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A test for the homoscedasticity of the residuals in fuzzy rule-based forecasters

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Abstract Heteroscedasticity is the property of having a changing variance throughout the time. Homoscedasticity is the converse, that is, having a constant variance. This is a key property for time series models which may have serious consequences when making inferences out of the errors of a given forecaster. Thus it has to be conveniently assessed in order to establish the quality of the model and its forecasts. This is important for every model including fuzzy rule-based systems, which have been applied to time series analysis for many years. Lagrange multiplier testing framework is used to evaluate wether the residuals of an FRBS are homoscedastic. The test robustness is thoroughly evaluated through an extensive experimentation. This is another important step towards a statistically sound modeling strategy for fuzzy rule-based systems.

Keywords Fuzzy rule-based systems · Heteroscedasticity · Residuals · Diagnostic checking

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1 Introduction

Time series are present in most human activities. In particular, they are frequent in Engineering, Science and Social Sciences. Their analysis and modeling is a central problem in those areas, where plenty of applications are found. In most cases this is a complex process and a number of steps should be followed in order to accomplish the task correctly. These steps include some kind of preprocessing, model structure identification, and parameter estimation. But as important as the model construction itself is the assessment of its performance.

This is quite a complex issue. Model evaluation aims to clarify if the model satisfies a set of quality criteria related to its ability to capture the interesting characteristics of the system under study. This set of evaluation criteria is heavily dependent on several considerations: the final use that the model is built for, the inner characteristics of the system that are to be captured and whether the emphasis is put on the empirical behavior of the model or if there are theoretical considerations that are regarded to be more important.

In addition, the approach adopted for model evaluation usually displays a different bias depending on the area in which it is deployed. A clear distinction is perceived in the procedures used in the Soft Computing field as opposed to those used in the statistical approach to time series analysis.

In the usually engineering-oriented Soft Computing framework, there has been an overwhelming preeminence of just one evaluation criterion: *goodness of fit*. Generally, the evaluation of a model implies the computation of the prediction (or classification) error produced when it is faced with previously unseen samples of the same type of the one used to estimate it. This measure, in its different flavors (mean squared error, mean average error and so on) is affected by some inherent limitations: it is not very meaningful for a single model unless compared against other models, and is usually range-dependent, which makes it difficult to compare the same model applied to different problems represented by data sets with different characteristics.

On the other hand, in the statistical approach to time series analysis, the evaluation is different. It has usually more to do with obtaining an estimate of the probability that the model is effectively capturing the interesting characteristics of the data set. This is achieved through developing hypothesis tests, also known as misspecification tests. These tests are deployed through the so-called diagnostic checking procedures.

There is a basic assumption behind modeling: a part of the system under study behaves according to a model but there is another part which cannot be explained by it and is usually considered to be white noise. This is the main idea encoded in the expression of the general model

$$y_t = \mathrm{G}(\mathbf{x}_t; \boldsymbol{\psi}) + \varepsilon_t, \qquad (1)$$

and it is also behind some of the diagnostic checking procedures.

The residuals series, $\{\varepsilon_t\}$, constitutes an important source of information on the quality of the model. For example it is essential to know whether its values are independent and normally distributed. If the residuals were not independent, that would mean that the model is failing to capture an important part of the behavior of the series, and hence it should be respecified. A test to address this in the framework of fuzzy rule-based models has been presented in [7].

Another desirable property that the model should satisfy refers to the variance of the series $\{\varepsilon_t\}$. If a model is properly capturing the inner behavior of the series, the residuals should have a constant variance throughout the time. This is called *homoscedasticity*. The converse, that is, if the variance is not constant along the time, is *heteroscedasticity*. If the residuals series is heteroscedastic, the model's precision depends on time, and hence there are parts of the state-space that are not properly modeled. Of course, this will affect very negatively the performance of the model. This situation should be detected so that convenient action is taken. Heteroscedasticity is very important in time series analysis, particularly in Econometrics. In fact, the econometrician Robert Engle won the 2003 Nobel Memorial Prize in Economics for his studies on regression analysis in the presence of heteroscedasticity, which led to his formulation of the Autoregressive conditional heteroscedasticity (ARCH) modeling technique.

A very influential model within the Soft Computing field are fuzzy rule-based models (FRBM), or fuzzy models for short [8, 14, 32]. They have been applied in a vast array of problems, the modeling of time series being one of them. The usual fuzzy models for time series analysis resemble those for their applications in other problems, especially regression tasks. However, few methods have considered the specific nature of the time series analysis. In particular, no attention has been paid so far to the heteroscedasticity issue. And certainly, to properly assess the good performance of a time series model, its homoscedasticy properties should be studied.

This paper addresses the detection of heteroscedasticity when fuzzy rule-based models are used to model time series. Through the study of the residuals in the Lagrange multiplier framework a heteroscedasticity detection test is defined.

The paper is structured as follows: after this general introduction, we present the concept of heteroscedasticity in Sect. 2. Then, in Sect. 3, the considered FRBM for time series modeling is exposed, together with the relations that link it with a family of statistical models. Section 4 develops the theory of the test for heteroscedasticity of the residuals and Sect. 5 shows an empirical evaluation including Montecarlo experiments as well as applications to real-world time series.

2 Heteroscedasticity in time series modeling

Etymologically, heteroscedasticity (also spelled as *heteroskedasticity*) means differing dispersion or variance. In statistics, a time series is called heteroscedastic if it has different variances throughout the time, and homoscedastic if it shows constant variance in the observable period.[1]

Let us suppose we have a time series $\{y_t\}_{t=1}^n$ and a vector of time series (explanatory variables) $\{\mathbf{x}_t\}_{t=1}^n$. When considering conditional expectations of y_t given $\mathbf{x}_t$, the time series $\{y_t\}_{t=1}^n$ is said to be heteroscedastic if the conditional variance of y_t given $\mathbf{x}_t$ changes with t. This is also referred to as conditional heteroscedasticity to emphasize the fact that it is the series of conditional variance that changes and not the unconditional variance.

The concept can be easily grasped from a graphical representation. The left part of Fig. 1 (which is adapted from [9]), depicts a classic picture of a homoscedastic situation. We can see a regression line estimated via orthogonal least squares in a simple, bivariate model. The vertical spread of the data around the predicted line appears to be fairly constant as X changes. In contrast, the right part of the figure shows a similar model with heteroscedasticity. The vertical spread of the data is large for small values of X and then gets smaller as X rises. If the spread of the data is not constant across the X values, heteroscedasticity is present.

In general, heteroscedasticity does not result in biased estimates for ordinary least squares coefficients, but it can

[1]Some definitions can be found on the web, e.g.: http://en.wikipedia.org/wiki/Heteroscedasticity. http://financial-dictionary.thefreedictionary.com/Autoregressive+conditional+heteroscedasticity.

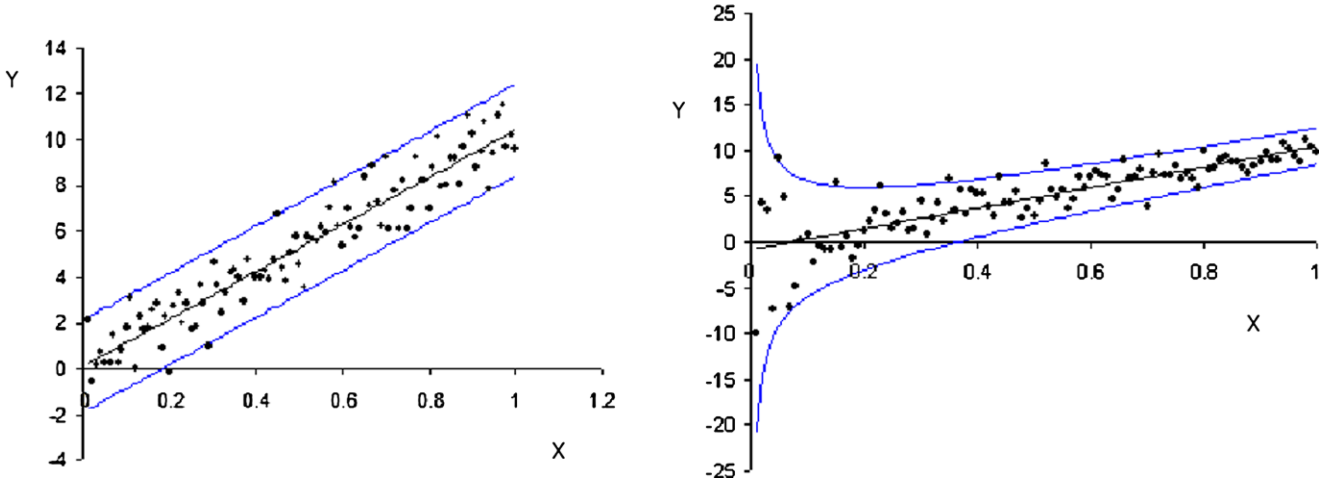

Fig. 1 Example of homoscedastic series (*left*) and heteroscedastic series (*right*)

cause the estimates of their variance to be above or below the true population variance. Hence analysis of outputs from models showing heteroscedasticity in their residual series are not faithful and can lead to erroneous inferences. For example this could lead to inappropriately rejecting a null hypothesis on a given sample.

The kind of forecasters we focus on in this paper is fuzzy rule-based models. In the case of fuzzy rule-based models applied to time series analysis, we are interested in studying the heteroscedasticity of the residual series in the state-space regions defined by the antecedent of the rules. If the model residual series shows smoothly changing variance between the rules, it is likely that some rules are failing to capture the behavior of the series in their state-space area. This represents an important source of diagnostic information about the goodness of the model.

3 Fuzzy rule-based models for time series analysis

Fuzzy sets were introduced by Lotfi A. Zadeh in his seminal paper published in 1965 [33]. After that starting milestone the number of researchers and practitioners interested in fuzzy sets have grown steadily. Fuzzy logic emerged initially as a means to represent and manage information with some kinds of uncertainty, mainly, imprecise, vague or ill-defined statements. Systems built out of fuzzy rules, so-called fuzzy rule-based models (FRBM) have been thoroughly studied. They display a host of “nice” properties which have entailed their applications in quite complex problems.

Depending on the intended goal, different kinds of FRBM can be built. If one is dealing with a problem for which precision is more important than interpretability, then the Takagi-Sugeno-Kang (TSK) paradigm [27, 28] is preferred over other variants of FRBM. This is the case when addressing time series analysis. When applied to model or forecast a univariate time series $\{y_t\}$, the rules of a TSK FRBM are expressed as:

$$\text{IF } y_{t-1} \text{ IS } A_1 \text{ AND } y_{t-2} \text{ IS } A_2 \text{ AND } \dots \text{ AND } y_{t-p} \text{ IS } A_p$$
$$\text{THEN } y_t = b_0 + b_1 y_{t-1} + b_2 y_{t-2} + \cdots + b_p y_{t-p}. \quad (2)$$

In this rule, all the variables y_{t-i} are lagged values of the time series, $\{y_t\}$. The inference engine of a TSK system works as follows:

The *firing strength* of the ith rule is obtained as the t-norm (usually, multiplication operator) of the membership values of the premise part terms of the linguistic variables:

$$\omega_i(\mathbf{x}) = \prod_{j=1}^{p} \mu_{A_j^i}(x_j), \quad (3)$$

where the shape of the membership function of the linguistic terms $\mu_{A_j^i}$ can be chosen from a wide range of functions. One of the most common is the Gaussian bell, although it can also be a logistic function and even non-derivable functions as a triangular or trapezoidal function.

The overall output is computed as a weighted average or weighted sum of the rules’ outputs. In the case of using the weighted sum, the system is called Fuzzy Additive System, and the output expression is:

$$y_t = G(\mathbf{x}_t; \boldsymbol{\psi}) + \varepsilon_t = \sum_{i=1}^{R} \omega_i(\mathbf{x}_t) \cdot \mathbf{b}_i \mathbf{x}_t + \varepsilon_t, \quad (4)$$

where G is the general nonlinear function with parameters $\boldsymbol{\psi}$, R denotes the number of fuzzy rules included in the system and ε_t is the series of the residuals as mentioned in the introduction. While many TSK FRBM perform a weighted average to compute the output, additive FRBM are also a common choice. They have been used in a large number of applications, for example [11, 17, 19, 31].

In the context of time series analysis there is a number of authors which have applied FRBM [15, 16, 18, 20, 23–25]. However, a key result concerning additive FRBM is that it has been proved [3] that this specification of the FRBM nests some models from the autoregressive regime switching family. More precisely, it is closely related with the Threshold Autoregressive model (TAR) [30], the Smooth Transition Autoregressive model (STAR) [29], the Linear Local-Global Neural Network (L^{2}GNN) [26] and the Neuro-Coefficient STAR [22].

This relation has given place to an ongoing exchange of knowledge and methods from the statistical framework to the fuzzy rule-based modelling of time series. For instance, a linearity test against FRBM has been developed [6], and more contributions are yet to come.

In this paper we will consider two types of membership functions: logistic, μ_{S}, and Gaussian, μ_{G}. The logistic function is the one used in [22], and although it is not so common in the fuzzy literature, we will use it here as an immediate result derived from the equivalences stated in [4]. It is defined as

$$\mu_{\mathsf{L}}(\mathbf{x}_t;\boldsymbol{\psi}) = \frac{1}{1+\exp(-\gamma(\boldsymbol{\omega}\mathbf{x}_t - c))}, \tag{5}$$

where $\boldsymbol{\psi} = (\gamma, \boldsymbol{\omega}, c)$.

On the other hand, Gaussian function will also be used because it is the most common membership function in fuzzy models. It is usually expressed as

$$\mu_{\mathsf{G}}(\mathbf{x}_t;\boldsymbol{\psi}) = \prod_i \exp\left(-\frac{(x_i - c_i)^2}{2\sigma^2}\right) \tag{6}$$

but we will rewrite it as

$$\mu_{\mathsf{G}}(\mathbf{x}_t;\boldsymbol{\psi}) = \prod_i \exp(-\gamma(x_i - c_i)^2), \tag{7}$$

where $\boldsymbol{\psi} = (\gamma, \mathbf{c})$.

4 Test of homoscedasticity of the residuals of an FRBM

When modeling a time series with an FRBM the model must be properly identified and estimated. Once the model is available, one might expect that the residuals are white noise, that is, they have a normal distribution, $\varepsilon_t \sim \mathrm{N}(0,\sigma^2)$. Moreover, it is expected that the residuals retain this distribution throughout time, that is, that the mean and the variance of ε_t remain constant through the changes of regime resulting from the prevalence of the different rules in different parts of the state-space.

It is thus interesting to develop a test to determine whether the variance σ^2 of the residual series changes when the model switches from one regime to another or not. Assuming that the variance does vary, we might note it as a time series $\{\sigma_t^2\}$, whose specification would be:

$$\sigma_t^2 = \sigma^2 + \sum_{i=1}^{r} \sigma_i^2 \mu_{\sigma,i}(\mathbf{x}_t;\boldsymbol{\psi}_{\mu_{\sigma,i}}), \tag{8}$$

where $\mu_{\sigma,i}$ are logistic or Gaussian functions satisfying the identifiability restrictions defined in [2]. This formulation allows the variance to change smoothly between regimes.

Following [21], to avoid complicated restrictions over the parameters to guarantee a positive variance, we rewrite (8) as

$$\begin{aligned}\sigma_t^2 &= \exp(G_\sigma(\mathbf{x}_t;\boldsymbol{\psi}_\sigma,\boldsymbol{\psi}_{\mu_{\sigma,i}}))\\ &= \exp\left(\varsigma + \sum_{i=1}^{r}\varsigma_i \mu_{\sigma,i}(\mathbf{x}_t;\boldsymbol{\psi}_{\mu_{\sigma,i}})\right),\end{aligned} \tag{9}$$

where $\boldsymbol{\psi}_\sigma = [\varsigma, \varsigma_1, \dots, \varsigma_r]'$ is a vector of real parameters.

To derive the test, let us consider $r = 1$. This is not a restrictive assumption because the test statistic remains unchanged if $r > 1$. We rewrite model (9) as

$$\sigma_t^2 = \exp(\varsigma + \varsigma_1 \mu_\sigma(\mathbf{x}_t;\boldsymbol{\psi}_{\mu_\sigma})), \tag{10}$$

where μ_σ is defined as (5) or as (7), depending on the membership function used by the model.

In both cases, logistic and Gaussian, the null hypothesis of homoscedasticity of the residuals is H_0: $\gamma_\sigma = 0$. As usual, model (10) is only identified under the alternative, $\gamma_\sigma \neq 0$, and we replace the membership function by its first-order Taylor expansion around $\gamma_\sigma = 0$. After this change and neglecting the remainder, both the logistic and the Gaussian cases result in

$$\sigma_t^2 = \exp\left(\rho + \sum_{i=1}^{q}\rho_i x_{i,t}\right), \tag{11}$$

so the null hypothesis becomes H_0: $\rho_1 = \rho_2 = \cdots = \rho_q = 0$. Under H_0, $\exp(\rho) = \sigma^2$.

The normal log-likelihood function in a neighbourhood of H_0 for observation t can be locally approximated by

$$\begin{aligned}l_t &= -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\left(\rho + \sum_{i=1}^{q}\rho_i x_{i,t}\right)\\ &\quad - \frac{\varepsilon_t^2}{2\exp(\rho + \sum_{i=1}^{q}\rho_i x_{i,t})}.\end{aligned} \tag{12}$$

In order to derive a Lagrange Multiplier (LM) type test, we need the partial derivatives of the log-likelihood:

$$\frac{\partial l_t}{\partial \rho} = -\frac{1}{2} + \frac{\varepsilon_t^2}{2\exp(\rho + \sum_{i=1}^{q} \rho_i x_{i,t})}, \tag{13}$$

$$\frac{\partial l_t}{\partial \rho_i} = -\frac{x_i}{2} + \frac{\varepsilon_t^2 x_i}{2\exp(\rho + \sum_{i=1}^{q} \rho_i x_{i,t})}, \tag{14}$$

and their consistent estimators under the null hypothesis:

$$\left.\frac{\partial \hat{l}_t}{\partial \rho}\right|_{\mathrm{H}_0} = \frac{1}{2}\left(\frac{\varepsilon_t^2}{\hat{\sigma}^2} - 1\right), \tag{15}$$

$$\left.\frac{\partial \hat{l}_t}{\partial \rho_i}\right|_{\mathrm{H}_0} = \frac{x_{i,t}}{2}\left(\frac{\varepsilon_t^2}{\hat{\sigma}^2} - 1\right), \tag{16}$$

where $\hat{\sigma}^2 = 1/T \sum_{t=1}^{T} \hat{\varepsilon}_t^2$. The LM statistic can then be written as

$$LM = \frac{1}{2}\left\{\sum_{t=!}^{T}\left(\frac{\varepsilon_t^2}{\hat{\sigma}^2} - 1\right)\tilde{\mathbf{x}}_t\right\}' \times \left\{\sum_{t=1}^{T} \tilde{\mathbf{x}}_t \tilde{\mathbf{x}}_t'\right\}^{-1} \times \left\{\sum_{t=!}^{T}\left(\frac{\varepsilon_t^2}{\hat{\sigma}^2} - 1\right)\tilde{\mathbf{x}}_t\right\}, \tag{17}$$

where $\tilde{\mathbf{x}}_t = [1, \mathbf{x}_t]'$. For details, see [21].

After the steps described above we are ready to formulate the test. It can be carried out as follows:

1. Estimate model (4) assuming homoscedasticity and compute the residuals $\hat{\varepsilon}_t$. Orthogonalize the residuals by regressing them on $\nabla \mathrm{G}(\mathbf{x}_t; \hat{\psi})$ and as before compute the $SSR_0 = \frac{1}{T}\sum_{t=1}^{T}(\frac{\tilde{\varepsilon}_t^2}{\hat{\sigma}_{\tilde{\varepsilon}_t}^2} - 1)^2$, where $\hat{\sigma}^2$ is the unconditional variance of $\tilde{\varepsilon}_t$.
2. Regress $(\frac{\tilde{\varepsilon}_t^2}{\hat{\sigma}_{\tilde{\varepsilon}_t}^2} - 1)$ on $\tilde{\mathbf{x}}_t$ and compute the residual sum of squares $SSR_1 = \frac{1}{T}\sum_{t=1}^{T} \tilde{\nu}_t^2$.
3. Compute the χ^2 statistic

$$LM_{\chi^2}^{\sigma} = T\frac{SSR_0 - SSR_1}{SSR_0}$$

or the F version of the test

$$LM_F^{\sigma} = \frac{(SSR_0 - SSR_1)}{q}\left(\frac{SSR_1}{(T-1-q)}\right)^{-1}.$$

Here T is the number of observations and q is the dimension of $\mathbf{x}_t$. Under H_0, $LM_{\chi^2}^{\sigma}$ is asymptotically distributed as a χ^2 with q degrees of freedom and LM_F^{σ} has approximately an F distribution with q and $T-1-q$ degrees of freedom.

The Lagrange Multiplier testing framework used in this paper is centered on the use of the product as a t-norm, which allows for the equivalence relations between FRBM and other statistical models. It would be very interesting indeed to consider an extension of the framework for other t-norms, especially as some studies show differences in the properties of FRBM with different t-norms [10, 12, 13].

5 Empirical evaluation

In order to assess the performance and power of the proposed test we have designed and conducted a thorough experiment. It can be divided into two different parts. In the first one, an extensive set of varied synthetic time series are generated and the corresponding FRBM identified and estimated. These experiments are designed to evaluate the robustness of the test against type I and type II errors. The second part illustrates the application of the test for modeling complex real-world time series.

5.1 Montecarlo experiment

To assess the power properties of the test, we devised two sets of experiments dealing with artificial series. On the one hand, we try to prove the robustness of the test against *type I errors* (rejecting the null hypothesis when it is actually true) by applying the test to a set of models whom we know that should produce homoscedastic residuals. On the other hand, we apply the test to a set of models whose residuals are most probably heteroscedastic, thus checking the robustness of the test against *type II errors* (not rejecting the null hypothesis when it is actually false). The use of synthetic series is very important because we know the exact underlying generating process and hence the adequacy of the forecasters can be correctly assessed.

In the first experiment (noted experiment E1) the idea was to build a statistically representative set of series (500 in our case) whose values were produced by iterating an FRBM with 3 rules:

IF y_{t-1} IS A_{11} AND y_{t-2} IS A_{12}

THEN $y_t = \mathbf{b}_1'\mathbf{x}_t = 0.2 + 0.3y_{t-1} - 0.9y_{t-2}$,

IF y_{t-1} IS A_{21} AND y_{t-2} IS A_{22}

THEN $y_t = \mathbf{b}_2'\mathbf{x}_t = -0.5 - 1.2y_{t-1} + 0.7y_{t-2}$,

IN ANY CASE $y_t = \mathbf{b}_3'\mathbf{x}_t = 0.5 + 0.8y_{t-1} - 0.2y_{t-2}$. (18)

This model can also be written as:

$$y_t = (0.2 + 0.3y_{t-1} - 0.9y_{t-2}) \times f(\mathbf{z}_t; \boldsymbol{\psi}_1) + (-0.5 - 1.2y_{t-1} + 0.7y_{t-2}) \times f(\mathbf{z}_t; \boldsymbol{\psi}_2) + 0.5 + 0.8y_{t-1} - 0.2y_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim NID(0, 0.2^2), \tag{19}$$

Table 1 Results of the F version of the test at the 0.05 significance level for heteroscedasticity of the residuals using the synthetic series generated by (18)

Experiment	# rules	Accept $\mathbb{H}_0$	Reject $\mathbb{H}_0$	% correct	Average p-value
E1	3	425	75	85	0.762
	2	105	395	79	0.044
E2	5	420	80	81	0.744
	4	103	397	74	0.047
	3	91	409	78	0.044
	2	89	411	78	0.038

where the membership functions are logistic (as in (7)) with parameters

$$\boldsymbol{\psi}_1 = [\gamma_1, \boldsymbol{\omega}_1, c_1] = [8.49, (0.7071, -0.7071), -1.0607],$$

$$\boldsymbol{\psi}_2 = [\gamma_2, \boldsymbol{\omega}_2, c_2] = [8.49, (0.7071, -0.7071), 1.0607].$$

Note that this model has three fuzzy rules, being the third one a *default rule*, i.e. a rule which applies to the whole input space as its antecedent membership function is always equal to 1.

In order to generate the 500 series, we used the skeleton of the model (19). By randomly fixing the starting values (y_0, y_1) we computed the value for $t = 2$ and then add a random value for ε_t. This process was repeated, iterating until $t = 1002$. Then, for each series, only the last 500 values were retained to avoid any initialization effects.

Once these 500 series were built, we applied the incremental building procedure presented in [1] to model each of them. Up to 94% of the models built were properly estimated to have 3 fuzzy rules, and from those models, the results of the test are shown in the first row of Table 1.

As we can see, the models presented homoscedastic residuals in 85% of the cases. This confirms that those models were properly built and that there is no need for respecification.

Next, we want to use the test in the case where heteroscedastic residuals are present. In order to do so, we used the same set of 500 series as before but, in this case, we used FRBM with only two rules to model them. With this limitation the complexity of the model is lower than the underlying generating process and the quality of the model should be poor. In addition, since the number of allowed regimes is smaller, we expect to end up with heteroscedastic residuals, and thus, the results of the test should verify a high rate of rejections of the null hypothesis.

The results are shown in the second row of Table 1. As expected, the null hypothesis was rejected in 79% of the cases, which shows the good properties of the test.

Now we want to check the behavior of the test with more complicated FRBM. In this new experiment (noted E2), we created a similar FRBM with 5 fuzzy rules, and used it as a generating process to produce 500 artificial time series. Then systems with 2, 3, 4 and 5 rules were estimated to model them. The results of applying the homoscedasticity test are shown in the second row group of Table 1, namely, the last four rows. As we can see the test is able to detect the expected heteroscedasticity of the simpler models and concludes that the more complex model with 5 rules is in general able to capture the inner behavior of the series.

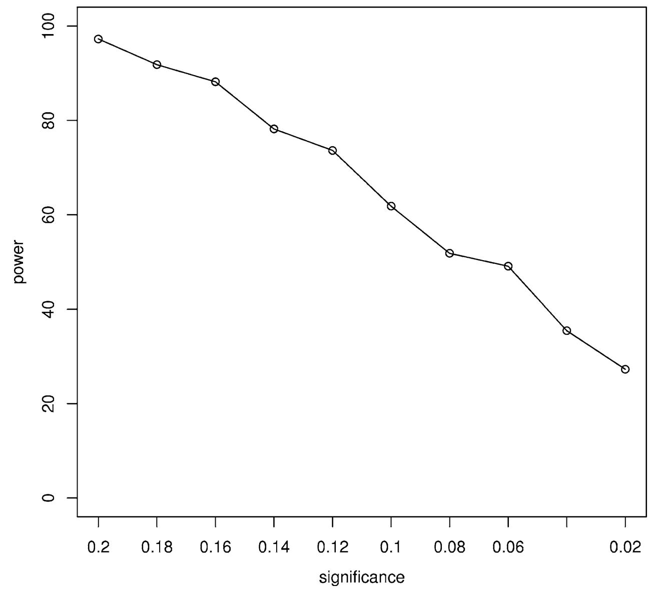

Fig. 2 Empirical power of the test with respect to the significance level

Another tool to assess the power of the test is to plot the percentage of proper rejections of the null hypothesis against the significance of the test. This is shown in Fig. 2, where we see that the test shows moderately good power for usual values of alpha.

5.2 Real-world series

One last step towards the assessment of the properties of the test is evaluating its performance in the modeling of real-world time series. We have considered three real time series, modeled them with FRBM and then proceeded to their analysis.

Table 2 Results of misspecification tests for three models facing real world cases (significance value: 0.95)

Model	# rules	Logistic membership function			Gaussian membership function		
		σ_{ε_t}	AIC	p-value	σ_{ε_t}	AIC	p-value
A	2	0.191	−313	0.179	0.205	−307	0.645
B	2	0.097	−6590	0.000	0.098	−6570	0.000
C	11	0.122	−24357	0.234	0.120	−24516	0.566

The studied cases are fully described in [1], and are a well known ecology problem (the Lynx series), a planning/management problem and a botanic problem.

The first series, commonly referred to as the Lynx series, is composed of the annual records of lynx captures in a certain part of Canada during a period spanning 113 years. It is a common benchmarking series used to test and compare time series models, and here we have used its logarithmic transformation. An FRBM with two rules (model A) was identified following the iterative procedure proposed in [1], and it was later estimated using a Genetic Algorithm.

The second considered series comes from an emergency call center and is the record of the number of calls received daily throughout four years. As the series is non-stationary and shows a high variability, it was differenced after applying a log-transformation. The identified FRBM (model B) was also composed of just two fuzzy rules, which were also fine tuned through a Genetic Algorithm.

Finally, the third series was a daily aerobiological log obtained over sixteen years in the city of Granada (Spain), containing daily counts of airborne olive tree pollen grains. This series was previously studied in [5].

Table 2 shows some information about the application of the FRBM, both in its logistic and Gaussian versions, to the three time series mentioned above. More precisely, for each model, the table shows, the number of rules of the model, the values for the variance of the residuals (σ_{ε_t}) and the Akaike information criterion (AIC), together with the p-value obtained with the test for homoscedasticity of the residuals.

By studying the p-values shown in columns 5 and 8 we can see how the null hypothesis of the test was rejected in all the six cases, which leads us to conclude that the variance of the residuals remained constant through time in every application.

As mentioned above, this is a necessary condition for considering that a model is properly capturing the behavior of a time series.

6 Conclusions and final remarks

In this paper, a new statistical tool to evaluate the residuals of a fuzzy rule-based model has been presented. It consists of a test against homoscedasticity of the residuals, that is, a test that allows the user to determine if the variance of the residual series remains constant throughout the time. In other words, this test is able to tell if a model errors are bigger in some parts of the state-space or not.

We have thoroughly evaluated the robustnest of the test against type I and type II errors. The experiments show a high robustness.

This represents a useful contribution and another step towards a statistically sound framework for the use of fuzzy rule-based models.

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