Applying Linguistic OWA Operators in Consensus Models under Unbalanced Linguistic Information

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Abstract. In Group Decision Making (GDM) the automatic consensus models are guided by different consensus measures which usually are obtained by aggregating similarities observed among experts' opinions. Most GDM problems based on linguistic approaches use symmetrically and uniformly distributed linguistic term sets to express experts' opinions. However, there exist problems whose assessments need to be represented by means of unbalanced linguistic term sets, i.e., using term sets which are not uniformly and symmetrically distributed. The aim of this paper is to present different Linguistic OWA Operators to compute the consensus measures in consensus models for GDM problems with unbalanced fuzzy linguistic information.

1 Introduction

In a classical Group Decision Making (GDM) situation there is a problem to solve, a solution set of possible alternatives, and a group of two or more experts, who express their opinions about this solution set of alternatives. These problems consists of multiple individuals interacting to reach a decision. Each expert may have unique

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motivations or goals and may approach the decision process from a different angle, but have a common interest in reaching eventual agreement on selecting the "best" option(s) [5, 14, 40].

Usually, many problems present quantitative aspects which can be assessed by means of precise numerical values [6, 27, 26, 37]. However, some problems present also qualitative aspects that are complex to assess by means of precise and exact values. In these cases, the fuzzy linguistic approach [15, 31, 46, 47, 53, 54, 55] can be used to obtain a better solution. This is the case, for instance, when experts try to evaluate the "comfort" of a car, where linguistic terms like "good", "fair", "poor" are used [38]. Many of these problems use linguistic variables assessed in linguistic term sets whose terms are uniformly and symmetrically distributed, i.e., assuming the same discrimination levels on both sides of mid linguistic term (see Fig. 1). However, there exist problems that need to assess their variables with linguistic term sets that are not uniformly and symmetrically distributed [17, 30]. This type of linguistic term sets are called *unbalanced linguistic term sets* (see Fig. 2).

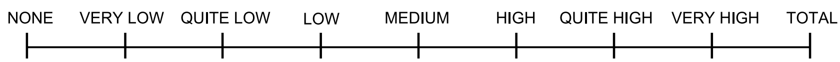

Fig. 1 Example of a linguistic term set of 9 labels

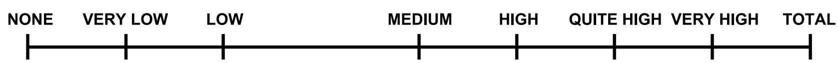

Fig. 2 Example of an unbalanced linguistic term set of 8 labels

To solve GDM problems, the experts are faced by applying two processes before obtaining a final solution [18, 22, 28, 36, 37]: *the consensus process* and *the selection process* (see Fig. 3). The former consists in obtaining the maximum degree of consensus or agreement between the set of experts on the solution set of alternatives. Normally, the consensus process is guided by a human figure called moderator [18, 22, 36], who is a person that does not participate in the discussion but monitors the agreement in each moment of the consensus process and is in charge of supervising and addressing the consensus process toward success, i.e., to achieve the maximum possible agreement and to reduce the number of experts outside of the consensus in each new consensus round. The latter refers to obtaining the solution set of alternatives from the opinions on the alternatives given by the experts. It involves two different steps [23, 41]: aggregation of individual opinions and exploitation of the collective opinion. Clearly, it is preferable that the set of experts achieves a great agreement among their opinions before applying the selection process and, therefore, in this paper we focus on the consensus process.

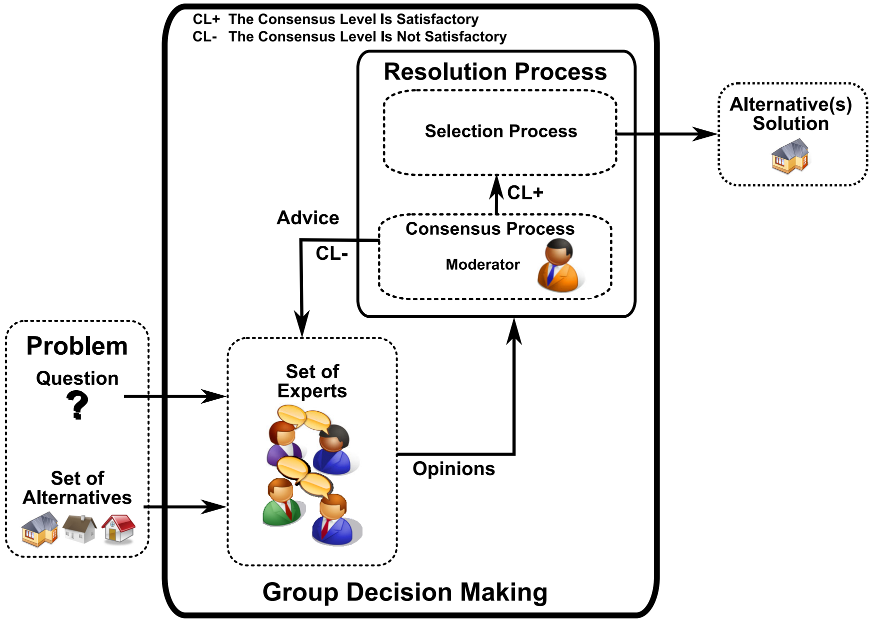

Fig. 3 Resolution process of a GDM problem

A consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator helping experts bring their opinions closer. The moderator uses a consensus measure to assess the consensus level existing among experts. If the consensus level is lower than a specified threshold, the moderator would urge experts to discuss their opinions further in an effort to bring them closer. On the contrary, when the consensus level is higher than the threshold, the moderator would apply the selection process in order to obtain the final consensus solution to the GDM problem.

In such a framework, we find different aspects to solve:

1. An important question is how to substitute the actions of the moderator in the group discussion process in order to automatically model the whole consensus process. Some automatic consensus approaches have been proposed in [4, 26, 28, 31, 39].
2. Most of these consensus models use only consensus measures to control and guide the consensus process. However, if a consensus process is seen as a type of persuasion model [12], other criteria could be used to guide consensus reaching processes as, for example, the *cooperation* or *consistency criterion*. Some fuzzy consensus approaches based on both consistency and consensus measures can be found in [11, 13, 21, 26].
3. On the other hand, a natural question in the consensus process is how to measure the closeness among experts' opinions in order to obtain the consensus

measure. To do so, different approaches have been proposed. For instance, several authors have introduced *hard consensus measures* varying between 0 (no consensus or partial consensus) and 1 (full consensus o complete agreement) [1, 2, 43, 44]. However, consensus as a full and unanimous agreement is far from being achieved in real situations, and even if it is, in such a situation, the consensus reaching process could be unacceptably costly. So, in practice, a more realistic approach is to use *softer consensus measures* [33, 34, 35], which assess the consensus degree in a more flexible way, and therefore reflect the large spectrum of possible partial agreements, and guide the consensus process until widespread agreement (not always full) is achieved among experts. The soft consensus measures are based on the concept of coincidence [22], measured by means of similarity criteria defined among experts' opinions.

4. Sometimes, we find problems when it is not possible to compute directly the similarity among opinions because experts provide incomplete preferences [27], or use different elements of preference representation [6], or different expression domains of preferences as multi-granular fuzzy linguistic contexts [25] or unbalanced fuzzy linguistic contexts [17, 30]. In [28, 31, 26], we have presented consensus models dealing with different elements of preference representation, multi-granular linguistic preferences and incomplete preferences, respectively, and, in this paper, we focus on consensus models under unbalanced fuzzy linguistic preferences.
5. Other aspect to study is how to obtain the consensus measures from closeness values measured among experts' opinions. Usually, this is done by aggregating those closeness values by means of adequate aggregation operators. The OWA type operators [48] are very useful to develop such aggregations because they allows us to include different semantics in the aggregation process, as for example, the concept of fuzzy majority [32] or consistency semantics [9].

The aim of this paper is to present some Linguistic OWA operators to compute consensus measures in GDM problems under unbalanced linguistic preferences. As in [18, 21], we assume a consensus model which is guided by two types of consensus measures, consensus degrees and proximity measures. We present two LOWA operators to compute those consensus measures in an unbalanced linguistic context: an unbalanced LOWA operator guided by the concept of fuzzy majority to compete the consensus degrees and an unbalanced Induced LOWA operator guided by the consistency semantics to compute the proximity measures.

In order to do this, the paper is structured as follows. In Section 2, we present some preliminaries. In Section 3 we define a methodology to manage unbalanced fuzzy linguistic information together with the unbalanced LOWA operators. Section 4 presents the application of those unbalanced LOWA operators in a consensus model for GDM problems with unbalanced fuzzy linguistic preferences. Finally, some concluding remarks are pointed out in Section 5.

2 Preliminaries

In this section, we make a review of the 2-tuple fuzzy linguistic representation model[24] and the concept of hierarchical linguistic contexts[25] which are used to define the methodology to manage unbalanced fuzzy linguistic information.

2.1 The 2-Tuple Fuzzy Linguistic Representation Model

The 2-tuple fuzzy linguistic representation model was introduced in [24] to carry out processes of computing with words in a precise way when the linguistic term sets are symmetrically and uniformly distributed and to improve several aspects of the ordinal fuzzy linguistic approach [15, 19, 20]. This model is based on the concept of *symbolic translation* and represents the linguistic information by means of a pair of values, (s,α), where s is a linguistic label and α is a numerical value that represents the value of the symbolic translation.

Definition 1. [24]Let $\beta \in [0,g]$ be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$, where g stands for cardinality of S, i.e., the result of a symbolic aggregation operation. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values, such that, $i \in [0,g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation.

This model defines a set of transformation functions to manage the linguistic information expressed by linguistic 2–tuples.

Definition 2. Let S be a linguistic term set and $\beta \in [0,g]$ a value supporting the result of a symbolic aggregation operation, then the 2–tuple that expresses the equivalent information to β is obtained with the following function:

$$\begin{array}{l} \Delta : [0,g] \longrightarrow S \times [-0.5, 0.5) \\ \quad \Delta(\beta) = (s_i, \alpha) \\ \quad i = round(\beta) \\ \quad \alpha = \beta - i \end{array} \tag{1}$$

where "round" is the usual round operation, s_i has the closest index label to β and α is the value of the symbolic translation.

Proposition 1. *Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and (s_i, α) be a linguistic 2-tuple. There is always a function Δ^{-1}, such that, from a 2-tuple value it returns its equivalent numerical value $\beta \in [0,g] \subset \mathscr{R}$:*

$$\begin{array}{c} \Delta^{-1} : S \times [-0.5, 0.5) \longrightarrow [0,g] \\ \Delta^{-1}(s_i, \alpha) = i + \alpha = \beta \end{array} \tag{2}$$

Remark 1. We should point out that a linguistic term can be seen as a linguistic 2–tuple by adding to it the value 0 as symbolic translation, $s_i \in S \Longrightarrow (s_i, 0)$.

The 2-tuples linguistic computational model presents different techniques to manage the linguistic information [24]:

1. *A 2-tuple comparison operator:* The comparison of linguistic information represented by 2-tuples is carried out according to an ordinary lexicographic order. Let (s_k, α_1) and (s_l, α_2) be two 2-tuples, with each one representing a counting of information:
   a. if $k < l$ then (s_k, α_1) is smaller than (s_l, α_2).
   b. if $k = l$ then
      i. if $\alpha_1 = \alpha_2$ then (s_k, α_1), (s_l, α_2) represent the same information.
      ii. if $\alpha_1 < \alpha_2$ then (s_k, α_1) is smaller than (s_l, α_2).
      iii. if $\alpha_1 > \alpha_2$ then (s_k, α_1) is bigger than (s_l, α_2).
2. *A 2-tuple negation operator:* It is defined as

$$Neg(s_i, \alpha) = \Delta(g - \Delta^{-1}(s_i, \alpha)). \tag{3}$$

3. *2-tuple aggregation operators:* Using the function Δ and Δ^{-1} any aggregation operator can be easily extended for dealing with linguistic 2-tuples, such as the Linguistic OWA operator [20], the weighted average operator, the OWA operator, etc., (see [24]).

2.2 Hierarchical Linguistic Contexts

In [25] the hierarchical linguistic contexts were introduced to improve the precision of processes of computing with words in multi-granular linguistic contexts [31]. In this work, we use them to manage the unbalanced fuzzy linguistic information.

A *Linguistic Hierarchy* is a set of levels, where each level represents a linguistic term set with different granularity from the remaining levels of the hierarchy. Each level is denoted as $l(t, n(t))$, where t is a number indicating the level of the hierarchy, and $n(t)$ is the cardinality of the linguistic term set of t. Moreover, we assume levels containing linguistic terms whose membership functions are triangular-shaped, uniformly and symmetrically distributed in $[0,1]$, and linguistic term sets having an odd value of granularity where the central label represents the value of *indifference*. A graphical example of a linguistic hierarchy is shown in Fig. 4.

The levels belonging to a linguistic hierarchy are ordered according to their granularity, i.e., for two consecutive levels t and $t+1$, $n(t+1) > n(t)$. Hence, the level $t+1$ could be considered as a refinement of the previous level t. Then, a linguistic hierarchy LH can be defined as the union of all levels t:

$$LH = \bigcup_t l(t, n(t)). \tag{4}$$

Given a LH, we denote as $S^{n(t)}$ the linguistic term set of LH corresponding to the level t of LH characterized by a granularity of uncertainty $n(t)$: $S^{n(t)} =$

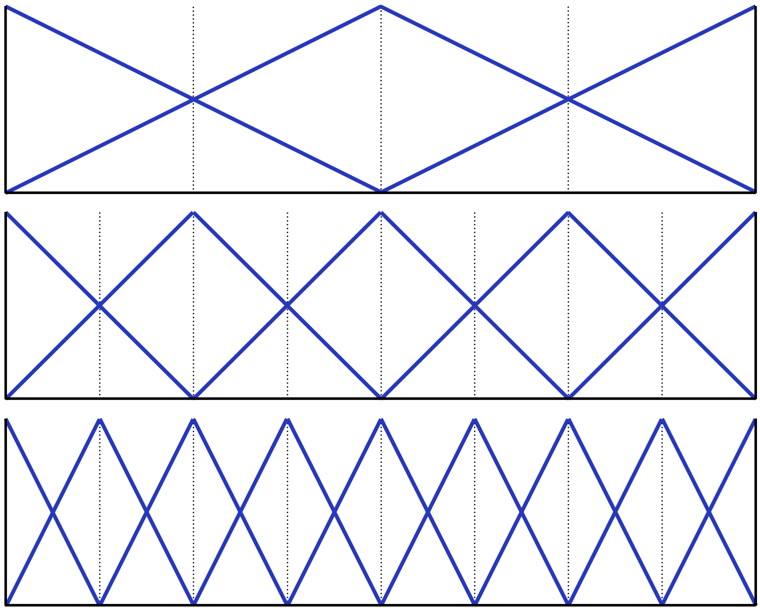

Fig. 4 Linguistic hierarchy of 3, 5 and 9 labels

$\{s_0^{n(t)},\ldots,s_{n(t)-1}^{n(t)}\}$. Furthermore, the linguistic term set of the level $t+1$ is obtained from its predecessor as:

$$l(t,n(t)) \rightarrow l(t+1, 2\cdot n(t)-1). \tag{5}$$

Transformation functions between labels from different levels to make processes of computing with words in multigranular linguistic information contexts without loss of information were defined in [25].

Definition 3. [25] Let $LH = \bigcup_t l(t,n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $S^{n(t)} = \{s_0^{n(t)},\ldots,s_{n(t)-1}^{n(t)}\}$, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic label in level t to a label in level t' is defined as $TF_{t'}^{t} : l(t,n(t)) \longrightarrow l(t',n(t'))$ such that

$$TF_{t'}^{t}(s_i^{n(t)},\alpha^{n(t)}) = \Delta_{t'}\left(\frac{\Delta_t^{-1}(s_i^{n(t)},\alpha^{n(t)})\cdot(n(t')-1)}{n(t)-1}\right). \tag{6}$$

3 A Model to Manage Unbalanced Fuzzy Linguistic Information

Following those results presented in [17, 30], a model to manage unbalanced fuzzy linguistic term sets based on the linguistic 2-tuple model is presented. It carries out computational operations of unbalanced fuzzy linguistic information using the 2-tuple computational model and different levels of a LH. This model presents two components:

- A *representation model of unbalanced fuzzy linguistic information.*
- A *computational model of unbalanced fuzzy linguistic information.*

3.1 An Unbalanced Fuzzy Linguistic Representation Model

The procedure to represent unbalanced fuzzy linguistic information defined in [30] works as follows:

1. Find a level t^- of LH to represent the subset of linguistic terms S_{un}^L on the left of the mid linguistic term of unbalanced fuzzy linguistic term set S_{un}. This level of LH should support the distribution of the labels of S_{un}^L on the discourse universe.
2. Find a level t^+ of LH to represent the subset of linguistic terms S_{un}^R on the right of the mid linguistic term of S_{un}.
3. Represent the mid term of S_{un} using the mid terms of the levels t^- and t^+.

The problem appears when there does not exist a level t^- or t^+ in LH to represent S_{un}^L or S_{un}^R, respectively. Then, we propose to overcome this problem by applying the following algorithm, which is defined assuming that there does not exist t^-, as it happens with the unbalanced fuzzy linguistic term set given in Fig. 2:

1. Represent S_{un}^L:
   a. Identify the mid term of S_{un}^L, called S_{mid}^L. To do so, we have to observe the distribution of the labels of S_{un}^L on the discourse universe.
   b. Find a level t_2^- of the left sets of LH^L to represent the left term subset of S_{un}^L, where LH^L represents the left part of LH.
   c. Find a level t_2^+ of the right sets of LH^L to represent the right term subset of S_{un}^L.
   d. Represent the mid term S_{mid}^L using the levels t_2^- and t_2^+.
2. Find a level t^+ of LH to represent the subset of linguistic terms S_{un}^R.
3. Represent the mid term of S_{un} using the levels t^+ and t_2^+.

For example, applying this algorithm, the representation of the unbalanced fuzzy linguistic term set $S_{un} = \{N, VL, L, M, H, QH, VH, T\}$ shown in Fig. 2 with the linguistic hierarchy LH shown in Fig. 4 would be as it is shown in Fig. 5. In this example,

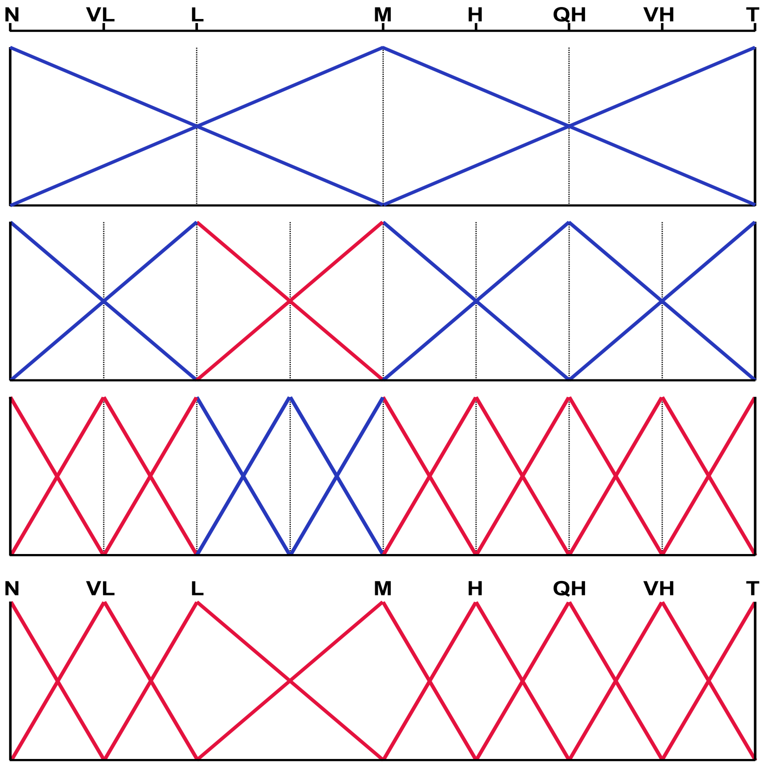

Fig. 5 Representation for an unbalanced term set of 8 labels

- $S^L_{un} = \{N, VL, L\}$,
- $S^L_{mid} = L$,
- $LH^L = \{s_0^{n(1)}\} \bigcup \{s_0^{n(2)}, s_1^{n(2)}\} \bigcup \{s_0^{n(3)}, s_1^{n(3)}, s_2^{n(3)}, s_3^{n(3)}\}$.

Thus, we have that $t_2^- = 3$, $t_2^+ = 2$, the mid label $S^L_{mid} = L$ (due to its position on the discourse universe) is represented using both levels, 3 and 2, and the mid term of S_{un} is represented using the levels 2 and 3.

3.2 An Unbalanced Fuzzy Linguistic Computational Model: Some Unbalanced Linguistic OWA Operators

In any fuzzy linguistic approach we need to define a computational model to manage and aggregate linguistic information. As in [24] we have to define three types of computation operators to deal with unbalanced fuzzy linguistic information: comparison operators, negation operator and aggregation operators. In an unbalanced linguistic context, previously to carry out any computation task of unbalanced fuzzy linguistic information we have to choose a level $t' \in \{t^-, t_2^-, t^+, t_2^+\}$, such that $n(t') = max\{n(t^-), n(t_2^-), n(t^+), n(t_2^+)\}$:

1. *An unbalanced linguistic comparison operator:* The comparison of linguistic information represented by two unbalanced linguistic 2-tuples $(s_k^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $(s_l^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ is similar to the usual comparison of two 2-tuples but acting on the values $TF_{t'}^t(s_k^{n(t)}, \alpha_1) = (s_v^{n(t')}, \beta_1)$ and $TF_{t'}^t(s_l^{n(t)}, \alpha_2) = (s_w^{n(t')}, \beta_2)$. Then, we have:

   a. if $v < w$ then $(s_v^{n(t')}, \beta_1)$ is smaller than $(s_w^{n(t')}, \beta_2)$.
   b. if $v = w$ then
      i. if $\beta_1 = \beta_2$ then $(s_v^{n(t')}, \beta_1)$, $(s_w^{n(t')}, \beta_2)$ represent the same information.
      ii. if $\beta_1 < \beta_2$ then $(s_v^{n(t')}, \beta_1)$ is smaller than $(s_w^{n(t')}, \beta_2)$.
      iii. if $\beta_1 > \beta_2$ then $(s_v^{n(t')}, \beta_1)$ is bigger than $(s_w^{n(t')}, \beta_2)$.

2. *An unbalanced linguistic 2-tuple negation operator.* Let $(s_k^{n(t)}, \alpha)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$ be an unbalanced linguistic 2-tuple, then:

$$NEG(s_k^{n(t)}, \alpha) = Neg(TF_{t''}^t(s_k^{n(t)}, \alpha)), \tag{7}$$

   where $t \neq t''$, $t'' \in \{t^-, t_2^-, t^+, t_2^+\}$.
3. *An unbalanced linguistic aggregation operator.* As aforementioned, in order to deal with unbalanced fuzzy linguistic information we have to represent it in a LH. Hence, any unbalanced linguistic aggregation operator must aggregate unbalanced fuzzy linguistic information by means of its representation in a LH. We use the aggregation processes designed in the 2-tuple computational model but acting on the unbalanced linguistic values transformed by means of $TF_{t'}^t$. Then, once a result is obtained, it is transformed to the correspondent level $t \in \{t^-, t_2^-, t^+, t_2^+\}$ by means of $TF_t^{t'}$ for expressing the result in the unbalanced linguistic term set S_{un}. In such a way, we define the following unbalanced linguistic OWA operators: the $LOWA_{un}$ operator which is an extension of the Linguistic Ordered Weighted Averaging operator proposed in [20] and the $ILOWA_{un}$ operator which is a linguistic extension of the Induced OWA operators [9, 50, 51, 52].

   • **Definition 4.** Let $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$ be a set of unbalanced linguistic assessments to aggregate, then the $LOWA_{un}$ operator ϕ_{un} is defined as:

$$\phi_{un}\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\} = W \cdot B^T = C_{un}^m\{w_k, b_k,\ k = 1, \ldots, m\} =$$

$$w_1 \otimes b_1 \oplus (1 - w_1) \otimes C_{un}^{m-1}\{\beta_h, b_h,\ h = 2, \ldots, m\}$$

   where $b_i = (a_i, \alpha_i) \in (S^{n(t)} \times [-0.5, 0.5))$, $W = [w_1, \ldots, w_m]$, is a weighting vector, such that, $w_i \in [0,1]$ and $\sum_i w_i = 1$, $\beta_h = \frac{w_h}{\sum_2^m w_k}$, $h = \{2, 3 \ldots, m\}$, and B is the associated ordered unbalanced 2-tuple vector. Each element $b_i \in B$ is the i-th largest unbalanced 2-tuple in the collection $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$, and C^m_{un} is the convex combination operator of m unbalanced 2-tuples. If $w_j = 1$ and $w_i = 0$ with $i \neq j$, $\forall i, j$ the convex combination is defined as: $C^m_{un}\{w_i, b_i,\ i = 1, \ldots, m\} = b_j$. And if $m = 2$ then it is defined as:

$$C^2_{un}\{w_l, b_l,\ l = 1,2\} = w_1 \otimes b_j \oplus (1 - w_1) \otimes b_i = TF^{t'}_t(s^{n(t')}_k, \alpha)$$

where $(s^{n(t')}_k, \alpha) = \Delta(\lambda)$ and $\lambda = \Delta^{-1}(TF^t_{t'}(b_i)) + w_1 \cdot (\Delta^{-1}(TF^t_{t'}(b_j)) - \Delta^{-1}(TF^t_{t'}(b_i)))$, $b_j, b_i \in (S^{n(t)} \times [-0.5, 0.5))$, $(b_j \geq b_i)$, $\lambda \in [0, n(t') - 1]$, $t \in \{t^-, t^-_2, t^+, t^+_2\}$.

In[48] it was defined an expression to obtain W by means of a fuzzy linguistic non-decreasing quantifier Q [56]:

$$w_i = Q(i/m) - Q((i-1)/m),\ \ i = \{1, 2, \ldots, m\}. \quad (8)$$

In such a way, it is possible to incorporate in the aggregation process the semantics of the fuzzy majority [32] represented by the quantifier. When the $LOWA_{un}$ operator uses a quantifier Q then it is called ϕ^Q_{un}.

- **Definition 5.** Let $\{(a_1, \alpha_1), \ldots, (a_m, \alpha_m)\}$ and $(u_1, \ldots, u_m)$ $u_i \in \mathscr{R}$ be a set of unbalanced linguistic assessments to aggregate and the set of values used to induce the ordering of the unbalanced linguistic assessments, respectively. Then, an $ILOWA_{un}$ operator Φ_{un} is defined as:

$$\Phi_{un}(\langle u_1, p_1\rangle, \ldots, \langle u_m, p_m\rangle) = TF^{t'}_t(\sum_{i=1}^{m} w_i \cdot \Delta^{-1}(TF^t_{t'} p_{\sigma(i)})), \quad (9)$$

being $p_i = (a_i, \alpha_i)$ and σ a permutation of $\{1, \ldots, m\}$ such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}$, $\forall i = 1, \ldots, m-1$, i.e., $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the pair with $u_{\sigma(i)}$ the i-th highest value in the set $\{u_1, \ldots, u_m\}$.

In the above definition, the reordering of the set of values to be aggregated, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated with them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable[9, 50, 51, 52]. A natural question in the definition of the unbalanced ILOWA operator is how to obtain the associated weighting vector. Following Yager's ideas on quantifier guided aggregation [49], we could compute the weighting vector of an IOWA operator using a linguistic quantifier Q [56] as:

$$w_i = Q\left(\frac{\sum_{k=1}^{i} u_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{i-1} u_{\sigma(k)}}{T}\right), \quad (10)$$

being $T = \sum_{k=1}^{n} u_k$ and σ the permutation used to produce the ordering of the values to be aggregated.

4 A Consensus Model for GDM Problems Based on Unbalanced Linguistic OWA Operators

In this section, we present a consensus model defined for GDM problems with unbalanced fuzzy linguistic preference relations providing support to the experts to reach consensus during the process of making a decision. This consensus model presents the following main characteristics:

1. It is designed to guide the consensus process of unbalanced fuzzy linguistic GDM problems.
2. It is based on two consensus criteria: *consensus degrees* and *proximity measures*. The first ones are used to measure the agreement amongst all the experts, while the second ones are used to learn how close the collective and individual expert's preference are. Both consensus criteria are calculated at three different levels: pair of alternatives, alternatives and relation.
3. It uses unbalanced linguistic OWA operators to compute the above consensus criteria.
4. A *feedback mechanism* is defined using the above consensus criteria. It substitutes the moderator's actions, avoiding the possible subjectivity that he/she can introduce, and gives advice to the experts to find out the changes they need to make in their opinions in order to obtain the highest degree of consensus possible.

This consensus model presents three phases:

1. *Computing consensus degrees.*
2. *Controlling the consensus state.*
3. *Feedback mechanism.*

In the following subsections, we describe them in detail.

4.1 Computing Consensus Degrees

A GDM problem based on preference relations is classically defined as a decision situation where there are a set of experts, $E = \{e_1, \ldots, e_m\}$ $(m \geq 2)$, and a finite set of alternatives, $X = \{x_1, \ldots, x_n\}$ $(n \geq 2)$, and each expert e_i provides his/her preferences about X by means of a preference relation, $P_{e_i} \subset X \times X$, where the value $\mu_{P_{e_i}}(x_j, x_k) = p_i^{jk}$ is interpreted as the preference degree of the alternative x_j over x_k for e_i. In this paper, we deal with unbalanced fuzzy linguistic preference relations, i.e., $P_{e_i} = (p_i^{jk}) \in S_{un}$, and therefore, p_i^{jk} represents the preference of alternative x_j over alternative x_k for the expert e_i assessed on an unbalanced fuzzy linguistic term set S_{un}.

Then, consensus degrees are used to measure the current level of consensus in the decision process. As aforementioned, they are given at three different levels: pairs of alternatives, alternatives and relations. To calculate them, some similarity or coincidence function are required to obtain the level of agreement amongst all the experts [18, 22, 31]. Moreover, these similarity functions detect how far each individual expert is from the rest. In such a way, the computation of the consensus degrees is carried out as follows:

1. For each pair of experts (e_i, e_j) $(i = 1,\ldots,m-1,\ j = i+1,\ldots,m)$, an unbalanced linguistic similarity matrix, $SM_{ij} = (sm_{ij}^{lk})$, $sm_{ij}^{lk} \in (S^{n(t)} \times [-0.5, 0.5))$, is defined as

$$sm_{ij}^{lk} = NEG(TF_t^{t'}(\Delta(|\Delta^{-1}(TF_{t'}^{t}(p_i^{lk})) - \Delta^{-1}(TF_{t'}^{t}(p_j^{lk}))|))). \quad (11)$$

   being $p_i^{lk} = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_j^{lk} = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.
2. An unbalanced linguistic consensus matrix, $CM = (cm^{lk})$, is calculated by aggregating all the similarity matrices using the $LOWA_{un}$ operator ϕ_{un}^{Q} as the aggregation function:

$$cm^{lk} = \phi_{un}^{Q}(sm_{ij}^{lk},\ i = 1,\ldots,m-1,\ j = i+1,\ldots,m). \quad (12)$$

3. Once the consensus matrix, CM, is computed, we proceed to calculate the consensus degrees at the three different levels:

   a. **Level 1.** *Unbalanced linguistic consensus degree on pairs of alternatives.* The consensus degree on a pair of alternatives (x_l, x_k), called cp^{lk}, is defined to measure the consensus degree amongst all the experts on that pair of alternatives. The closer $\frac{\Delta^{-1}(cp^{lk})}{n(t)-1}$ to 1, the greater the agreement amongst all the experts on the pair of alternatives (x_l, x_k). Thus, this measure is used to identify those pairs of alternatives with a poor level of consensus and it coincides is with the element (l,k) of the consensus matrix CM:

$$cp^{lk} = cm^{lk};\ \forall\, l,k = \{1,2,\ldots,n\} \wedge l \neq k. \quad (13)$$

   b. **Level 2.** *Unbalanced linguistic consensus degree on alternatives.* The consensus degree on an alternative x_l, called ca^l, is defined to measure the consensus degree amongst all the experts on that alternative:

$$ca^l = \phi_{un}^{Q}(cp^{l1},\ldots,cp^{ln}). \quad (14)$$

   c. **Level 3.** *Unbalanced linguistic consensus degree on the relation.* The consensus degree on the relation, called cr, is defined to measure the global consensus degree amongst all the experts' opinions and is used by the consensus model to control the consensus situation. It is calculated as:

$$cr = \phi_{un}^{Q}(ca^1,\ldots,ca^n). \quad (15)$$

4.2 Controlling the Consensus State

The consensus state control process involves deciding if the feedback mechanism should be applied to provide advice to the experts or if the consensus process should be finished. To do so, a minimum consensus threshold, $\gamma \in [0,1]$, is fixed before applying the consensus model. When the consensus measure, cr, satisfies the minimum consensus threshold, γ, the consensus model finishes and the selection process is applied to obtain the solution. Additionally, the consensus model should avoid situations in which the global consensus measure may not converge to the minimum consensus threshold. To do that, a maximum number of rounds *MaxRounds* should be fixed and compared to the current number of round of the consensus model *NumRound*.

Then, the operation of the consensus state control process is as follows: Firstly, the global consensus measure, cr, is checked against the minimum consensus threshold, γ. If $\frac{\Delta^{-1}(cr)}{n(t)-1} > \gamma$, the consensus process finishes and the selection process is applied. Otherwise, it will check if the maximum number of rounds, *MaxRounds*, has been reached. If so, it finishes and the selection process is applied too, and if not, it activates the feedback mechanism.

4.3 Feedback Mechanism

If the global consensus measure is lower than the minimum consensus threshold then the experts' opinions must be modified. The goal of the feedback mechanism is to provide recommendations to support the experts in changing their opinions. The feedback mechanism uses proximity measures to identify those experts furthest away from the collective opinion. In the following, both the computation of the proximity measures and the production of advice are explained in detail.

4.3.1 Computation of Proximity Measures

These measures evaluate the agreement between the individual experts' opinions and the group opinion. To compute them for each expert, we need to obtain the collective unbalanced fuzzy linguistic preference relation, $P_{e_c} = (p_c^{lk})$, which summarizes preferences given by all the experts. We compute it by means of the aggregation of the set of individual unbalanced fuzzy linguistic preference relations $\{P_{e_1}, \ldots, P_{e_m}\}$ using an $IOWA_{un}$ operator, Φ_{un}^{Q}, which allows to obtain each collective preference degree p_{ik}^c according to the most consensual individual preference degrees using the consensus scores of each expert e_h for each pair of alternatives x_i and x_k

$$\{z_{ik}^1, z_{ik}^2, \ldots, z_{ik}^m\}$$

$$z_{ik}^h = \frac{\sum_{l=h+1}^{n}(\Delta^{-1}(sm_{ik}^{hl})/(n(t)-1)) + \sum_{l=1}^{h-1}(\Delta^{-1}(sm_{ik}^{lh})/(n(t)-1))}{n-1}. \tag{16}$$

as the values of the order inducing variable, i.e.,

$$p_{ik}^{c} = \Phi_{un}^{Q}(\langle z_{ik}^{1}, \bar{p}_{ik}^{1}\rangle, \ldots, \langle z_{ik}^{m}, \bar{p}_{ik}^{m}\rangle) = TF_{t}^{t'}(\Delta(\sum_{h=1}^{m} w_h \cdot \Delta^{-1}(TF_{t'}^{t}(\bar{p}_{ik}^{\sigma(h)})))), \quad (17)$$

where

- σ is a permutation of $\{1, \ldots, m\}$ such that $z_{ik}^{\sigma(h)} \geq z_{ik}^{\sigma(h+1)}$, $\forall h = 1, \ldots, m-1$, i.e., $\langle z_{ik}^{\sigma(h)}, \bar{p}_{ik}^{\sigma(h)}\rangle$ is the pair of the linguistic unbalanced 2-tuples with $z_{ik}^{\sigma(h)}$ the h-th highest consensus score in the set $\{z_{ik}^{1} \ldots, z_{ik}^{m}\}$;
- the weighting vector is computed according to the following expression:

$$w_h = Q\left(\frac{\sum_{j=1}^{h} z_{ik}^{\sigma(j)}}{T}\right) - Q\left(\frac{\sum_{j=1}^{h-1} z_{ik}^{\sigma(j)}}{T}\right), \quad (18)$$

  with $T = \sum_{j=1}^{m} z_{ik}^{j}$ and Q a fuzzy linguistic quantifier.

Once P_{e_c} is obtained, we can compute the proximity measures carrying out the following two steps:

1. For each expert, e_i, a proximity matrix, $PM_i = (pm_i^{lk})$, is obtained where

$$pm_i^{lk} = NEG(TF_{t}^{t'}(\Delta(\left|\Delta^{-1}(TF_{t'}^{t}(p_i^{lk})) - \Delta^{-1}(TF_{t'}^{t}(p_c^{lk}))\right|))). \quad (19)$$

   being $p_i^{lk} = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_c^{lk} = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.
2. Computation of proximity measures at three different levels:

   a. **Level 1.** *Unbalanced linguistic proximity measure on pairs of alternatives.* The proximity measure of an expert e_i on a pair of alternatives (x_l, x_k) to the group's one, called pp_i^{lk}, is expressed by the element (l,k) of the proximity matrix PM_i:

$$pp_i^{lk} = pm_i^{lk};\ \forall\, l,k = 1, \ldots, n \wedge l \neq k. \quad (20)$$

   b. **Level 2.** *Unbalanced linguistic proximity measure on alternatives.* The proximity measure of an expert e_i on an alternative x_l to the group's one, called pa_i^{l}, is calculated as follows:

$$pa_i^{l} = \phi_{un}^{Q}(pp_i^{l1}, \ldots, pp_i^{ln}). \quad (21)$$

   c. **Level 3.** *Unbalanced linguistic proximity measure on the relation.* The proximity measure of an expert e_i on his/her unbalanced fuzzy linguistic preference relation to the group's one, called pr_i, is calculated as the average of all proximity measures on the alternatives:

$$pr_i = \phi_{un}^{Q}(pa_i^{1}, \ldots, pa_i^{n}). \quad (22)$$

Then, we can use them to provide advice to the experts to change their opinions and to find out which direction that change has to follow in order to obtain the highest degree of consensus possible.

4.3.2 Production of Advice

The production of advice to achieve a solution with the highest degree of consensus possible is carried out in two steps: *Identification rules* and *Direction rules*.

1. **Identification rules (IR).** We must identify the experts, alternatives and pairs of alternatives that are contributing less to reach a high degree of consensus and, therefore, should participate in the change process.

   a. *Identification rule of experts (IR.1).* It identifies the set of experts that should receive advice on how to change some of their preference values. This set of experts, called $EXPCH$, that should change their opinions are those whose proximity measure on the relation, pr_i, is lower than the minimum consensus threshold γ. Therefore, the identification rule of experts, IR.1, is the following:

   $$EXPCH = \{i \mid (\frac{\Delta^{-1}(pr_i)}{n(t)-1}) < \gamma\} \tag{23}$$

   b. *Identification rule of alternatives (IR.2).* It identifies the alternatives whose associated assessments should be taken into account by the above experts in the change process of their preferences. This set of alternatives is denoted as ALT_i. The identification rule of alternatives, IR.2, is the following:

   $$ALT_i = \{x_l \in X \mid (\frac{\Delta^{-1}(ca^l)}{n(t)-1}) < \gamma \wedge e_i \in EXPCH\} \tag{24}$$

   c. *Identification rule of pairs of alternatives (IR.3).* It identifies the particular pairs of alternatives (x_l, x_k) whose respective associated assessments p_i^{lk} the expert e_i should change. This set of pairs of alternatives is denoted as $PALT_i$. The identification rule of pairs of alternatives, IR.3, is the following:

   $$PALT_i = \{(x_l, x_k) \mid x_l \in ALT \wedge e_i \in EXPCH \wedge (\frac{\Delta^{-1}(pp_i^{lk})}{n(t)-1}) < \gamma\} \tag{25}$$

2. **Direction rules (DR).** We must find out the direction of the change to be recommended in each case, i.e., the direction of change to be applied to the preference assessment p_i^{lk}, with $(x_l, x_k) \in PALT_i$. To do this, we define the following two direction rules.

   a. *DR.1.* If $p_i^{lk} > p_c^{lk}$, the expert e_i should decrease the assessment associated to the pair of alternatives (x_l, x_k), i.e., p_i^{lk}.
   b. *DR.2.* If $p_i^{lk} < p_c^{lk}$, the expert e_i should increase the assessment associated to the pair of alternatives (x_l, x_k), i.e., p_i^{lk}.

Remark 2. These direction rules will not be produced when a decrease or increase are suggested to an assessment represented by the first or last label of the unbalanced linguistic term set, respectively.

5 Concluding Remarks

In this paper we have presented an application of LOWA operators in a consensus model for GDM problems with unbalanced fuzzy linguistic preference relations. We have defined two unbalanced LOWA operators to aggregate unbalanced linguistic information to compute the consensus criteria in order to guide the consensus process. In such a way we can incorporate different semantics in computation of the consensus criteria, as the concept of fuzzy majority or consensus semantics.

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