Hesitant Fuzzy Linguistic Term Sets for Decision Making

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***Abstract*—Dealing with uncertainty is always a challenging problem, and different tools have been proposed to deal with it. Recently, a new model that is based on hesitant fuzzy sets has been presented to manage situations in which experts hesitate between several values to assess an indicator, alternative, variable, etc. Hesitant fuzzy sets suit the modeling of quantitative settings; however, similar situations may occur in qualitative settings so that experts think of several possible linguistic values or richer expressions than a single term for an indicator, alternative, variable, etc. In this paper, the concept of a hesitant fuzzy linguistic term set is introduced to provide a linguistic and computational basis to increase the richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars by using comparative terms. Then, a multicriteria linguistic decision-making model is presented in which experts provide their assessments by eliciting linguistic expressions. This decision model manages such linguistic expressions by means of its representation using hesitant fuzzy linguistic term sets.**

***Index Terms*—Context-free grammar, fuzzy linguistic approach, hesitant fuzzy sets, linguistic decision making, linguistic information.**

I. Introduction

PROBLEMS that are defined under uncertain conditions are common in real-world decision-making problems but are quite challenging because of the difficulty of modeling and coping with such uncertainty. Different tools have been used to solve problems, such as probability; however, in many situations, uncertainty is not probabilistic in nature but, rather, imprecise or vague. Hence, other models, such as fuzzy logic and fuzzy sets theory [6], [39], have been successfully applied to handle imperfect, vague, and imprecise information [26]. Nevertheless, to handle vague and imprecise information whereby two or more sources of vagueness appear simultaneously, the modeling tools of ordinary fuzzy sets are limited. For this reason, different generalizations and extensions of fuzzy sets have been introduced.

1) *Type-2 fuzzy sets* [6], [24] and *type-n fuzzy sets* [6] that incorporate uncertainty about the membership function in their definition.
2) *Nonstationary fuzzy sets* [8] that introduce into the membership functions a connection that expresses a slight variation in the membership function.
3) *Intuitionistic fuzzy sets* [1] that extend fuzzy sets by an additional degree, which is called the degree of uncertainty.
4) *Fuzzy multisets* [37] based on multisets that allow repeated elements in the set.
5) *Hesitant fuzzy sets (HFS)* that have been recently introduced in [32] provide a very interesting extension of fuzzy sets. They try to manage those situations, where a set of values are possible in the definition process of the membership of an element.

The previous fuzzy tools suit problems that are defined as quantitative situations, but uncertainty is often because of the vagueness of meanings that are used by experts in problems whose nature is rather qualitative. In such situations, the fuzzy linguistic approach [40]–[42] has provided very good results in many fields and applications [2], [13], [18], [20], [27], [36]. However, in a similar way to the fuzzy sets, the use of the fuzzy linguistic approach presented some limitations, mainly regarding information modeling and computational processes, which are called processes of computing with words (CW) [9], [15], [21], [23]. Different linguistic models have tried to extend and improve the fuzzy linguistic approach from both points of view.

1) *The linguistic model based on type-2 fuzzy sets* representation [22], [33], [43] that represents the semantics of the linguistic terms by type-2 membership functions and using interval type-2 fuzzy sets for CW.
2) *The linguistic 2-tuple model* [12] that adds a parameter to the linguistic representation that is known as *symbolic translation,* which keeps the accuracy in the processes of CW.
3) *The proportional 2-tuple model* [34] that generalizes and extends the 2-tuple model by using two linguistic terms with their proportion to model the information and performs the processes of CW more accurately.
4) Other extensions that are based on previous ones introduced in [5] and [17].

By the revision of the fuzzy linguistic approach and the different linguistic extensions and generalizations, it is observed that the modeling of linguistic information is still quite *limited*, mainly because it is based on the elicitation of single and very simple terms that should encompass and express the information provided by the experts regarding a linguistic variable.

Manuscript received January 31, 2011; revised April 27, 2011 and August 2, 2011; accepted August 19, 2011. Date of publication September 29, 2011; date of current version February 7, 2012. This work was supported in part under the Research Project TIN-2009-08286 and P08-TIC-3548 and by the European fund for regional development.

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Digital Object Identifier 10.1109/TFUZZ.2011.2170076

However, in different situations, the experts that are involved in the problems defined under uncertainty cannot easily provide a single term as an expression of his/her knowledge, because he/she is thinking of several terms at the same time or looking for a more complex linguistic term that is not usually defined in the linguistic term set.

Therefore, we work with a view to overcome such limitations, taking into account the idea under the concept of HFS introduced in [32] to deal with several values in a membership function in a quantitative setting. In this paper, we propose the concept of *hesitant fuzzy linguistic term set* (HFLTS), based on the fuzzy linguistic approach, that will serve as the basis of increasing the flexibility of the elicitation of linguistic information by means of linguistic expressions. Additionally, different computational functions and properties of HFLTS are introduced, and we then present how they can be used to improve the elicitation of linguistic information by using the fuzzy linguistic approach and context-free grammars. This is very important because it allows us to use different expressions to represent decision makers' knowledge/preferences in decision making.

In order to answer the question:

How is the concept of HFLTS and its use in decision making justified?

We present a multicriteria linguistic decision-making model in which experts provide their assessments by means of linguistic expressions based on comparative terms close to the expressions used by human beings. This decision model manages the linguistic expressions that are represented by HFLTS. We propose the use of two symbolic aggregation operators that allow us to obtain a linguistic interval, which is associated with each alternative, and an exploitation process based on the application of the nondominance choice degree to a preference relation that is obtained from the previous linguistic intervals.

We are only aware of two papers on linguistic decision making that use linguistic expressions instead of single terms [19], [30]. In [30], the authors presented a linguistic model that dealt with linguistic expressions generated by applying logical connectives to the linguistic terms. In [19], the authors introduced the concepts of determinacy and consistency of linguistic terms in multicriteria decision-making problems and presented a model based on a fuzzy set in which decision makers could provide their assessments by using several linguistic terms and the reliability degree of each term. These proposals are not very close to human beings' cognitive processes and they are simpler than the model proposed in this paper, that uses linguistic expressions based on comparative terms.

The paper is organized as follows. In Section II, we briefly review some preliminary concepts that will be used in the HFLTS proposal. In Section III, we introduce the concept of HFLTS and several basic properties and operations to carry out the processes of CW. In Section IV, we present the use of HFLTS to facilitate and increase flexibility to elicit linguistic information. In Section V, we present a multicriteria linguistic decision-making model and define two symbolic aggregation operators to accomplish the processes of CW by using linguistic intervals. An illustrative example is also introduced in this section. In Section VI, we make some concluding remarks and suggest future research in this area. Appendix A contains a brief review of several necessary concepts to compare HFLTS, and Appendix B contains some definitions to build a preference relation between numeric intervals.

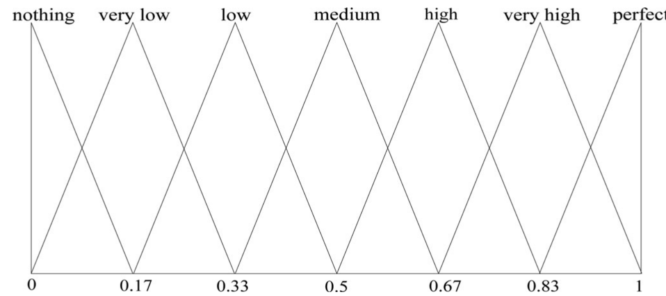

Fig. 1. Set of seven terms with its semantics.

II. PRELIMINARIES

Due to the fact that our proposal is based on the fuzzy linguistic approach [40]–[42] and HFS [32], in this section, we review their main concepts, necessary to understand the proposal of HFLTS and its use.

A. Fuzzy Linguistic Approach

In many real decision situations, the use of linguistic information is suitable and straightforward because of the nature of different aspects of the problem. In such situations, one common approach to model the linguistic information is the fuzzy linguistic approach [40]–[42] that uses the fuzzy sets theory [39] to manage the uncertainty and model the information.

In [40]–[42], Zadeh introduced the concept of linguistic variable as "a variable whose values are not numbers but words or sentences in a natural or artificial language." A linguistic value is less precise than a number, but it is closer to human cognitive processes that are used to successfully solve problems dealing with uncertainty. A linguistic variable is formally defined as follows.

Definition 1: [40]. A linguistic variable is characterized by a quintuple $(H,T(H),U,G,M)$ in which H is the name of the variable; $T(H)$ (or simply T) denotes the term set of H, i.e., the set of names of linguistic values of H, with each value being a fuzzy variable that is denoted generically by X and ranging across a universe of discourse U, which is associated with the base variable u; G is a syntactic rule (which usually takes the form of a grammar) for the generation of the names of values of H; and M is a semantic rule for associating its meaning with each H, $M(X)$, which is a fuzzy subset of U.

To deal with linguistic variables, it is necessary to choose the linguistic descriptors for the term set and their semantics. Fig. 1 shows a linguistic term set with the syntax and semantics of their terms.

There are different approaches to selecting the linguistic descriptors and different ways to define their semantics [38], [40]–[42]. The selection of the linguistic descriptors can be performed by means of the following.

1) *An ordered structure approach*: This defines the linguistic term set by means of an ordered structure providing the term set that is distributed on a scale at which a total order has been defined [10], [38]. For example, a set of seven terms, S, could be given as follows:

$$S = \{s_0 : \text{nothing}, s_1 : \text{very\_low}, s_2 : \text{low}, s_3 : \text{medium} \\ s_4 : \text{high}, s_5 : \text{very\_high}, s_6 : \text{perfect}\}.$$

In these cases, the existence of the following is usually required.
   a) A negation operator $\text{Neg}(s_i) = s_j$ so that $j = g - i$ ($g + 1$ is the granularity of the term set).
   b) A maximization operator: $\text{Max}(s_i, s_j) = s_i$ if $s_i \geq s_j$.
   c) A minimization operator: $\text{Min}(s_i, s_j) = s_i$ if $s_i \leq s_j$.
2) *A context-free grammar approach*: This defines the linguistic term set by means of a context-free grammar G so that the linguistic terms are the sentences that are generated by G [3], [4], [40]–[42]. A grammar G is a 4-tuple (V_N, V_T, I, P), where V_N is the set of nonterminal symbols, V_T is the set of terminals' symbols, I is the starting symbol, and P is the production rules that are defined in an extended Backus–Naur form [4]. Among the terminal symbols of G, we can find primary terms (e.g., low, medium, high), hedges (e.g., not, much, very), relations (e.g., lower than, higher than), conjunctions (e.g., and, but), and disjunctions (e.g., or). Thus, choosing I as any nonterminal symbol and using P could be generated linguistic expressions, such as, $\{\text{lower than medium}, \text{greater than high}, \ldots\}$.

The definition of their semantics can be accomplished as in [38] and [40]–[42] as follows.

1) *Semantics based on membership functions and a semantic rule*: This approach assumes that the meaning of each linguistic term is given by means of a fuzzy subset that is defined in the interval [0, 1], which is described by membership functions [4]. This semantic approach is used when the linguistic descriptors are generated by means of a context-free grammar. Thus, it contains two elements: a) the primary fuzzy sets that are associated with the primary linguistic terms and b) a semantic rule M that provides the fuzzy sets of the nonprimary linguistic terms [40]–[42].
2) *Semantics based on an ordered structure of the linguistic term set*: It introduces the semantics from the structure that is defined over the linguistic term set. Therefore, the users provide their assessments by using an ordered linguistic term set [31], [38]. The distribution of a linguistic term set on a scale [0, 1] can be distributed symmetrically [38] or nonsymmetrically [11], [31].
3) *Mixed semantics*: This assumes elements from the aforementioned semantic approaches.

B. Hesitant Fuzzy Sets

In [32], the author introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set. Although this situation might be modeled by fuzzy multisets, they are not completely adequate for these situations.

An HFS is defined in terms of a function that returns a set of membership values for each element in the domain [32].

Definition 2: Let X be a reference set, an HFS on X is a function h that returns a subset of values in [0, 1]:

$$h : X \to \{[0, 1]\}.$$

Therefore, given a set of fuzzy sets an HFS is defined as the union of their membership functions.

Definition 3: Let $M = \{\mu_1, \mu_2, \ldots, \mu_n\}$ be a set of n membership functions. The HFS that is associated with M, h_M, is defined as

$$h_M : M \to \{[0, 1]\}$$
$$h_M(x) = \bigcup_{\mu \in M} \{\mu(x)\}.$$

Some basic operations with the HFS were defined [32] as follows.

Definition 4: Given an HFS h, its lower and upper bounds are

$$h^-(x) = \min h(x)$$
$$h^+(x) = \max h(x).$$

Definition 5: Let h be an HFS, its complement is defined as

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}.$$

Proposition 1: [32]. The complement is involutive.

$$(h^c)^c = h.$$

Definition 6: Let h_1 and h_2 be two HFSs, their union is defined as

$$(h_1 \cup h_2)(x) = \{h \in (h_1(x) \cup h_2(x)) / h \geq \max(h_1^-, h_2^-)\}.$$

Definition 7: Let h_1 and h_2 be two HFS, their intersection is defined as

$$(h_1 \cap h_2)(x) = \{h \in (h_1(x) \cap h_2(x)) / h \leq \min(h_1^+, h_2^+)\}.$$

Definition 8: Let h be an HFS, the envelope of h, $A_{\text{env}(h)}$, is defined as

$$A_{\text{env}(h)} = \{x, \mu_A(x), \nu_A(x)\}$$

with $A_{\text{env}(h)}$ being the intuitionistic fuzzy set [1] of h, and μ and v are, respectively, defined as

$$\mu_A(x) = h^-(x)$$

and

$$v_A(x) = 1 - h^+(x).$$

III. Hesitant Fuzzy Linguistic Term Sets

Similarly to the situations that are described and managed by HFS in [32], where an expert may consider several values to define a membership function, in the qualitative setting, it may occur that experts hesitate among several values to assess a linguistic variable. The fuzzy linguistic approach is, however, aimed at statically assessing single linguistic terms for the linguistic variables. Hence, it is clear that, when experts hesitate about several values for a linguistic variable, the fuzzy linguistic approach is very limited. As pointed out in Section I, there are two proposals that use linguistic expressions instead of single terms [19], [30]. However, neither of them is adequate to fulfill the necessities and requirements of experts in hesitant situations.

Consequently, bearing in mind the idea under the HFS [32], in this section the concept of HFLTS, that is based on the fuzzy linguistic approach and the HFS is introduced. Some basic operations of HFLTS are then defined and some properties of such operations are revised.

A. Concept and Basic Operations

Definition 9: Let S be a linguistic term set, $S = \{s_0, \ldots, s_g\}$, an HFLTS, H_S, is an ordered finite subset of the consecutive linguistic terms of S.

Let S be a linguistic term set, $S = \{s_0, \ldots, s_g\}$, we then define the empty HFLTS and the full HFLTS for a linguistic variable ϑ as follows.

1) empty HFLTS: $H_S(\vartheta) = \{\}$,
2) full HFLTS: $H_S(\vartheta) = S$.

Any other HFLTS is formed with at least one linguistic term in S.

Example 1: Let S be a linguistic term set, $S = \{s_0 : \text{nothing}, s_1 : \text{very\_low}, s_2 : \text{low}, s_3 : \text{medium}, s_4 : \text{high}, s_5 : \text{very\_high}, s_6 : \text{perfect}\}$, a different HFLTS might be

$H_S(\vartheta) = \{s_1 : \text{very\_low}, s_2 : \text{low}, s_3 : \text{medium}\}$
$H_S(\vartheta) = \{s_3 : \text{medium}, s_4 : \text{high}, s_5 : \text{very\_high}\ s_6 : \text{perfect}\}$.

Once the concept of HFLTS has been defined, it is necessary to introduce the computations and operations that can be performed on them.

Let S be a linguistic term set, $S = \{s_0, \ldots, s_g\}$, and H_S, H_S^1, and H_S^2 be the three HFLTS.

Definition 10: The upper bound H_{S^+} and lower bound H_{S^-} of the HFLTS H_S are defined as

1) $H_{S^+} = \max(s_i) = s_j$, $s_i \in H_S$ and $s_i \leq s_j \forall i$;
2) $H_{S^-} = \min(s_i) = s_j$, $s_i \in H_S$ and $s_i \geq s_j \forall i$.

Definition 11: The complement of HFLTS, H_S, is defined as

$$H_S^c = S - H_S = \{s_i / s_i \in S \text{ and } s_i \notin H_S\}.$$

Proposition 2: The complement of an HFLTS is involutive:

$$(H_S^c)^c = H_S.$$

Proof: By the use of the definition of a *complement of an HFLTS*

$$(H_S^c)^c = S - H_S^c = S - (S - H_S) = H_S.$$

Definition 12: The union between two HFLTS, H_S^1 and H_S^2, is defined as

$$H_S^1 \cup H_S^2 = \{s_i / s_i \in H_S^1 \text{ or } s_i \in H_S^2\}$$

and the result will be another HFLTS.

Definition 13: The intersection of two HFLTS, H_S^1 and H_S^2, is

$$H_S^1 \cap H_S^2 = \{s_i / s_i \in H_S^1 \text{ and } s_i \in H_S^2\}$$

and the result of this operation is another HFLTS.

The comparison of linguistic terms is necessary in many problems, and it has always been defined in different linguistic approaches. An HFLTS is a linguistic term subset, and the comparison among these elements is not simple. Therefore, we introduce the concept of *envelope* for an HFLTS in order to simplify these operations as shown later in the text.

Definition 14: The envelope of the HFLTS, $\text{env}(H_S)$, is a linguistic interval whose limits are obtained by means of upper bound (max) and lower bound (min). Hence

$$\text{env}(H_S) = [H_{S^-}, H_{S^+}], H_{S^-} H_{S^+}.$$

Example 2: Let $S = \{\text{nothing}, \text{very\_low}, \text{low}, \text{medium}, \text{high}, \text{very\_high}, \text{perfect}\}$ be a linguistic term set, and $H_S = \{\text{high}, \text{very\_high}, \text{perfect}\}$ be an HFLTS of S, its envelope is

$H_{S^-}(\text{high}, \text{very\_high}, \text{perfect}) = \text{high}$
$H_{S^+}(\text{high}, \text{very\_high}, \text{perfect}) = \text{perfect}$

$$\text{env}(H_S) = [\text{high}, \text{perfect}].$$

Definition 15: The definition of the comparison between two HFLTS is based on the concept of the envelope of the HFLTS, $\text{env}(H_S)$. Hence, the comparison between H_S^1 and H_S^2 is defined as follows:

$$H_S^1(\vartheta) > H_S^2(\vartheta) \text{ iff } \text{env}(H_S^1(\vartheta)) > \text{env}(H_S^2(\vartheta))$$
$$H_S^1(\vartheta) = H_S^2(\vartheta) \text{ iff } \text{env}(H_S^1(\vartheta)) = \text{env}(H_S^2(\vartheta)).$$

Consequently, the comparison is conducted by interval values. In Appendix A, different approaches to comparing intervals are briefly reviewed and how to compare HFLTS is then clarified.

B. Properties

To conclude this section, some relevant properties of the HFLTS operations are reviewed.

Let H_S^1, H_S^2, and H_S^3 be three HFLTS, and $S = \{s_0, \ldots, s_g\}$. Then

1) *Commutativity*

$$H_S^1 \cup H_S^2 = H_S^2 \cup H_S^1$$
$$H_S^1 \cap H_S^2 = H_S^2 \cap H_S^1.$$

Proof of the union:

$\subseteq$

Let $s_i \in S$ be a linguistic value, $s_i \in H_S^1 \cup H_S^2$, then, by the definition of union, $s_i \in H_S^1$ or $s_i \in H_S^2$; if $s_i \in H_S^2$ or $s_i \in H_S^1$, then $s_i \in H_S^2 \cup H_S^1$.

$\supseteq$

Let $s_i \in H_S^2 \cup H_S^1$, then, $s_i \in H_S^2$ or $s_i \in H_S^1$; if $s_i \in H_S^1$ or $s_i \in H_S^2$, then $s_i \in H_S^1 \cup H_S^2$.

The demonstration of the intersection would be similar to the union.

2) *Associative*

$$H_S^1 \cup (H_S^2 \cup H_S^3) = (H_S^1 \cup H_S^2) \cup H_S^3$$
$$H_S^1 \cap (H_S^2 \cap H_S^3) = (H_S^1 \cap H_S^2) \cap H_S^3.$$

Proof of the union:
$\subseteq$
Let $s_i \in S$ be a linguistic value, $s_i \in H_S^1 \cup (H_S^2 \cup H_S^3)$, then, $s_i \in H_S^1$ or $s_i \in H_S^2 \cup H_S^3$. In the second case, $s_i \in H_S^2$ or $s_i \in H_S^3$; therefore, if $s_i \in H_S^1 \cup H_S^2$ or $s_i \in H_S^3$, then $s_i \in (H_S^1 \cup H_S^2) \cup H_S^3$.
$\supseteq$
Let $s_i \in (H_S^1 \cup H_S^2) \cup H_S^3$ then, $s_i \in H_S^1 \cup H_S^2$ or $s_i \in H_S^3$. In the first case, $s_i \in H_S^1$ or $s_i \in H_S^2$; therefore, if $s_i \in H_S^1$ or $s_i \in H_S^2 \cup H_S^3$, then $s_i \in H_S^1 \cup (H_S^2 \cup H_S^3)$.
In a similar way, the associative property of the intersection can be demonstrated.

3) *Distributive*

$$H_S^1 \cap (H_S^2 \cup H_S^3) = (H_S^1 \cap H_S^2) \cup (H_S^1 \cap H_S^3)$$
$$H_S^1 \cup (H_S^2 \cap H_S^3) = (H_S^1 \cup H_S^2) \cap (H_S^1 \cup H_S^3).$$

Proof of the union:
$\subseteq$
Let $s_i \in (H_S^1 \cup H_S^2) \cap H_S^3$ then, $s_i \in H_S^1 \cup H_S^2$ and $s_i \in H_S^3$. Therefore, $s_i \in H_S^1$ or $s_i \in H_S^2$.
If $s_i \in H_S^1$, then $s_i \in H_S^1 \cap H_S^3$.
If $s_i \in H_S^2$, then $s_i \in H_S^2 \cap H_S^3$.
Thus, $s_i \in H_S^1 \cap H_S^3$ or $s_i \in H_S^2 \cap H_S^3$, this means that $s_i \in (H_S^1 \cap H_S^3) \cup (H_S^2 \cap H_S^3)$.
$\supseteq$
Let $s_i \in (H_S^1 \cap H_S^3) \cup (H_S^2 \cap H_S^3)$. Then, $s_i \in H_S^1 \cap H_S^3$ or $s_i \in H_S^2 \cap H_S^3$. On the first case, as $s_i \in H_S^1$, then $s_i \in H_S^1 \cup H_S^2$; therefore, $s_i \in (H_S^1 \cup H_S^2) \cap H_S^3$. In the second case, as $s_i \in H_S^2$, then $s_i \in H_S^1 \cup H_S^2$; therefore, $s_i \in (H_S^1 \cup H_S^2) \cap H_S^3$.
Similarly to the property of the union, the distributive property of the intersection can be demonstrated.

IV. Elicitation of Linguistic Information Based on Hesitant Fuzzy Linguistic Term Sets

Throughout the paper, it has been pointed out that the aim of the HFLTS is to improve the elicitation of linguistic information, mainly when experts hesitate among several values to assess linguistic variables.

The concept of HFLTS has been introduced as something that can be directly used by the experts to elicit several linguistic values for a linguistic variable, but such elements are not similar to human beings' way of thinking and reasoning. Therefore, in this section, the definition of simple but elaborated linguistic expressions that are more similar to human beings' expressions is proposed to be semantically represented by means of HFLTS and generated by a context-free grammar.

A simple context-free grammar G_H is introduced to support the type of linguistic information that we want to allow the experts to elicit in order to increase the flexibility and expressiveness of linguistic information, which is denoted by ll.

Besides the previous grammar G_H, it is also necessary to define how its linguistic expressions will be represented and managed in processes of CW. To do so, a function $E(ll)$ is presented that transforms such linguistic expressions into HFLTS.

The context-free grammar G_H and the transformation function $E(\cdot)$ are further detailed in the following sections.

A. Context-Free Grammar for Eliciting Linguistic Information Based on HFLTS

A context-free grammar G provides a way to generate linguistic terms and linguistic expressions by means of its different elements. Our objective is to define a context-free grammar G_H that generates simple but rich linguistic expressions that can be easily represented by means of HFLTS. Therefore, the context-free grammar G_H is defined to generate the type of linguistic expressions that we want to model in hesitant situations.

Definition 16: Let G_H be a context-free grammar, and $S = \{s_0, \ldots, s_g\}$ be a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:
$V_N = \{\langle$primary term$\rangle$, $\langle$composite term$\rangle$
$\langle$unary relation$\rangle$, $\langle$binary relation$\rangle$, $\langle$conjunction$\rangle\}$
$V_T = \{$lower than, greater than, between, and, $s_0, s_1, \ldots, s_g\}$
$I \in V_N$.

The production rules are defined in an extended Backus–Naur form so that the brackets enclose optional elements and the symbol "|" indicates alternative elements [4]. For the context-free grammar G_H, the production rules are as follows:
$P = \{I ::= \langle$primary term$\rangle|\langle$composite term$\rangle$
$\langle$composite term$\rangle ::= \langle$unary relation$\rangle\langle$primary term$\rangle|$
$\langle$binary relation$\rangle\langle$primary term$\rangle\langle$conjunction$\rangle\langle$primary term$\rangle$
$\langle$primary term$\rangle ::= s_0|s_1|\ldots|s_g$
$\langle$unary relation$\rangle ::=$ lower than|greater than
$\langle$binary relation$\rangle ::=$ between
$\langle$conjunction$\rangle ::=$ and$\}$.

Remark 1: The *unary relation* has some limitations. If the nonterminal symbol is "lower than," then the "primary term" cannot be s_0, and if the nonterminal symbol is "greater than," then the "primary term" cannot be s_g.

Remark 2: In the "binary relation," the "primary term" on the left-hand side must be less than the "primary term" on the right-hand side.

Example 3: Let $S = \{$nothing, very_low, low, medium, high, very_high, perfect$\}$ be a linguistic term set; some linguistic expressions that are obtained by means of the context-free grammar G_H might be

$$\text{ll}_1 = \text{high}$$
$$\text{ll}_2 = \text{lower than medium}$$
$$\text{ll}_3 = \text{greater than high}$$
$$\text{ll}_4 = \text{between medium and very\_high}.$$

These linguistic expressions are close to the linguistic structures used by human beings to provide their assessments in real-world problems, where they are not sure about one single value to assess the criteria or the alternatives. Therefore, the

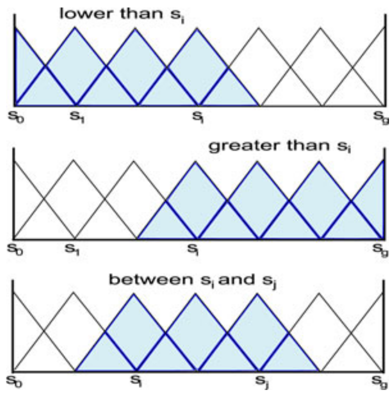

Fig. 2. HFLTS associated with the linguistic expressions.

hesitant situation is modeled by means of linguistic structures that are generated by the production rules, $P \in G_H$, being necessary to model semantically such information. To do so, the use of HFLTS is proposed.

B. Transforming Linguistic Expressions of G_H into HFLTS

The transformation of the linguistic expressions ll that are produced by G_H into HFLTS is done by means of the transformation function E_{G_H}.

Definition 17: Let E_{G_H} be a function that transforms linguistic expressions ll, which are obtained by G_H, into HFLTS H_S, where S is the linguistic term set that is used by G_H:

$$E_{G_H} : \text{ll} \longrightarrow H_S.$$

The linguistic expressions that are generated by using the production rules will be transformed into HFLTS in different ways according to their meaning:

1) $E_{G_H}(s_i) = \{s_i / s_i \in S\}$;
2) $E_{G_H}(\text{less than } s_i) = \{s_j / s_j \in S \text{ and } s_j \leq s_i\}$;
3) $E_{G_H}(\text{greater than } s_i) = \{s_j / s_j \in S \text{ and } s_j \geq s_i\}$;
4) $E_{G_H}(\text{between } s_i \text{ and } s_j) = \{s_k / s_k \in S \text{ and } s_i \leq s_k \leq s_j\}$.

With the previous definition of E_{G_H}, it is easy to figure out the representation of the initial linguistic expressions ll into HFLTS. Fig. 2 shows these transformations graphically.

Example 4: By the use of the linguistic expressions that are obtained in Example 3, i.e., $\text{ll}_1, \text{ll}_2, \text{ll}_3$, and ll_4 their transformation into HFLTS by the transformation function E_{G_H} is

$E_{G_H}(\text{high}) = \{\text{high}\}$

$E_{G_H}(\text{lower than medium}) = \{\text{nothing, very\_low, low medium}\}$

$E_{G_H}(\text{greater than high}) = \{\text{high, very\_high, perfect}\}$

$E_{G_H}(\text{between medium and very\_high}) = \{\text{medium high, very\_high}\}$.

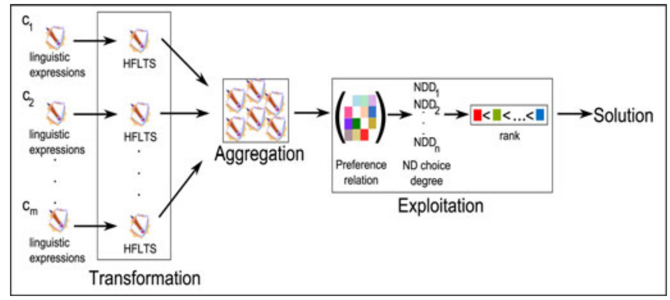

Fig. 3. Schema of the decision-making model.

V. Multicriteria Linguistic Decision-Making Model With Linguistic Expressions Based on Comparative Terms

In this section, we present a multicriteria linguistic decision-making model in which decision makers can provide their assessments by means of linguistic expressions based on comparative terms close to the expressions used by human beings or by means of single linguistic terms. This decision model manages such linguistic expressions by its representation using HFLTS. To fuse these linguistic expressions, we propose two symbolic aggregation operators, min_upper and max_lower, that provide a linguistic interval for each alternative. Finally, an exploitation process based on the application of the nondominance choice degree to obtain the solution set of alternatives is proposed.

An example of a decision-making problem is also introduced to easily understand the proposed model.

A. Multicriteria Linguistic Decision-Making Problem

A multicriteria linguistic decision-making problem consists of a finite set of alternatives, $X = \{x_1, \ldots, x_n\}$, where each alternative is defined by means of a finite set of criteria, $C = \{c_1, \ldots, c_m\}$, which is assessed by using linguistic expressions.

In this decision-making problem, we suppose a linguistic term set, $S = \{s_0, \ldots, s_g\}$, and a context-free grammar G_H, which produces the linguistic expressions $\text{ll}(x_i, c_j)$ based on comparative terms to assess the criteria, $C = \{c_1, \ldots, c_m\}$, for each alternative, $X = \{x_1, \ldots, x_n\}$.

B. Multicriteria Linguistic Decision Making Model

The proposed decision-making model consists mainly of the following three phases (see Fig. 3).

1) *Transformation phase*: The linguistic expressions provided by experts are transformed into HFLTS by using the transformation function E_{G_H}.
2) *Aggregation phase*: The assessments represented by HFLTS are aggregated by using two symbolic aggregation operators that obtain a linguistic interval, which is used to rank the alternatives in the following phase.
3) *Exploitation phase*: The linguistic intervals obtained in the previous phase are used to build a preference relation between alternatives, and a nondominance choice degree is applied to obtain a solution set of alternatives for the decision problem.

TABLE I
ASSESSMENTS THAT ARE PROVIDED FOR THE DECISION PROBLEM

		criteria		
	ll	c_1	c_2	c_3
alternatives	x_1	*between vl and m*	*between h and vh*	*h*
	x_2	*between l and m*	*m*	*lower than l*
	x_3	*greater than h*	*between vl and l*	*greater than h*

TABLE II
ASSESSMENTS TRANSFORMED INTO HFLTS

		criteria		
	$H_S^j(x_i)$	c_1	c_2	c_3
alternatives	x_1	$\{vl, l, m\}$	$\{h, vh\}$	$\{h\}$
	x_2	$\{l, m\}$	$\{m\}$	$\{n, vl, l\}$
	x_3	$\{h, vh, p\}$	$\{vl, l\}$	$\{h, vh, p\}$

TABLE III
UPPER BOUND FOR EACH HFLTS

		criteria		
	$H_{S+}^j(x_i)$	c_1	c_2	c_3
alternatives	x_1	$\{m\}$	$\{vh\}$	$\{h\}$
	x_2	$\{m\}$	$\{m\}$	$\{l\}$
	x_3	$\{p\}$	$\{l\}$	$\{p\}$

TABLE IV
MINIMUM LINGUISTIC TERM OF THE SET OF CRITERIA

$alternatives/H_{S^+_{min}}(x_i)$		
x_1	x_2	x_3
$\{m\}$	$\{l\}$	$\{l\}$

These phases are explained in further detail later, forming an illustrative example to easily understand the decision model.

1) *Transformation of the linguistic expressions into HFLTS*: Our aim is to facilitate the expressiveness of the experts in linguistic decision-making problems, providing a linguistic modeling close to human beings by using linguistic expressions that are generated by a context-free grammar G_H. The linguistic expressions that are provided by experts must be transformed into HFLTS to accomplish the aggregation process by means of the transformation function E_{G_H}, which is introduced in Definition 17.
   Example 5: Let $X = \{x_1, x_2, x_3\}$ be a set of alternatives, $C = \{c_1, c_2, c_3\}$ be a set of criteria defined for each alternative, and $S = \{s_0$: nothing$(n), s_1$: very low$(vl), s_2$: low$(l), s_3$: medium$(m), s_4$:high(h), s_5 : very high$(vh), s_6$: perfect$(p)\}$ be the linguistic term set that is used by the context-free grammar G_H to generate the linguistic expressions. The assessments that are provided in such a problem are shown in Table I.
   The transformation of such expressions into HFLTS by means of the transformation function E_{G_H} is shown in Table II.
2) *Aggregation of the assessments represented by HFLTS*: Once the assessments are represented by HFLTS, it is necessary to fuse the set of criteria for each alternative by using symbolic aggregation operators. In decision making, it is common to use two different points of view [29], the *pessimistic* and the *optimistic*. To find a balance between both approximations, we shall define two aggregation operators, min_upper and max_lower, which carry out the aggregation by using HFLTS. The "min_upper" operator selects the worst of the superior values and the "max_lower" selects the best of the inferior values. The result of application of these operators is two linguistic terms that will be used to build a linguistic interval. To do so, the minimum linguistic term will be the left limit of the interval and the maximum one will be the right. Each alternative has an associated linguistic interval that represents the core information of the HFLTS aggregated.
   a) Min_upper operator: This is a symbolic aggregation operator that is introduced to combine HFLTS, and it obtains the worst of the maximum linguistic terms.
   Definition 18: Let $X = \{x_1, \ldots, x_n\}$ be a set of alternatives, $C = \{c_1, \ldots, c_m\}$ be a set of criteria, $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, and $\{H_S^j(x_i)/i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ be a set of HFLTS. The min_upper operator consists of the following two steps.
      i) Apply the upper bound H_{S^+} for each HFLTS that is associated with each alternative: $H_{S^+}(x_i) = \{H_{S^+}^1(x_i), \ldots, H_{S^+}^m(x_i)\}, i \in \{1, \ldots, n\}$.
      ii) Obtain the minimum linguistic term for each alternative: $H_{S^+_{\min}}(x_i) = \min\{H_{S^+}^j(x_i)/j \in \{1, \ldots, m\}\}$, $i \in \{1, \ldots, n\}$.
   According to Example 5, the aggregation with such an operator is carried out as follows.
      i) Apply the upper bound for each HFLTS (see Table III).
      ii) Obtain the minimum linguistic term of the set of criteria for each alternative (see Table IV).
   b) Max_lower operator: This symbolic operator is also introduced to combine HFLTS, but it is opposite to the previous one, because it obtains the best of the minimum linguistic terms.
   Definition 19: Let $X = \{x_1, \ldots, x_n\}$ be a set of alternatives, $C = \{c_1, \ldots, c_m\}$ be a set of criteria; $S = \{s_0, \ldots, s_g\}$ be a linguistic term set, and $\{H_S^j(x_i)/i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$ be a set of HFLTS. The max_lower operator also consists of the following two steps.
      i) Apply the lower bound for each HFLTS that is associated with each alternative:

$$H_{S^-}(x_i) = \{H_{S^-}^1(x_i), \ldots, H_{S^-}^m(x_i)\}$$
$$i \in \{1, \ldots, n\}.$$

      ii) Obtain the maximum linguistic term for each alternative:

TABLE V
LOWER BOUND FOR EACH HFLTS

		criteria		
	$H^j_{S^-}(x_i)$	c_1	c_2	c_3
alternatives	x_1	$\{vl\}$	$\{h\}$	$\{h\}$
	x_2	$\{l\}$	$\{m\}$	$\{n\}$
	x_3	$\{h\}$	$\{vl\}$	$\{h\}$

TABLE VI
MAXIMUM LINGUISTIC TERM OF THE SET OF CRITERIA

$alternatives/H_{S^-_{max}}(x_i)$		
x_1	x_2	x_3
$\{h\}$	$\{m\}$	$\{h\}$

TABLE VII
LINGUISTIC INTERVALS FOR THE ALTERNATIVES

$alternatives/H'(x_i)$		
x_1	x_2	x_3
$[m,h]$	$[l,m]$	$[l,h]$

$$H_{S^-_{\max}}(x_i) = \max\{H^j_{S^-}(x_i)/j \in \{1,\ldots,m\}\} \quad i \in \{1,\ldots,n\}.$$

Following Example 5, the results obtained by the application of the max_lower operator are as follows.

i) Apply the lower bound for each HFLTS (see Table V).

ii) Obtain the maximum linguistic term of the set of criteria for each alternative (see Table VI).

c) The linguistic terms that are obtained from the previous aggregation operators are used to build a linguistic interval for each alternative that represents the core information of the HFLTS aggregated. The left limit is the minimum of them, and the right limit is the maximum:

$$H'_{\max}(x_i) = \max\{H_{S^+_{\min}}(x_i), H_{S^-_{\max}}(x_i)\}$$

$$H'_{\min}(x_i) = \min\{H_{S^+_{\min}}(x_i), H_{S^-_{\max}}(x_i)\}$$

$$H'(x_i) = [H'_{\min}(x_i), H'_{\max}(x_i)].$$

Following Example 5, the linguistic intervals that are obtained are shown in Table VII.

3) *Exploitation phase*: Once the linguistic information has been aggregated, the exploitation phase is carried out, where the set of alternatives will be ordered to select the best one(s) according to the following steps.

a) Building of a preference relation: Now, the aggregated information regarding each alternative is expressed by a linguistic interval. Hence, to order such alternatives, first, a binary preference relation is built [7], [25] between alternatives. This preference relation is obtained by adapting the method that is proposed in [35]. In Appendix B, one such method is revised.

b) Application of a choice degree: For ranking alternatives from the preference relation, different choice functions could be applied [25], [28]. Here, we propose the use of a nondominance choice degree NDD, which indicates the degree to which the alternative x_i is not dominated by the remaining ones. Its definition is given as follows.

Definition 20 [25]: Let $P = [p_{ij}]$ be a preference relation that is defined over a set of alternatives X. For the alternative x_i, its nondominance degree NDD_i is obtained as

$$\text{NDD}_i = \min\{1 - p^S_{ji}, j \neq i\}$$

where $p^S_{ji} = \max\{p_{ji} - p_{ij}, 0\}$ represents the degree to which x_i is strictly dominated by x_j.

c) Finally, we obtain the set of nondominated alternatives as follows:

$$X^{\text{ND}} = \{x_i/x_i \in X, \text{NDD}_i = \max_{x_j \in X}\{\text{NDD}_j\}\}.$$

Following Example 5, the exploitation phase consists of the following steps.

a) Computing the preference degrees by using the definition that is introduced in [35]. This function must be adapted to deal with linguistic intervals; therefore, $\text{Ind}(s_i) = i$ (it provides the index associated with the label), $s_i \in S = \{s_0,\ldots,s_g\}$, $(P(x_1 > x_2))$, $(P(x_2 > x_1))$, $(P(x_1 > x_3))$, $(P(x_3 > x_1))$, $(P(x_2 > x_3))$, $(P(x_3 > x_2))$, and (P_D), shown at the bottom of the next page.

b) A nondominance choice degree NDD_i is applied to the preference relation

$$P^S_D = \begin{pmatrix} - & 1 & 0.334 \\ 0 & - & 0 \\ 0 & 0.334 & - \end{pmatrix}$$

$$\text{NDD}_1 = \min\{(1-0),(1-0)\} = 1$$

$$\text{NDD}_2 = \min\{(1-1),(1-0.334)\} = 0$$

$$\text{NDD}_3 = \min\{(1-0.334),(1-0)\} = 0.664.$$

c) Finally, the solution set of alternatives is

$$\mathbf{X}^{\mathbf{ND}} = \{\mathbf{x_1}\}$$

VI. CONCLUDING REMARKS AND FUTURE WORKS

In this paper, the concept of HFLTS has been introduced to increase the flexibility and richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars to support the elicitation of linguistic information by experts in hesitant situations under qualitative settings. In addition, different computational functions and properties of HFLTS have been presented. Afterwards, a multicriteria linguistic decision-making model in which experts provide their assessments by using linguistic expressions based on comparative terms has been presented and applied to a decision-making problem to show the usefulness of the HFLTS in decision making.

In the future, the application of HFLTS to group decision-making problems that are defined with uncertainty will be explored, where the experts will be able to provide their assessments by means of preference relations by using linguistic expressions based on HFLTS.

Appendix A

Due to the fact that the comparison of HFLTS is based on their envelope, which are intervals, in this Appendix, a brief review is made of several methods to compare numeric intervals that could be used in the comparison of HFLTS, but first, the concept of numeric interval is revised.

Definition 21: [14]. An interval is defined by an ordered pair in brackets as

$$A = [a_L, a_R] = \{a : a_L \leqq a \leqq a_R\}$$

where a_L is the left limit, and a_R is the right limit of A.

Definition 22: [14]. The interval is also denoted by its center and width as

$$A = \langle a_C, a_W \rangle = \{a : a_C - a_W \leqq a \leqq a_c + a_W\}$$

where a_C is the center, and a_W is the width of A.

From definitions 21 and 22, the center and width of an interval may be calculated as

$$a_C = \frac{1}{2}(a_R + a_L)$$
$$a_W = \frac{1}{2}(a_R - a_L).$$

Different approaches to comparing intervals have been introduced in the literature. Two order relations are presented in [14]. One of them is defined by the left and right limits of an interval. This order relation is partial, and there are many pairs of intervals that cannot be compared with such a relation. To overcome this limitation, the authors defined a second-order relation by the center and width of the interval, but it is also a partial-order relation. In [16], the author defined a fuzzy preference relation between two intervals on the real line by means of a formula that uses probability relations. The disadvantage of this approach is that it does not take into account the width of the intervals, and it could, therefore, find that two intervals are equal, although their widths were different.

Afterwards, in [29], the authors presented two approaches to compare any two interval numbers. In the following, we present one of them, which we consider suitable to accomplish the comparison of HFLTS by using their envelopes, because it overcomes the drawbacks of Tanaka, Ishibuchi, and Kundu's approaches, in further detail. Such a method introduces an acceptability function that indicates the grade of acceptability regarding *the first interval is inferior to the second interval* and is defined as follows.

Definition 23: [29]. Let I be the set of all closed intervals on the real line $\Re$, and A and B are the two intervals, $A, B \in I$. The acceptability function, $A_< : I \times I \longrightarrow [0, \infty)$, is defined as

$$A_< = \frac{b_C - a_C}{b_W + a_W}$$

where $b_W + a_W \neq 0$; $a_C \leq b_C$; and a_C, b_C, a_W, and b_W are the centers and widths of the intervals A and B.

This grade of acceptability is a real number that represents that the grade of acceptance of the interval A is inferior to the interval B and is interpreted as follows:

1) If $A_< = 0$, then it is not accepted that the interval A is inferior to B.
2) If $0 < A_< < 1$, then $A_<$ is accepted with different grades of satisfaction from 0 to 1.
3) If $A_< \geqq 1$, then it is absolutely true that the interval A is inferior to B.

Appendix B

Appendix A revises the comparison between numeric intervals by means of a grade of acceptability that indicates if the first interval is inferior to the second one, but it does enable

$$P(x_1 > x_2) = \frac{\max(0, \mathrm{Ind}(s_4) - \mathrm{Ind}(s_2)) - \max(0, \mathrm{Ind}(s_3 - \mathrm{Ind}(s_3))}{(\mathrm{Ind}(s_4) - \mathrm{Ind}(s_3)) + (\mathrm{Ind}(s_3 - \mathrm{Ind}(s_2))} = 1$$

$$P(x_2 > x_1) = \frac{\max(0, \mathrm{Ind}(s_3) - \mathrm{Ind}(s_3)) - \max(0, \mathrm{Ind}(s_2 - \mathrm{Ind}(s_4))}{(\mathrm{Ind}(s_4) - \mathrm{Ind}(s_3)) + (\mathrm{Ind}(s_3 - \mathrm{Ind}(s_2))} = 0$$

$$P(x_1 > x_3) = \frac{\max(0, \mathrm{Ind}(s_4) - \mathrm{Ind}(s_2)) - \max(0, \mathrm{Ind}(s_3 - \mathrm{Ind}(s_4))}{(\mathrm{Ind}(s_4) - \mathrm{Ind}(s_3)) + (\mathrm{Ind}(s_4 - \mathrm{Ind}(s_2))} = 0.667$$

$$P(x_3 > x_1) = \frac{\max(0, \mathrm{Ind}(s_4) - \mathrm{Ind}(s_3)) - \max(0, \mathrm{Ind}(s_2 - \mathrm{Ind}(s_4))}{(\mathrm{Ind}(s_4) - \mathrm{Ind}(s_3)) + (\mathrm{Ind}(s_4 - \mathrm{Ind}(s_2))} = 0.333$$

$$P(x_2 > x_3) = \frac{\max(0, \mathrm{Ind}(s_3) - \mathrm{Ind}(s_2)) - \max(0, \mathrm{Ind}(s_2 - \mathrm{Ind}(s_4))}{(\mathrm{Ind}(s_3) - \mathrm{Ind}(s_2)) + (\mathrm{Ind}(s_4 - \mathrm{Ind}(s_2))} = 0.333$$

$$P(x_3 > x_2) = \frac{\max(0, \mathrm{Ind}(s_4) - \mathrm{Ind}(s_2)) - \max(0, \mathrm{Ind}(s_2 - \mathrm{Ind}(s_3))}{(\mathrm{Ind}(s_3) - \mathrm{Ind}(s_2)) + (\mathrm{Ind}(s_4 - \mathrm{Ind}(s_2))} = 0.667$$

$$P_D = \begin{pmatrix} - & 1 & 0.667 \\ 0 & - & 0.333 \\ 0.333 & 0.667 & - \end{pmatrix}.$$

us to discover the reciprocal preference degree between both intervals. Therefore, in this Appendix, the method that is proposed in [35] is reviewed to obtain a preference relation from a vector of intervals, and it is used in the exploitation phase of the multicriteria linguistic decision-making model presented in Section V.

Definition 24: [35]. Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ be two interval utilities; the preference degree of A over B (or $A > B$) is defined as

$$P(A > B) = \frac{\max(0, a_2 - b_1) - \max(0, a_1 - b_2)}{(a_2 - a_1) + (b_2 - b_1)}$$

and the preference degree of B over A (or $B > A$) is defined as

$$P(B > A) = \frac{\max(0, b_2 - a_1) - \max(0, b_1 - a_2)}{(a_2 - a_1) + (b_2 - b_1)}.$$

It is obvious that $P(A < B) + P(B > A) = 1$ and $P(A > B) = P(B > A) = 0.5$, when $A = B$, $a_1 = b_1$ and $a_2 = b_2$.

Therefore, the preference relation for the alternatives is obtained as follows.

Definition 25: [35]. Let P_D be a preference relation

$$P_D = \begin{pmatrix} - & p_{12} & \cdots & p_{1n} \\ p_{21} & - & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & - \end{pmatrix}$$

where

$$p_{ij} = P(x_i > x_j) = \frac{\max(0, x_{iR} - x_{jL}) - \max(0, x_{iL} - x_{jR})}{(x_{iR} - x_{iL}) - (x_{jR} - x_{jL})}$$

is the preference degree of the alternative x_i over x_j; $i, j \in \{1, \ldots, n\}$; $i \neq j$, and $x_i = [x_{iL}, x_{iR}]$, $x_j = [x_{jL}, x_{jR}]$.

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