - 0.5$, we obtain the estimate

$$cp_{ik}^{j2} = p_{jk} - p_{ji} + 0.5. \quad (4)$$

3) From $p_{ij} = p_{ik} + p_{kj} - 0.5$, we obtain the estimate

$$cp_{ik}^{j3} = p_{ij} - p_{kj} + 0.5. \quad (5)$$

The values

$$\varepsilon p_{ik}^1 = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} \left| cp_{ik}^{j1} - p_{ik} \right|}{n-2} \quad (6)$$

$$\varepsilon p_{ik}^2 = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} \left| cp_{ik}^{j2} - p_{ik} \right|}{n-2} \quad (7)$$

$$\varepsilon p_{ik}^3 = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} \left| cp_{ik}^{j3} - p_{ik} \right|}{n-2} \quad (8)$$

represent average deviations of all $n-2$ possible estimates $cp_{ik}^{jl} (l \in \{1,2,3\})$ with respect to the actual value p_{ik}. When the information provided in a fuzzy preference relation is completely consistent then all $cp_{ik}^{jl} \in [0,1] (l \in \{1,2,3\};\ \forall j \in \{1,\ldots,n\})$ coincide with p_{ik}. However, because experts are not always fully consistent, the information given by an expert may not verify (2) and some of the estimated preference degree values cp_{ik}^{jl} may not belong to the unit interval [0, 1]. We note, from (3)–(5), what the maximum value of any of the preference degrees $cp_{ik}^{jl} (l \in \{1,2,3\})$ is 1.5 while the minimum one is -0.5, and therefore as $p_{ik} \in [0,1]$, $|cp_{ik}^{jl} - p_{ik}| \in [0, 1.5]$.

The value

$$\varepsilon p_{ik} = \frac{2}{3} \cdot \frac{\varepsilon p_{ik}^1 + \varepsilon p_{ik}^2 + \varepsilon p_{ik}^3}{3} \quad (9)$$

can be used to measure the error in [0, 1] expressed in a preference degree between two alternatives. Thus, it can be used to define the consistency level between the preference degree p_{ik} and the rest of the preference values of the fuzzy preference relation.

Definition 4: The consistency level associated with a preference value p_{ik} is defined as

$$\mathrm{CL}_{ik} = 1 - \varepsilon p_{ik}. \quad (10)$$

When $\mathrm{CL}_{ik} = 1$ then $\varepsilon p_{ik} = 0$ and there is no inconsistency at all. The lower the value of CL_{ik}, the higher the value of εp_{ik} and the more inconsistent is p_{ik} with respect to the rest of information.

In the following, we define the consistency level of the whole fuzzy preference relation.

Definition 5: The consistency level of a fuzzy preference relation P is defined as follows:

$$\mathrm{CL}_P = \frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \mathrm{CL}_{ik}}{n^2 - n} \quad (11)$$

with $\mathrm{CL}_P \in [0,1]$. When $\mathrm{CL}_P = 1$, the preference relation P is fully consistent; otherwise, the lower CL_P the more inconsistent P.

When working with the incomplete fuzzy preference relation, (6)–(11) cannot be used to estimate its consistency level, and therefore the above definitions of CL_{ik} and CL_P have to be extended. To do this, the following sets are introduced:

$$A = \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\}$$
$$\mathrm{MV} = \{(i,j) \in A \mid p_{ij} \text{ is unknown}\}$$
$$\mathrm{EV} = A \setminus \mathrm{MV}. \quad (12)$$

MV is the set of pairs of alternatives for which the preference degree of the first alternative over the second one is unknown or missing; EV is the set of pairs of alternatives for which the expert provides preference values. We do not take into account the preference value of one alternative over itself as this is always assumed to be equal to 0.5.

The following sets are also needed:

$$H_{ik}^1 = \{j \neq i,k \mid (i,j),(j,k) \in \mathrm{EV}\} \quad (13)$$
$$H_{ik}^2 = \{j \neq i,k \mid (j,i),(j,k) \in \mathrm{EV}\} \quad (14)$$
$$H_{ik}^3 = \{j \neq i,k \mid (i,j),(k,j) \in \mathrm{EV}\}. \quad (15)$$

H_{ik}^1, H_{ik}^2, H_{ik}^3 are the sets of intermediate alternative $x_j (j \neq i,k)$ that can be used to estimate the preference value $p_{ik} (i \neq k)$ using (3)–(5), respectively.

For a particular preference degree $p_{ik} ((i,k) \in \mathrm{EV})$, (6)–(9) can be redefined as

$$\varepsilon p_{ik}^1 = \begin{cases} \frac{\sum_{j \in H_{ik}^l} |cp_{ik}^{jl} - p_{ik}|}{\# H_{ik}^l}, & \text{if } \left(\# H_{ik}^l \neq 0\right); l \in \{1,2,3\} \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

and

$$\varepsilon p_{ik} = \frac{2}{3} \cdot \frac{\varepsilon p_{ik}^1 + \varepsilon p_{ik}^2 + \varepsilon p_{ik}^3}{\mathcal{K}} \quad (17)$$

with

$$\mathcal{K} = \begin{cases} 3, & \text{if } \left(\# H_{ik}^1 \neq 0\right) \wedge \left(\# H_{ik}^2 \neq 0\right) \wedge \left(\# H_{ik}^3 \neq 0\right) \\ 2, & \text{if } \left(\# H_{ik}^t = 0\right) \wedge \left(\left(\# H_{ik}^v \neq 0\right) \wedge \left(\# H_{ik}^w \neq 0\right)\right) \\ & \quad t,v,w \in \{1,2,3\}, t \neq v \neq w \\ 1, & \text{otherwise.} \end{cases} \quad (18)$$

In decision-making situations with incomplete information, the notion of completeness is also an important factor to take into account when analyzing the consistency. Clearly, the higher the number of preference values provided by an expert the higher the chance of inconsistency. Therefore, a degree of completeness associated with the number of preference values provided should also be taken into account to produce a fairer measure of consistency of the incomplete fuzzy preference relation.

Given the incomplete fuzzy preference relation, we can easily characterize two completeness levels, the completeness level of a relation, and the completeness level of an alternative. For the incomplete fuzzy preference relation P, its completeness level CP_P can be defined as the ratio of the number of preference values known $\#\mathrm{EV}$ to the total possible number of preference values $n^2 - n$

$$\mathrm{CP}_P = \frac{\#\mathrm{EV}}{n^2 - n}. \tag{19}$$

For an alternative x_i, we can define its completeness level CP_i as the ratio between the actual number of preference values known for x_i, $\#\mathrm{EV}_i(\mathrm{EV}_i \subseteq \mathrm{EV})$, and the total number of possible preference values in which x_i is involved with a different alternative $2(n-1)$

$$\mathrm{CP}_i = \frac{\#\mathrm{EV}_i}{2(n-1)}. \tag{20}$$

The consistency level CL_{ik}, associated with a preference value p_{ik}, $(i,k) \in \mathrm{EV}$, is defined as a linear combination of its associated error εp_{ik} and the average of the completeness values associated to the two alternatives involved in that preference degree CP_i and CP_k using a parameter $\alpha_{ik} \in [0,1]$ to control the influence of completeness in the evaluation of the consistency levels. This parameter α_{ik} should decrease with respect to the number of preference values known, in such a way that it takes the value of 0 when all the preference values in which x_i and x_k are involved are known, in which case the completeness concept lacks any meaning and, therefore, should not be taken into account; and it takes the value of 1 when no values are known.

The total number of different preference values involving the alternatives x_i and x_k is equal to $4(n-1)-2$: the total number of possible preference values involving $x_i(2(n-1))$ plus the total number of possible preference values involving $x_k(2(n-1))$ minus the common preference value involving x_i and x_k, p_{ik} and p_{ki}. The number of different preference values known for x_i and x_k is $\#\mathrm{EV}_i + \#\mathrm{EV}_k - \#(\mathrm{EV}_i \cap \mathrm{EV}_k)$. Thus, we claim that $\alpha_{ik} = f(\#\mathrm{EV}_i + \#\mathrm{EV}_k - \#(\mathrm{EV}_i \cap \mathrm{EV}_k))$, being f a decreasing function with $f(0) = 1$ and $f(4(n-1)-2) = 0$. This is summarized in the following definition.

Definition 6: The consistency level CL_{ik}, associated with a preference value p_{ik}, $(i,k) \in \mathrm{EV}$, is defined as a linear combination of its associated error εp_{ik} and the average of the completeness values associated to the two alternatives involved in that preference degree CP_i and CP_k

$$\mathrm{CL}_{ik} = (1-\alpha_{ik}) \cdot (1-\varepsilon p_{ik}) + \alpha_{ik} \cdot \frac{\mathrm{CP}_i + \mathrm{CP}_k}{2}, \quad \alpha_{ik} \in [0,1] \tag{21}$$

with $\alpha_{ik} = f(\#\mathrm{EV}_i + \#\mathrm{EV}_k - \#(\mathrm{EV}_i \cap \mathrm{EV}_k))$, being f a decreasing function with $f(0) = 1$ and $f(4(n-1)-2) = 0$.

In the above definition, the simple linear solution could be used to obtain the parameter α_{ik}

$$\alpha_{ik} = 1 - \frac{\#\mathrm{EV}_i + \#\mathrm{EV}_k - \#(\mathrm{EV}_i \cap \mathrm{EV}_k)}{4(n-1)-2}. \tag{22}$$

In the following, we define the consistency level of the incomplete fuzzy preference relation.

Definition 7: The consistency level of the incomplete fuzzy preference relation P is defined as follows:

$$\mathrm{CL}_P = \frac{\sum_{(i,k)\in \mathrm{EV}} \mathrm{CL}_{ik}}{\#\mathrm{EV}}. \tag{23}$$

Clearly, this redefinition of CL_P is an extension of (11), because when P is complete both EV and A coincide and thus: $\#\mathrm{EV} = n^2 - n$, $\#H_{ik}^1 = \#H_{ik}^2 = \#H_{ik}^3 = n-2$, and $\alpha_{ik} = 0\ \forall i,k$.

IV. Estimating Missing Values in Incomplete Fuzzy Preference Relations Using AC

As we have already mentioned, missing information is a problem that has to be addressed because experts are not always able to provide preference degrees between every pair of possible alternatives. Nevertheless, in this section, we will show that these values can be estimated from the existing information.

Usual procedures for GDM problems correct this lack of knowledge of a particular expert using the information provided by the rest of experts together with aggregation procedures [23]. These kind of approaches have several disadvantages. Among them, we can cite the requirement of multiple experts in order to estimate the missing value of a particular expert. Another drawback is that these procedures do not usually take into account the differences between experts' preferences, which could lead to the estimation of a missing value that would not naturally be compatible with the rest of the preference values given by that expert. Finally, some of these missing information retrieval procedures are interactive, that is, they need experts to collaborate in "real time," an option which is not always possible.

Our proposal is quite different to the above approaches. We put forward a procedure that estimates missing information in an expert's incomplete fuzzy preference relation using only the rest of the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by the expert.

Next, we present the iterative procedure to estimate missing values of the incomplete fuzzy preference relations and sufficient conditions to guarantee the successful estimation of all the missing values.

A. Iterative Procedure to Estimate Missing Values

In order to develop the iterative procedure to estimate missing values, the following two different tasks have to be carried out.

1) Establish the elements that can be estimated in each step of the procedure.
2) Produce the particular expression that will be used to estimate a particular missing value.

1) Elements to Be Estimated in Step h: The subset of missing values MV that can be estimated in step h of our procedure is denoted by EMV_h (estimated missing values) and defined as follows:

$$\mathrm{EMV}_h = \left\{ (i,k) \in \ \mathrm{MV} \ \setminus \bigcup_{l=0}^{h-1} \mathrm{EMV}_l \mid i \neq k \ \wedge \ \exists j \in \left\{ H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3} \right\} \right\} \tag{24}$$

with

$$H_{ik}^{h1} = \left\{ j \mid (i,j),(j,k) \in \left\{ \mathrm{EV} \bigcup_{l=0}^{h-1} \mathrm{EMV}_l \right\} \right\} \tag{25}$$

$$H_{ik}^{h2} = \left\{ j \mid (j,i),(j,k) \in \left\{ \mathrm{EV} \bigcup_{l=0}^{h-1} \mathrm{EMV}_l \right\} \right\} \tag{26}$$

$$H_{ik}^{h3} = \left\{ j \mid (i,j),(k,j) \in \left\{ \mathrm{EV} \bigcup_{l=0}^{h-1} \mathrm{EMV}_l \right\} \right\} \tag{27}$$

and $\mathrm{EMV}_0 = \emptyset$ (by definition). When $\mathrm{EMV}_{\mathrm{maxIter}} = \emptyset$ with $\mathrm{maxIter} > 0$ the procedure will stop as there will not be any more missing values to be estimated. Moreover, if $\bigcup_{l=0}^{\mathrm{maxIter}} \mathrm{EMV}_l = \mathrm{MV}$ then all missing values are estimated, and consequently, the procedure is said to be successful in the completion of the incomplete fuzzy preference relation.

2) Expression to Estimate a Particular Value p_{ik} in Step h: In order to estimate a particular value p_{ik} with $(i,k) \in \mathrm{EMV}_h$, we propose the application of the following function:

function estimate_$p(i,k)$

1) $cp_{ik}^1 = 0$, $cp_{ik}^2 = 0$, $cp_{ik}^3 = 0$.
2) $cp_{ik}^1 = ((\sum_{j \in H_{ik}^{h1}} cp_{ik}^{j1}) / \#H_{ik}^{h1})$ if $\#H_{ik}^{h1} \neq 0$.
3) $cp_{ik}^2 = ((\sum_{j \in H_{ik}^{h2}} cp_{ik}^{j2}) / \#H_{ik}^{h2})$ if $\#H_{ik}^{h2} \neq 0$.
4) $cp_{ik}^3 = ((\sum_{j \in H_{ik}^{h3}} cp_{ik}^{j3}) / \#H_{ik}^{h3})$ if $\#H_{ik}^{h3} \neq 0$.
5) Calculate $cp_{ik} = (1/\mathcal{K})(cp_{ik}^1 + cp_{ik}^2 + cp_{ik}^3)$.

end function.

The function estimate_$p(i,k)$ computes the final estimated value of the missing value cp_{ik} as the average of all estimated values that can be calculated using all the possible intermediate alternatives x_j and using the three possible expressions (3)–(5).

We should point out that some estimated values of the incomplete fuzzy preference relation could lie outside the unit interval, i.e., for some (i,k), we may have $cp_{ik} < 0$ or $cp_{ik} > 1$. In order to normalize the expression domains in the decision model, the following function is used

$$f(y) = \begin{cases} 0, & \text{if } y < 0 \\ 1, & \text{if } y > 1 \\ y, & \text{otherwise.} \end{cases}$$

We also point out that the consistency level CL_{ik} of an estimated preference value cp_{ik}, if necessary, may be defined as the average of the consistency levels of all the preference values used to estimate it and computed easily inside the function estimate_$p(i,k)$.

The iterative estimation procedure pseudocode is as follows:

```
ITERATIVE ESTIMATION PROCEDURE
EMV_0 = ∅
h = 1
while EMV_h ≠ ∅ {
    for every (i,k) ∈ EMV_h {
            estimate_p(i,k)
    }
    h++
}
```

B. Sufficient Conditions to Estimate All Missing Values

It is very important to establish conditions that guarantee that all the missing values of the incomplete fuzzy preference relation can be estimated. We assume that experts provide their judgements freely by means of the incomplete fuzzy preference relations with preferences degrees $p_{ik} \in [0,1]$ and $p_{ii} = 0.5$, without any other restriction, as for example, that of imposing the additive reciprocity property.

In the following, we provide sufficient conditions that guarantee the success of the above iterative estimation procedure.

1) If for all $p_{ik} \in \mathrm{MV}(i \neq k)$ there exists at least a $j \in \{H_{ik}^1 \cup H_{ik}^2 \cup H_{ik}^3\}$, then all missing preference values can be estimated in the first iteration of the procedure $(\mathrm{EMV}_1 = \mathrm{MV})$.
2) Under the assumption of AC property, a different sufficient condition was given in [17]. This condition states that any incomplete fuzzy preference relation can be completed when the set of $n-1$ preference values $\{p_{12}, p_{23}, \ldots, p_{(n-1)n}\}$ is known.
3) A more general condition than the previous one, which includes that when a complete row or column of preference values is known, is given in the following proposition.

Proposition 1: The incomplete fuzzy preference relation can be completed if a set of $n-1$ nonleading diagonal preference values, where each one of the alternatives is compared at least once, is known.

Proof: Proof by induction on the number of alternatives will be used.

1) *Basis:* For $n = 3$, we suppose that two preference degrees involving the three alternatives are known. These degrees can be provided in three different ways.

a) p_{ij} and $p_{jk} (i \neq j \neq k)$ are given: In this first case, all the possible combinations of the two preference values are: $\{p_{12}, p_{23}\}$, $\{p_{13}, p_{32}\}$, $\{p_{21}, p_{13}\}$, $\{p_{23}, p_{31}\}$, $\{p_{31}, p_{12}\}$, and $\{p_{32}, p_{21}\}$. In any of these cases, we can find the remaining preference degrees of the relation $\{p_{ik}, p_{kj}, p_{ji}, p_{ki}\}$ as follows:

$$p_{ik} = p_{ij} + p_{jk} - 0.5 \quad p_{kj} = p_{ik} - p_{ij} + 0.5$$
$$p_{ji} = p_{jk} - p_{ik} + 0.5 \quad p_{ki} = p_{kj} - p_{ij} + 0.5.$$

b) p_{ji} and $p_{jk} (i \neq j \neq k)$ are given: In this second case, all the possible combinations of the two preference values are: $\{p_{21}, p_{23}\}$, $\{p_{31}, p_{32}\}$, and $\{p_{12}, p_{13}\}$. In any of these cases, we can find the remaining preference degrees of the relation $\{p_{ik}, p_{ki}, p_{kj}, p_{ij}\}$ as follows:

$$p_{ik} = p_{jk} - p_{ji} + 0.5 \quad p_{ki} = p_{ji} - p_{jk} + 0.5$$
$$p_{kj} = p_{ki} - p_{ji} + 0.5 \quad p_{ij} = p_{kj} - p_{ki} + 0.5.$$

c) p_{ij} and $p_{kj} (i \neq j \neq k)$ are given: In this third case, all the possible combinations of the two preference values are: $\{p_{12}, p_{32}\}$, $\{p_{13}, p_{23}\}$ and $\{p_{21}, p_{31}\}$. In any of these cases, we can find the remaining preference degrees of the relation $\{p_{ik}, p_{ki}, p_{ji}, p_{jk}\}$ as follows:

$$p_{ik} = p_{ij} - p_{kj} + 0.5 \quad p_{ki} = p_{kj} - p_{ij} + 0.5$$
$$p_{ji} = p_{ki} - p_{kj} + 0.5 \quad p_{jk} = p_{ik} - p_{ij} + 0.5.$$

2) *Induction hypothesis:* Let us assume that the proposition is true for $n = q - 1$.
3) *Induction step:* Let us suppose that the expert provides only $q - 1$ preference degrees where each one of the q alternatives is compared at least once.

In this case, we can select a set of $q - 2$ preference degrees where $q - 1$ different alternatives are involved. Without loss of generality, we can assume that these $q - 1$ alternatives are $x_1, x_2, \ldots, x_{q-1}$, and therefore the remaining preference degree involving the alternative x_q could be $p_{qi} (i \in \{1, \ldots, q - 1\})$ or $p_{iq} (i \in \{1, \ldots, q - 1\})$.

By the induction hypothesis, we can estimate all the preference values of the fuzzy preference relation of order $(q - 1) \times (q - 1)$ associated with the set of alternatives $\{x_1, x_2, \ldots, x_{q-1}\}$. Therefore, we have estimates for the following set of preference degrees:

$$\{p_{ij},\ i, j = 1, \ldots, q - 1,\ i \neq j\}.$$

If the value we know is $p_{qi}, i \in \{1, \ldots, q - 1\}$, then we can estimate $\{p_{qj}, j = 1, \ldots, q - 1, i \neq j\}$ and $\{p_{jq}, j = 1, \ldots, q - 1\}$ using $p_{qj} = p_{qi} + p_{ij} - 0.5, \forall j$ and $p_{jq} = p_{ji} - p_{qi} + 0.5, \forall j$, respectively. If the value we know is $p_{iq}, i \in \{1, \ldots, q - 1\}$ then $\{p_{qj}, j = 1, \ldots, q - 1\}$ and $\{p_{jq}, j = 1, \ldots, q - 1, i \neq j\}$, are estimated by means of $p_{qj} = p_{ij} - p_{iq} + 0.5, \forall j$ and $p_{jq} = p_{ji} + p_{iq} - 0.5, \forall j$, respectively. ■

V. SELECTION PROCESS

The selection process we present consists of two phases: 1) aggregation and 2) exploitation. The aggregation phase defines a collective fuzzy preference relation, which indicates the global preference between every ordered pair of alternatives, while the exploitation phase transforms the global information about the alternatives into a global ranking of them, from which a selection set of alternatives is derived.

A. Aggregation: The Collective Fuzzy Preference Relation

Once we have estimated all the missing values in every incomplete fuzzy preference relation, we have a set of m individual fuzzy preference relations $\{P^1, \ldots, P^m\}$. From this set, a collective fuzzy preference relation $P^c = (p^c_{ik})$ must be derived by means of an aggregation procedure. In our case, each value $p^c_{ik} \in [0, 1]$ will represent the preference of alternative x_i over alternative x_k according to the majority of the most consistent experts' opinions.

Clearly, a rational assumption in the resolution process of a GDM is that of associating more importance to the experts who provide the most consistent information. This assumption implies that GDM problems should be viewed as heterogeneous. Indeed, in any GDM problem with the incomplete fuzzy preference relations, each expert e_h can have an importance degree associated with him/her, which, for example, can be his/her own consistency level of the relation CL_{P^h} or consistency levels of the preference values CL^h_{ik} in each preference value p_{ik}.

Usually, procedures for the inclusion of these importance values in the aggregation process involve the transformation of the preference values p^h_{ik} under the importance degree I^h, to generate a new value, $\overline{p}^h_{ik}$ [11], [14]. This activity is carried out by means of a transformation function g, $\overline{p}^h_{ik} = g(p^h_{ik}, I^h)$. Examples of functions g used in these cases include the minimum operator [14], the exponential function $g(x, y) = x^y$ [35], or generally any t-norm operator. In our case, we apply an alternative approach that consists of using importance degrees or consistency levels as the order inducing values of the IOWA operator [41] to be applied in the aggregation stage of the selection process. In the next sections, we explain how this is done.

1) Ordered Weighted Averaging (OWA) and IOWA Operators: Yager in [37] introduced the OWA operator, which is commutative, idempotent, continuous, monotonic, neutral, compensative, and stable for positive linear transformations. A fundamental aspect of the OWA operator is the reordering of the arguments to be aggregated, based upon the magnitude of their respective values.

Definition 8 [37]: An OWA operator of dimension n is a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, which has a set of weights cor weighting vector associated with it, $W = (w_1, \ldots, w_n)$, with $w_i \in [0, 1]$, $\sum_{i=1}^n w_i = 1$, and it is defined to aggregate a list of values $\{p_1, \ldots, p_n\}$ according to the following expression:

$$\phi_W(p_1, \ldots, p_n) = \sum_{i=1}^n w_i \cdot p_{\sigma(i)} \tag{28}$$

being $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ a permutation such that $p_{\sigma(i)} \geq p_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $p_{\sigma(i)}$ is the i highest value in the set $\{p_1, \ldots, p_n\}$.

A natural question in the definition of the OWA operator is how to obtain the associated weighting vector. In [37], Yager proposed two ways to obtain it. The first approach is to use some kind of learning mechanism using some sample data; while the second one tries to give some semantics or meaning to the weights. The latter possibility has allowed multiple applications in the area we are interested in, quantifier-guided aggregation [36].

In the process of quantifier-guided aggregation, given a collection of n criteria represented as fuzzy subsets of the alternatives X, the OWA operator is used to implement the concept of fuzzy majority in the aggregation phase by means of a fuzzy linguistic quantifier [43], which indicates the proportion of satisfied criteria "necessary for a good solution" [39] (see the Appendix for more details). This implementation is done by using the quantifier to calculate the OWA weights. In the case of a regular increasing monotone (RIM) quantifier Q, the procedure to evaluate the overall satisfaction of Q criteria (or experts) (e_k) by the alternative x_j is carried out calculating the OWA weights as follows:

$$w_i = Q(i/n) - Q\left((i-1)/n\right), \quad i = 1, \ldots, n. \tag{29}$$

When a fuzzy quantifier Q is used to compute the weights of the OWA operator ϕ, then it is symbolized by ϕ_Q. We make note that this type of aggregation "is very strongly dependent upon the weighting vector used" [39], and consequently also upon the function expression used to represent the fuzzy linguistic quantifier.

In [39], Yager also proposed a procedure to evaluate the overall satisfaction of Q important (u_k) criteria (or experts) (e_k) by the alternative x_j. In this procedure, once the satisfaction values to be aggregated have been ordered, the weighting vector associated with an OWA operator using a linguistic quantifier Q are calculated following the expression

$$w_i = Q\left(\frac{\sum_{k=1}^{i} u_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{i-1} u_{\sigma(k)}}{T}\right) \tag{30}$$

being $T = \sum_{k=1}^{n} u_k$ the total sum of importance, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with a zero importance degree. In our case, the consistency levels of the fuzzy preference relations are used to derive the "importance" values associated with the experts.

Inspired by the work of Mitchell and Estrakh [26], Yager and Filev in [41] defined the IOWA operator as an extension of the OWA operator to allow a different reordering of the values to be aggregated.

Definition 9 [41]: An IOWA operator of dimension n is a function $\Phi_W : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, with $w_i \in [0,1]$, $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n two-tuples $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ according to the following expression:

$$\Phi_W\left(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\right) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)} \tag{31}$$

being σ a permutation of $\{1, \ldots, n\}$ such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the two-tuple with $u_{\sigma(i)}$, the ith highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition, the reordering of the set of values to be aggregated $\{p_1, \ldots, p_n\}$ is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated with them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable and $\{p_1, \ldots, p_n\}$ the values of the argument variable [40]–[42].

Clearly, the aforementioned approaches to calculate the weighting vector of an OWA operator can also be applied to the case of IOWA operators. When a fuzzy linguistic quantifier Q is used to compute the weights of the IOWA operator Φ, then it is symbolized by Φ_Q.

2) AC-Based IOWA Operator: Definition 9 allows the construction of many different operators. Indeed, the set of consistency levels of the relations, $\{\mathrm{CL}_{P^1}, \ldots, \mathrm{CL}_{P^m}\}$, or the set of consistency levels of the preference values, $\{\mathrm{CL}_{ik}^1, \ldots, \mathrm{CL}_{ik}^m\}$, may be used not just to associate "importance" values to the experts $E = \{e_1, \ldots, e_m\}$ but also to define an IOWA operator, i.e., the ordering of the preference values to be aggregated $\{p_{ik}^1, \ldots, p_{ik}^m\}$ can be induced by ordering the experts from the most to the least consistent one. In this case, we obtain an IOWA operator that we call the AC-IOWA operator and denote it as Φ_W^{AC}. This new operator can be viewed as an extension of the consistency IOWA (C-IOWA) operator defined in [4].

Definition 10: The AC-IOWA operator of dimension m, Φ_W^{AC}, is an IOWA operator whose set of order inducing values is $\{\mathrm{CL}_{P^1}, \ldots, \mathrm{CL}_{P^m}\}$ or $\{\mathrm{CL}_{ik}^1, \ldots, \mathrm{CL}_{ik}^m\}$.

Because an expert may be consistent in some of his preferences and inconsistent in others, our aggregation process is carried out using an AC-IOWA operator guided by the set of consistency levels of preference values, i.e., $\{\mathrm{CL}_{ik}^1, \ldots, \mathrm{CL}_{ik}^m\}$. Therefore, the collective fuzzy preference relation is obtained as follows:

$$p_{ik}^c = \Phi_Q^{\mathrm{AC}}\left(\left\langle \mathrm{CL}_{ik}^1, p_{ik}^1\right\rangle, \ldots, \langle \mathrm{CL}_{ik}^m, p_{ik}^m\rangle\right) \tag{32}$$

where Q is the fuzzy quantifier used to implement the fuzzy majority concept and, using (30), to compute the weighting vector of the AC-IOWA operator Φ_Q^{AC}.

B. Exploitation: Choosing the Alternative(s)

At this point, in order to select the alternative(s) "best" acceptable for the majority (Q) of the most consistent experts, we propose two quantifier-guided choice degrees of alternatives, a dominance, and a nondominance degree, which can be applied according to different selection policies.

1) Choice Degrees of Alternatives:

QGDD_i: The quantifier-guided dominance degree quantifies the dominance that one alternative has over all the others in a fuzzy majority sense and is defined as follows:

$$\mathrm{QGDD}_i = \phi_Q \left(p^c_{i1}, p^c_{i2}, \ldots, p^c_{i(i-1)}, p^c_{i(i+1)}, \ldots, p^c_{in} \right). \quad (33)$$

QGNDD_i: The quantifier-guided nondominance degree gives the degree in which each alternative is not dominated by a fuzzy majority of the remaining alternatives. Its expression being

$$\mathrm{QGNDD}_i = \phi_Q \left(1 - p^s_{1i}, 1 - p^s_{2i}, \ldots, 1 - p^s_{(i-1)i}, 1 - p^s_{(i+1)i}, \ldots, 1 - p^s_{ni} \right) \quad (34)$$

where $p^s_{ji} = \max\{p^c_{ji} - p^c_{ij}, 0\}$ represents the degree in which x_i is strictly dominated by x_j. When the fuzzy quantifier represents the statement "all," whose algebraic aggregation corresponds to the conjunction operator min, this nondominance degree coincides with Orlovski's nondominated alternative concept [29].

2) Selection Policies: The application of the above choice degrees of alternatives over X may be carried out according to two different policies.

— *Sequential policy:* One of the choice degrees is selected and applied to X according to the preference of the experts, obtaining a selection set of alternatives. If there is more than one alternative in this selection set, then the other choice degree is applied to select the alternative of this set with the best second choice degree.

— *Conjunctive policy:* Both choice degrees are applied to X, obtaining two selection sets of alternatives. The final selection set of alternatives is obtained as the intersection of these two selection sets of alternatives.

The latter conjunction selection process is more restrictive than the former sequential selection process because it is possible to obtain an empty selection set. Therefore, in a complete selection process the choice degrees can be applied in three steps.

Step 1) The application of each choice degree of alternatives over X to obtain the following sets of alternatives:

$$X^{\mathrm{QGDD}} = \left\{ x_i \in X \mid \mathrm{QGDD}_i = \sup_{x_j \in X} \mathrm{QGDD}_j \right\} \quad (35)$$

$$X^{\mathrm{QGNDD}} = \left\{ x_i \in X \mid \mathrm{QGNDD}_i = \sup_{x_j \in X} \mathrm{QGNDD}_j \right\} \quad (36)$$

whose elements are called maximum dominance elements on the fuzzy majority of X quantified by Q and maximal nondominated elements by the fuzzy majority of X quantified by Q, respectively.

Step 2) The application of the conjunction selection policy, obtaining the following set of alternatives:

$$X^{\mathrm{QGCP}} = X^{\mathrm{QGDD}} \bigcap X^{\mathrm{QGNDD}}. \quad (37)$$

If $X^{\mathrm{QGCP}} \neq \emptyset$, then End. Otherwise, continue.

Step 3) The application of the one of the two sequential selection policies, according to either a dominance or nondominance criterion, i.e., the following.

a) Dominance-based sequential selection process QG-DD-NDD. To apply the quantifier-guided dominance degree over X, and obtain X^{QGDD}. If $\#(X^{\mathrm{QGDD}}) = 1$, then End, and this is the solution set. Otherwise, continue obtaining

$$X^{\mathrm{QG-DD-NDD}} = \left\{ x_i \in X^{\mathrm{QGDD}} | \mathrm{QGNDD}_i = \sup_{x_j \in X^{\mathrm{QGDD}}} \mathrm{QGNDD}_j \right\}. \quad (38)$$

This is the selection set of alternatives.

b) Nondominance-based sequential selection process QG-NDD-DD. To apply the quantifier-guided nondominance degree over X, and obtain X^{QGNDD}. If $\#(X^{\mathrm{QGNDD}}) = 1$, then End, and this is the solution set. Otherwise, continue obtaining

$$X^{\mathrm{QG-NDD-DD}} = \left\{ x_i \in X^{\mathrm{QGNDD}} | \mathrm{QGDD}_i = \sup_{x_j \in X^{\mathrm{QGNDD}}} |\mathrm{QGDD}_j \right\}. \quad (39)$$

This is the selection set of alternatives.

VI. Illustrative Example

For the sake of simplicity, we will assume a low number of experts and alternatives. Let us suppose that four different experts $\{e_1, e_2, e_3, e_4\}$ provide the following fuzzy preference relations over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ x & - & x & x \\ x & x & - & x \\ x & x & x & - \end{pmatrix}$$

$$P^2 = \begin{pmatrix} - & x & \mathbf{0.7} & x \\ \mathbf{0.4} & - & x & \mathbf{0.6} \\ \mathbf{0.3} & x & - & x \\ x & \mathbf{0.4} & x & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & \mathbf{0.3} & \mathbf{0.5} & \mathbf{0.75} \\ \mathbf{0.7} & - & \mathbf{0.7} & \mathbf{0.9} \\ \mathbf{0.5} & \mathbf{0.3} & - & \mathbf{0.7} \\ \mathbf{0.25} & \mathbf{0.1} & \mathbf{0.3} & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & x & \mathbf{0.6} & \mathbf{0.3} \\ \mathbf{0.4} & - & \mathbf{0.4} & \mathbf{0.3} \\ \mathbf{0.4} & \mathbf{0.6} & - & \mathbf{0.3} \\ \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & - \end{pmatrix}.$$

A. Estimation of Missing Values

Three given preference relations are incomplete $\{P^1, P^2, P^4\}$. For P^1, there are just three known values; because they involve all four alternatives, then all the missing values can be successfully estimated:

Step 1) The set of elements that can be estimated are

$$\mathrm{EMV}_1 = \{(2,3),(2,4),(3,2),(3,4),(4,2),(4,3)\}\}.$$

After these elements have been estimated, we have

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ x & - & 0.9 & 0.7 \\ x & 0.1 & - & 0.3 \\ x & 0.3 & 0.7 & - \end{pmatrix}.$$

As an example, to estimate p_{43}, the procedure is as follows:

$$\begin{aligned} H_{43}^{11} &= \emptyset \quad \Rightarrow cp_{43}^1 = 0 \\ H_{43}^{12} &= \{1\} \Rightarrow cp_{43}^2 = cp_{43}^{12} = p_{13} - p_{14} + 0.5 = 0.7 \\ H_{43}^{23} &= \emptyset \quad \Rightarrow cp_{43}^3 = 0 \\ \mathcal{K} &= 1 \quad \Rightarrow cp_{43} = \frac{0+0.7+0}{1} = 0.7. \end{aligned}$$

Step 2) The set of elements that can be estimated are

$$\mathrm{EMV}_1 = \{(2,1),(3,1),(4,1)\}\}.$$

After these elements have been estimated, we have the following completed fuzzy preference relation:

$$P^1 = \begin{pmatrix} - & \mathbf{0.2} & \mathbf{0.6} & \mathbf{0.4} \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix}.$$

As an example, to estimate p_{41}, the procedure is as follows:

$$\begin{aligned} H_{41}^{21} &= \emptyset \quad \Rightarrow cp_{41}^1 = 0 \\ H_{41}^{22} &= \emptyset \quad \Rightarrow cp_{41}^2 = 0 \\ H_{41}^{23} &= \{2,3\} \Rightarrow cp_{41}^{23} = cp_{41}^{33} = 0.6 \Rightarrow cp_{41}^3 = 0.6 \\ \mathcal{K} &= 1 \quad \Rightarrow cp_{41} = \frac{0+0+0.6}{1} = 0.6. \end{aligned}$$

The corresponding consistency level matrix associated with the incomplete fuzzy preference relation P^1 is calculated as follows:

$$\mathrm{EV}_1 = \{(1,2),(1,3),(1,4)\}; \quad \mathrm{EV}_2 = \{(1,2)\}$$
$$\mathrm{EV}_3 = \{(1,3)\}; \quad \mathrm{EV}_4 = \{(1,4)\}$$
$$\mathrm{CP}_1 = 3/6;\ \mathrm{CP}_2 = \mathrm{CP}_3 = \mathrm{CP}_4 = 1/6$$
$$\alpha_{12} = \alpha_{13} = \alpha_{12} = \alpha_{12} = 1 - \frac{3+1-1}{10} = 0.7.$$

For p_{12}, we have that there is no intermediate alternative to calculate an estimated value and, consequently, we have

$$\varepsilon p_{12} = 0 \Rightarrow \mathrm{CL}_{12}^1 = (1-0.7)\cdot(1-0) + 0.7 \cdot \frac{\frac{3}{6}+\frac{1}{6}}{2} \approx 0.53.$$

The same result is obtained for p_{13} and p_{14}, i.e., $\mathrm{CL}_{13}^1 = \mathrm{CL}_{14}^1 \approx 0.53$. This means that the consistency level for each one of the estimated values is also 0.53, as they are calculated as the average of the consistency values used to estimate them. Consequently, we have

$$\mathrm{CL}^1 = \begin{pmatrix} - & 0.53 & 0.53 & 0.53 \\ 0.53 & - & 0.53 & 0.53 \\ 0.53 & 0.53 & - & 0.53 \\ 0.53 & 0.53 & 0.53 & - \end{pmatrix}.$$

For P^2, P^3, and P^4, we get

$$P^2 = \begin{pmatrix} - & 0.6 & \mathbf{0.7} & 0.7 \\ \mathbf{0.4} & - & 0.6 & \mathbf{0.6} \\ \mathbf{0.3} & 0.4 & - & 0.5 \\ 0.3 & \mathbf{0.4} & 0.5 & - \end{pmatrix}$$

$$\mathrm{CL}^2 = \begin{pmatrix} - & 0.62 & 0.59 & 0.67 \\ 0.75 & - & 0.64 & 0.59 \\ 0.59 & 0.67 & - & 0.62 \\ 0.64 & 0.59 & 0.62 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & \mathbf{0.3} & \mathbf{0.5} & \mathbf{0.75} \\ \mathbf{0.7} & - & \mathbf{0.7} & \mathbf{0.9} \\ \mathbf{0.5} & \mathbf{0.3} & - & \mathbf{0.7} \\ \mathbf{0.25} & \mathbf{0.1} & \mathbf{0.3} & - \end{pmatrix}$$

$$\mathrm{CL}^3 = \begin{pmatrix} - & 0.98 & 0.98 & 0.97 \\ 0.98 & - & 1.0 & 0.98 \\ 0.98 & 1.0 & - & 0.98 \\ 0.97 & 0.98 & 0.98 & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & 0.6 & \mathbf{0.6} & \mathbf{0.3} \\ \mathbf{0.4} & - & \mathbf{0.4} & \mathbf{0.3} \\ \mathbf{0.4} & \mathbf{0.6} & - & \mathbf{0.3} \\ \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & - \end{pmatrix}$$

$$\mathrm{CL}^4 = \begin{pmatrix} - & 0.93 & 0.93 & 0.93 \\ 0.92 & - & 0.93 & 0.93 \\ 0.93 & 0.93 & - & 0.93 \\ 0.93 & 0.93 & 0.93 & - \end{pmatrix}.$$

As an example of how the iterative estimation procedure has worked over P^4, we show the estimate for p_{12}, the only missing value in this preference relation

$$\begin{aligned} H_{12}^{11} &= \{3,4\} \Rightarrow cp_{12}^{31} = 0.7 \quad cp_{12}^{41} = 0.5 \Rightarrow cp_{12}^1 = 0.6 \\ H_{12}^{12} &= \{3,4\} \Rightarrow cp_{12}^{32} = 0.7 \quad cp_{12}^{42} = 0.5 \Rightarrow cp_{12}^2 = 0.6 \\ H_{12}^{13} &= \{3,4\} \Rightarrow cp_{12}^{33} = 0.7 \quad cp_{12}^{43} = 0.5 \Rightarrow cp_{12}^3 = 0.6 \end{aligned}$$
$$cp_{12} = \frac{0.6+0.6+0.6}{3} = 0.6.$$

The following calculations are needed to obtain the consistency level CL_{21}.

1) Computation of εp_{21}

$$H_{21}^{11} = \{3,4\} \Rightarrow cp_{21}^{31} = 0.3 \quad cp_{21}^{41} = 0.5 \Rightarrow \varepsilon p_{21}^{1} = 0.1$$
$$H_{21}^{12} = \{3,4\} \Rightarrow cp_{21}^{32} = 0.3 \quad cp_{21}^{42} = 0.5 \Rightarrow \varepsilon p_{21}^{1} = 0.1$$
$$H_{21}^{13} = \{3,4\} \Rightarrow cp_{21}^{33} = 0.3 \quad cp_{21}^{43} = 0.5 \Rightarrow \varepsilon p_{21}^{1} = 0.1$$
$$\varepsilon p_{12} = \frac{2}{3} \cdot \frac{0.1+0.1+0.1}{3} = \frac{0.2}{3}.$$

2) Computation of CP_1, CP_2, and α_{21}

$$EV_1 = \{(1,3),(1,4),(2,1),(3,1),(4,1)\} \Rightarrow P_1 = \frac{5}{6}$$
$$EV_2 = \{(2,1),(2,3),(2,4),(3,2),(4,2)\} \Rightarrow CP_2 = \frac{5}{6}$$
$$EV_1 \cap EV_2 = \{(2,1)\} \Rightarrow \alpha_{21} = 1 - \frac{5+5-1}{10} = \frac{1}{10}.$$

3) Computation of CL_{21}

$$CL_{21} = \left(1 - \frac{1}{10}\right) \cdot \left(1 - \frac{0.2}{3}\right) + \frac{1}{10} \cdot \frac{\frac{5}{6}+\frac{5}{6}}{2} \approx 0.92.$$

We should point out that because the third and fourth experts have been very consistent when expressing their preferences and have provided an almost complete preference relation, the consistency levels for their preference values are very high.

B. Aggregation Phase

Once the fuzzy preference relations are completed, we aggregate them by means of the AC-IOWA operator and using the consistency level of the preference values as the order-inducing variable. We make use of the linguistic quantifier "most of," represented by the RIM quantifier $Q(r) = r^{1/2}$ (see the Appendix), which applying (30) generates a weighting vector of four values to obtain each collective preference value p_{ik}^c.

As an example, the collective preference value p_{12}^c with two decimal places is obtained as follows:

$$CL_{12}^1 = 0.53 \quad CL_{12}^2 = 0.62 \quad CL_{12}^3 = 0.98 \quad CL_{12}^4 = 0.93$$
$$p_{12}^1 = 0.2 \quad p_{12}^2 = 0.6 \quad p_{12}^3 = 0.3 \quad p_{12}^4 = 0.6$$
$$\sigma(1) = 3 \quad \sigma(2) = 4 \quad \sigma(3) = 2 \quad \sigma(4) = 1$$
$$T = CL_{12}^1 + CL_{12}^2 + CL_{12}^3 + CL_{12}^4$$
$$Q(0) = 0 \quad Q\left(\frac{CL_{12}^4}{T}\right) = 0.57$$
$$Q\left(\frac{CL_{12}^4 + CL_{12}^3}{T}\right) = 0.79 \quad Q\left(\frac{CL_{12}^4 + CL_{12}^3 + CL_{12}^2}{T}\right) = 0.91$$
$$Q\left(\frac{CL_{12}^4 + CL_{12}^3 + CL_{12}^2 + CL_{12}^1}{T}\right) = Q(1) = 1$$
$$w_1 = 0.57 \quad w_2 = 0.22 \quad w_3 = 0.12 \quad w_4 = 0.09$$
$$p_{12}^c = w_1 \cdot p_{12}^3 + w_2 \cdot p_{12}^4 + w_3 \cdot p_{12}^2 + w_4 \cdot p_{12}^1$$
$$= 0.57 \cdot 0.3 + 0.22 \cdot 0.6 + 0.12 \cdot 0.6 + 0.09 \cdot 0.2$$
$$= 0.39.$$

The collective fuzzy preference relation is

$$P^c = \begin{pmatrix} - & 0.39 & 0.55 & 0.61 \\ 0.6 & - & 0.64 & 0.71 \\ 0.45 & 0.36 & - & 0.55 \\ 0.39 & 0.29 & 0.45 & - \end{pmatrix}.$$

C. Exploitation Phase

Using again the same fuzzy quantifier "most of," and (29), we obtain the weighting vector $W = (w_1, w_2, w_3)$

$$w_1 = Q\left(\frac{1}{3}\right) - Q(0) = 0.58 - 0 = 0.58$$
$$w_2 = Q\left(\frac{2}{3}\right) - Q\left(\frac{1}{3}\right) = 0.82 - 0.58 = 0.24$$
$$w_3 = Q(1) - Q\left(\frac{2}{3}\right) = 1 - 0.82 = 0.18$$

and the following quantifier-guided dominance and nondominance degrees of all the alternatives:

	x_1	x_2	x_3	x_4
$QGDD_i$	0.57	0.67	0.49	0.41
$QGNDD_i$	0.96	1.00	0.93	0.81.

Clearly, the maximal sets are

$$X^{QGDD} = \{x_2\} \; X^{QGNDD} = \{x_2\}.$$

Finally, applying the conjunction selection policy, we obtain

$$X^{QGCP} = X^{QGDD} \cap X^{QGNDD} = \{x_2\}$$

which means that alternative x_2 is the best alternative according to "most of" the most consistent experts.

VII. Conclusion

We have looked at the issue of GDM problems with the incomplete fuzzy preference relations, and we have presented a new GDM model based on the AC property to deal with such situations. This new model is composed of two phases, the estimation of missing preference values and the selection of the best alternatives, both guided by the concept of AC.

To model the estimation phase, we have proposed an iterative procedure to estimate missing preference values. We have given sufficient conditions to guarantee the direct application of our procedure. Our proposal attempts to estimate the missing information in an expert's incomplete fuzzy preference relation using only the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by that expert. Because an important objective in the design of our procedure was to maintain experts' consistency levels, the procedure is guided by the expert's AC, and this is measured taking into account only the available preference values.

Based also on the AC, we have presented a selection process to solve GDM problems with the incomplete fuzzy preference relations. To model the selection phase, we have defined a new IOWA operator guided by the AC, the AC-IOWA operator. This operator permits the aggregation of experts' preferences in a collective fuzzy preference relation and the exploitation of such collective relation to achieve the solution set of alternatives according to the "fuzzy" majority of the most consistent experts' opinions.

In the future, we will address the extension of the management procedure of incomplete information to the case of linguistic preference relations [13], [18], [34] and its application to model users' preferences or desires in different problems like Internet business [27], technology selection [7], learning evaluation [24], and Web quality evaluation [19].

APPENDIX
FUZZY QUANTIFIERS AND THEIR USE TO MODEL FUZZY MAJORITY

The majority is traditionally defined as a threshold number of individuals. Fuzzy majority is a soft majority concept expressed by a fuzzy quantifier, which is manipulated via a fuzzy logic-based calculus of linguistically quantified propositions. Therefore, using fuzzy-majority-guided aggregation operators, we can incorporate the concept of majority into the computation of the solution.

Quantifiers can be used to represent the amount of items satisfying a given predicate. Classic logic is restricted to the use of the two quantifiers, there exists and for all, which are closely related, respectively, to the OR and AND connectives. Human discourse is much richer and more diverse in its quantifiers, e.g., about five, almost all, a few, many, most of, as many as possible, nearly half, at least half. In an attempt to bridge the gap between formal systems and natural discourse and, in turn, provide a more flexible knowledge representation tool, Zadeh introduced the concept of fuzzy quantifiers [43].

Zadeh suggested that the semantics of a fuzzy quantifier can be captured by using fuzzy subsets for its representation. He distinguished between two types of fuzzy quantifiers: absolute and relative. Absolute quantifiers are used to represent amounts that are absolute in nature such as about two or more than five. These absolute linguistic quantifiers are closely related to the concept of the count or number of elements. He defined these quantifiers as fuzzy subsets of the nonnegative real numbers, $\mathbb{R}^+$. In this approach, an absolute quantifier can be represented by a fuzzy subset Q, such that for any $r \in \mathbb{R}^+$ the membership degree of r in Q, $Q(r)$, indicates the degree in which the amount r is compatible with the quantifier represented by Q. Relative quantifiers, such as most, at least half, can be represented by fuzzy subsets of the unit interval [0, 1]. For any $r \in [0,1]$, $Q(r)$ indicates the degree in which the proportion r is compatible with the meaning of the quantifier it represents. Any quantifier of natural language can be represented as a relative quantifier or, given the cardinality of the elements considered, as an absolute quantifier.

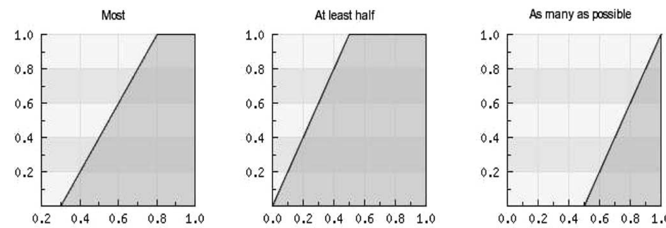

Fig. 2. Examples of relative fuzzy linguistic quantifiers.

A relative quantifier, $Q : [0,1] \rightarrow [0,1]$, satisfies

$$Q(0) = 0 \quad \exists r \in [0,1] \text{ such that } Q(r) = 1.$$

Yager in [39] distinguishes two categories of these relative quantifiers: RIM quantifiers such as all, most, many, at least α; and regular decreasing monotone (RDM) quantifiers such as at most one, few, at most α.

A RIM quantifier satisfies

$$\forall a, b \text{ if } a > b \text{ then } Q(a) \geq Q(b).$$

A widely used membership function for RIM quantifiers is [20]

$$Q(r) = \begin{cases} 0, & \text{if } r < a \\ \frac{r-a}{b-a}, & \text{if } a \leq r < b \\ 1, & \text{if } r \geq b \end{cases} \tag{40}$$

with $a, b, r \in [0,1]$. Some examples of relative quantifiers are shown in Fig. 2, where the parameters, (a, b) are (0.3, 0.8), (0, 0.5), and (0.5, 1), respectively.

The particular RIM function with parameters (0.3, 0.8) used to represent the fuzzy linguistic quantifier "most of" when applied with an OWA or IOWA operator associates a low weighting value to the most important/consistent experts because it assigns a value of 0 to the first 30% of experts. To overcome this problem, a different RIM function to represent the fuzzy linguistic quantifier "most of" should be used. To guarantee that all the important/consistent experts have associated a nonzero weighting value, and therefore all of them contribute to the final aggregated value, a strictly increasing RIM function should be used. On the other hand, in order to associate a high weighting value to those values with a high consistency level, a RIM function with a rate of increase in the unit interval inversely proportional to the value of the variable r seems to be adequate.

Yager in [39] considers the parameterized family of RIM quantifiers

$$Q(r) = r^a, \qquad a \geq 0$$

and the particular function with $a = 2$ to represent fuzzy linguistic quantifier "most of." This function is strictly increasing but, when used with an OWA or IOWA operators, associates high weighting values to low consistent values. In order to overcome this drawback, either of the following two approaches could be adopted.

1) The values are ordered using the opposite criteria, i.e., the first one being the one with lowest consistency degree.
2) A RIM function with $a < 1$ is used.

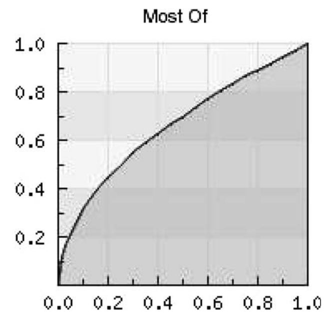

Fig. 3. Relative fuzzy linguistic quantifier "most of."

We have opted for the second one, and in particular in this paper, we use RIM function $Q(r) = r^{1/2}$ given in Fig. 3 to represent fuzzy linguistic quantifier "most of."

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