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INTEGRATION OF A CONSISTENCY CONTROL MODULE WITHIN A CONSENSUS MODEL

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In group decision making (GDM) processes, prior to the selection of the best alternative(s), it would be desirable that experts achieve a high degree of consensus or agreement between them. Due to the complexity of most decision making problems, individuals' preferences may not satisfy formal properties. 'Consistency' is one of such properties, and it is associated with the *transitivity property.* Obviously, when carrying out a rational decision making, consistent information, i.e. information which does not imply any kind of contradiction, is more appropriate than information containing some contradictions. Therefore, in a GDM process, consistency should also be sought after. In this paper we present a consensus model for GDM problems that proceeds from consistency to consensus. This model integrates a novel consistency reaching module based on consistency measures. In particular, the model generates advice on how experts should change their preferences in order to increase their consistency. Also, the consensus model is considered adaptive because the search for consensus is adapted to the level of agreement achieved at each consensus round.

Keywords: Decision-making; fuzzy preferences; consistency; transitivity; consensus; recommendations.

1. Introduction

Any decision making problem includes a selection process which involves, as part of it, the choice between the various alternative solutions to the problem.[1] In GDM problems, however, it may happen that some experts from the group would not accept the group choice if they consider that their opinions have not been take into account 'properly'. Indeed, group choice should be based on the desires or preferences of 'all' the individuals in the group, a premise on which democratic theory is based on Ref. 2. Therefore, a consensus process to obtain the maximum degree of agreement between all the experts on the solution set of alternatives seems necessary in any GDM situation.

Preference relations are usually assumed to model experts' preferences in group decision making problems.[3–6] The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time which facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception of the alternatives and, as a consequence, the provided preferences could be not rational. Usually, rationality is related to consistency, which is associated with the *transitivity property*. Many properties have been suggested to model transitivity of a fuzzy preference relation and, consequently, consistency may be measured according to which of these different properties is required to be satisfied. One of these properties is the "additive transitivity", which, as shown in Ref. 7 can be seen as the parallel concept of Saaty's consistency property in the case of multiplicative preference relations.[8]

In any 'rational' decision making process, consistent information, i.e. information which does not imply any kind of contradiction, is more relevant or important than information containing some contradictions. As a consequence, in a GDM context, consistency should also be sought after in order to make rational choices. To do this, it would help if the experts knew how consistent they are. By letting the experts know their associated consistency measures at any moment, they could judge whether or not it is high enough. Also, with this information, experts would be able to analyze their preferences and make the necessary changes to their most inconsistent preference values to increase their global consistency.

In GDM situations, consensus between experts is usually searched using the basic rationality principles that each expert presents. Thus, consistency criteria should be first applied to fix the rationality of each expert and only afterwards experts' agreement should be obtained. If we were to secure consensus and only thereafter consistency, we could destroy the consensus in favor of the individual consistency and the final solution might not be acceptable for the group of experts.

In Refs. 9 and 10, consensus models were proposed for GDM problems which used two types of measurements to guide the consensus reaching process: *consensus degrees* to evaluate the agreement of all the experts, and *proximity degrees* to evaluate the distance between the experts' individual preferences and the group or collective ones.[11] In Refs. 12–14 consensus models which use a recommendation module to help experts to change their preferences were presented. In Ref. 15 a consensus model which adapts the search of preferences to be changed to the current level of agreement in each one of the consensus rounds was defined. In this paper, we continue improving that consensus model: the adaptive consensus module is refined with the introduction of a new recommendation module; a new consistency control module is being integrated within it; and, when necessary, experts are presented with recommendations on how to become more consistent. As a result, in this new adaptive consensus model once the experts provide their individual preference relations, consistency measures for each one are computed. These consistency measures are used to generate a consistency feedback mechanism that generates advice to the most inconsistent experts on the necessary changes to their most inconsistent preference values to increase their global consistency. When this consistency control module has been applied, consensus is sought after.

The rest of the paper is set out as follows. Section 2 presents the structure of a new adaptive consensus model integrating a consistency control module. Section 3 describes in detail the consistency control module. The adaptive consensus model is described in Sec. 4. Finally, Sec. 5 draws our conclusions.

2. An Adaptive Consensus Model with Consistency Control

In this section we present the structure of a new adaptive consensus model with consistency control. The structure of this new consensus model is depicted in Fig. 1. It is composed of two processes:

(i) **Consistency Control Process.** Once experts' preferences are given, their consistency degrees are computed. If an expert is not consistent enough, that expert will receive appropriate recommendations on the changes to his/her preference values in order to increase his/her global consistency to an acceptable/agreed threshold level one. We should point out that the consistency control process is applied only in the first round of the consensus reaching process, because, as we shall show in the following section, when all the individual preference relations have associated a consistency degree above a particular threshold value then any weighted average collective preference relation will also have associated a consistency degree above that threshold value. Adding to this the fact that the consensus process tends to make the individual opinions closer to the collective ones,[12] we conclude that individual consistency degrees will tend towards the collective one and therefore above the threshold value. Therefore, it is unnecessary to control the consistency level of each expert in each consensus round.

(ii) **Adaptive Consensus Reaching Process.** Once the experts have changed their opinions according to the consistency recommendations, an adaptive consensus reaching process is carried out. This process is considered adaptive because the search for consensus is adapted to the level of agreement achieved in each consensus round. The model will follow different recommendation policies

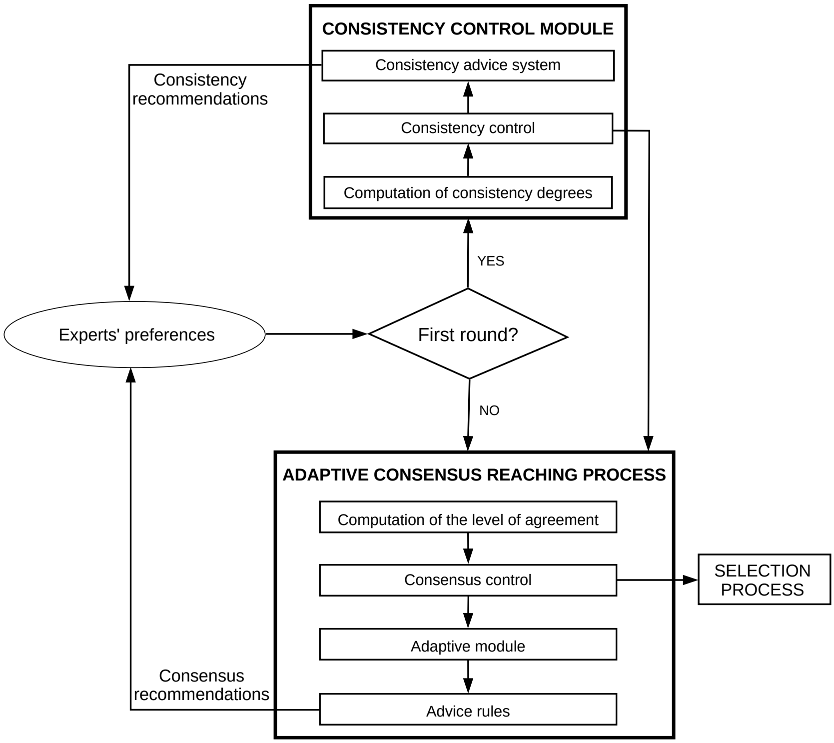

Fig. 1. Adaptive consensus model with consistency control.

depending on a rough classification of the consensus level in 'low', 'medium' and 'high'. In the first case, the model will recommend changes to 'all' experts on 'all' preference values in which disagreement has been identified; in the second case, the alternatives and their associated preference values in which disagreement still exists are identified, and only those experts furthest from the group (as a collective) on those alternatives will be advised to make changes on the identified preference values; while in the last case, only those experts furthest from the group on the alternatives and on the preference values in which disagreement still exists will be advised to change the identified preference values.

Both processes are explained in detail in Secs. 3 and 4, respectively.

3. Consistency Control Module

The purpose of the consistency control module is to measure the level of consistency of each individual preference relation (expert) in order to identify the experts, alternatives and preference values most/more inconsistent within the GDM problem. This inconsistency identification is also used to suggest new 'consistent' preference

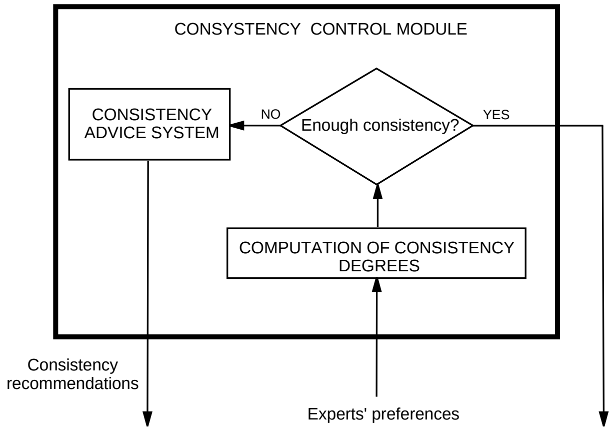

Fig. 2. Consistency control module.

values. The consistency control module develops its activity by means of three processes as illustrated in Fig. 2, which will be covered in the following subsections.

3.1. *Computation of consistency degrees*

In GDM problems with fuzzy preference relations some properties about the preferences expressed by the experts are usually assumed and desirable in order to avoid contradictions in their opinions, i.e, inconsistent opinions. One of these properties is associated with the transitivity in the pairwise comparison among any three alternatives. For fuzzy preference relations, transitivity has been modeled in many different ways due to the role the intensities of preference have.[7] In this paper, we make use of the *additive transitivity property.*

Being $P = (p_{ij})$ a fuzzy preference relation, the mathematical formulation of the *additive transitivity* was given by Tanino in Ref. 16:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \quad \forall i, j, k \in \{1, \ldots, n\}\,. \tag{1}$$

Because additive transitivity implies additive reciprocity $(p_{ij} + p_{ji} = 1 \quad \forall i, j)$, it can be rewritten as:

$$p_{ik} = p_{ij} + p_{jk} - 0.5 \quad \forall i, j, k \in \{1, \ldots, n\}\,. \tag{2}$$

We will consider a fuzzy preference relation P to be "*additive consistent*" when for every three alternatives in the problem $x_i, x_j, x_k \in X$ their associated preference degrees p_{ij}, p_{jk}, p_{ik} fulfil (2).

Given a reciprocal fuzzy preference relation, (2) can be used to calculate an estimated value of a preference degree using other preference degrees. Indeed, using an intermediate alternative x_j, the following estimated value of p_{ik} $(i \neq k)$ is obtained:

$$ep_{ik}^{j} = p_{ij} + p_{jk} - 0.5\,. \tag{3}$$

The overall estimated value ep_{ik} of p_{ik} is obtained as the average of all possible values ep^j_{ik}, i.e.,

$$ep_{ik} = \sum_{\substack{j=1 \\ j \neq i,k}}^{n} \frac{ep^j_{ik}}{n-2} . \tag{4}$$

The value $|ep_{ik} - p_{ik}|$ can be used as a measure of the error between a preference value and its estimated one.[17]

The estimated value of an estimated value, $e^2 p_{ik}$, is:

$$e^2 p_{ik} = ep_{ik} + \frac{2}{n-2} \cdot (p_{ik} - ep_{ik}) . \tag{5}$$

This result implies that this process of estimating preference values converges toward perfect consistency, which is expressed in the following proposition:

Proposition 1. *Let P be a reciprocal fuzzy preference relation. The following holds:*

$$\left| e^r p_{ik} - e^{r-1} p_{ik} \right| = \left(\frac{2}{n-2} \right)^{r-1} |ep_{ik} - p_{ik}| , \; r > 1 . \tag{6}$$

Proof. Proof by induction on the number of estimated steps, r, will be used.

Basis: $r = 2$ In this case (6) derives from (5).
Induction hypothesis: Assume the proposition is true for $r = n$, i.e.

$$\left| e^n p_{ik} - e^{n-1} p_{ik} \right| = \left(\frac{2}{n-2} \right)^{n-1} |ep_{ik} - p_{ik}| .$$

Induction step: Applying (5) we have

$$e^2(e^{n-1} p_{ik}) = e(e^{n-1} p_{ik}) + \frac{2}{n-2} \cdot \left(e^{n-1} p_{ik} - e(e^{n-1} p_{ik}) \right) .$$

Because $e^2(e^{n-1} p_{ik}) = e^{n+1} p_{ik}$ and $e(e^{n-1} p_{ik}) = e^n p_{ik}$ we have that

$$e^{n+1} p_{ik} = e^n p_{ik} + \frac{2}{n-2} \cdot \left(e^{n-1} p_{ik} - e^n p_{ik} \right)$$

and therefore

$$\left| e^{n+1} p_{ik} - e^n p_{ik} \right| = \frac{2}{n-2} \cdot \left| e^n p_{ik} - e^{n-1} p_{ik} \right| .$$

Applying the induction hypothesis, we obtain:

$$\left| e^{n+1} p_{ik} - e^n p_{ik} \right| = \left(\frac{2}{n-2} \right)^n |ep_{ik} - p_{ik}| . \qquad \square$$

When the information provided is completely consistent then $ep^j_{ik} = p_{ik} \; \forall j$. However, because experts are not always fully consistent, the information given by an expert may not verify (2) and some of the estimated preference degree values ep^j_{ik} may not belong to the unit interval $[0, 1]$. We note, from (3), that the maximum value of any of the preference degrees ep_{ik}^j is 1.5 while the minimum one is -0.5. In order to normalize the expression domains in the decision model the final estimated value of p_{ik} $(i \neq k)$, cp_{ik}, is defined as the median of the values 0, 1 and ep_{ik}:

$$cp_{ik} = med\{0, 1, ep_{ik}\}. \tag{7}$$

Example 1. The following are a (reciprocal) fuzzy preference relation and its (reciprocal) estimated fuzzy preference relation

$$P = \begin{pmatrix} - & 0.7 & 0.9 & 0.5 \\ 0.3 & - & 0.6 & 0.7 \\ 0.1 & 0.4 & - & 0.8 \\ 0.5 & 0.3 & 0.2 & - \end{pmatrix} \longrightarrow CP = \begin{pmatrix} - & 0.55 & 0.5 & 1.0 \\ 0.45 & - & 0.55 & 0.6 \\ 0.5 & 0.45 & - & 0.35 \\ 0.0 & 0.4 & 0.65 & - \end{pmatrix}.$$

The value $cp_{14} = 1$ has been obtained as follows:

$$ep_{14} = \frac{ep_{14}^2 + ep_{14}^3}{2} = \frac{0.9 + 1.2}{2} = 1.05 \Rightarrow cp_{14} = med\{0, 1, 1.05\} = 1.$$

3.1.1. *Individual reciprocal fuzzy preference relation*

The error in $[0, 1]$ between a preference value, p_{ik}, and its final estimated one, cp_{ik}, is:

$$\varepsilon p_{ik} = |cp_{ik} - p_{ik}|\,. \tag{8}$$

Reciprocity of $P = (p_{ij})$ implies reciprocity of $CP = (cp_{ik})$, therefore $\varepsilon p_{ik} = \varepsilon p_{ki}$. We interpret $\varepsilon p_{ik} = 0$ as a situation of total consistency between p_{ik} (p_{ki}) and the rest of information in P. Obviously, the higher the value of εp_{ik} the more inconsistent is p_{ik} (p_{ki}) with respect to the rest of information in P.

This interpretation allows us to evaluate the consistency in each one of the three different levels of a reciprocal fuzzy preference relation P:

Level 1. *Consistency degree associated to a pair of alternatives* (x_i, x_k),

$$cd_{ik} = 1 - \varepsilon p_{ik}\,. \tag{9}$$

Level 2. *Consistency degree associated to an alternative* x_i,

$$cd_i = \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{cd_{ik}}{n-1}\,. \tag{10}$$

When $cd_i = 1$ all the preference values involving the alternative x_i are fully consistent, otherwise, the lower cd_i the more inconsistent these preference values are with respect to the rest of information in P.

Level 3. *Consistency degree of the reciprocal fuzzy preference relation*,

$$cd = \sum_{i=1}^{n} \frac{cd_i}{n}\,. \tag{11}$$

When $cd = 1$ the reciprocal fuzzy preference relation P is fully consistent, otherwise, the lower cd the more inconsistent P.

The estimated values and consistency degrees for a reciprocal preference relation are illustrated in the following example:

Example 2. (Example 1 continuation) The consistency degrees at the three levels of the preference relation P are:

Pairs of alternatives	Alternatives	Relation
$CD = \begin{pmatrix} - & 0.85 & 0.6 & 0.5 \\ 0.85 & - & 0.95 & 0.9 \\ 0.6 & 0.95 & - & 0.55 \\ 0.5 & 0.9 & 0.55 & - \end{pmatrix}$	$cd_1 = 0.65$; $cd_2 = 0.9$ $cd_3 = 0.7$; $cd_4 = 0.65$	$cd = 0.73$

3.1.2. *Weighted mean collective preference relation*

Let $P^c = (p^c_{ij})$ be the weighted mean collective preference relation obtained from a set of reciprocal fuzzy preference relations $\{P^1, \ldots, P^m\}$. The estimated value of the collective preference value $p^c_{ij} = \sum_{l=1}^{n} w_l \cdot p^l_{ij}$, with $w_l \geq 0\ \forall l$ and $\sum_l w_l = 1$, is $ep^c_{ij} = \sum_{\substack{k=1 \\ j\neq i,k}}^{n} \frac{(ep^c_{ij})^k}{n-2}$, with $(ep^c_{ij})^k = p^c_{ik} + p^c_{kj} - 0.5$. Putting these expressions together we get:

$$ep^c_{ij} = \sum_{\substack{k=1 \\ j\neq i,k}}^{n} \frac{\sum_{l=1}^{n} w_l \cdot p^l_{ik} + \sum_{l=1}^{n} w_l \cdot p^l_{kj} - 0.5}{n-2} = \sum_{\substack{k=1 \\ j\neq i,k}}^{n} \frac{\sum_{l=1}^{m} w_l \cdot (p^l_{ik} + p^l_{kj} - 0.5)}{n-2}$$

$$= \sum_{l=1}^{m} w_l \frac{\sum_{\substack{k=1 \\ j\neq i,k}}^{n} p^l_{ik} + p^l_{kj} - 0.5}{n-2} = \sum_{l=1}^{m} w_l \cdot ep^l_{ij}\,.$$

This result is summarized in the following proposition:

Proposition 2. *The estimated collective preference relation of a weighted mean collective preference obtained from a set of reciprocal fuzzy preference relations is also the weighted mean of the estimated individual preference relations.*

The error between a collective preference value and its estimated one verifies

$$\left|ep^c_{ij} - p^c_{ij}\right| = \left|\sum_{l=1}^{m} w_l \cdot ep^l_{ij} - \sum_{l=1}^{n} w_l \cdot p^l_{ij}\right| = \left|\sum_{l=1}^{m} w_l \cdot (ep^l_{ij} - p^l_{ij})\right|$$

$$\leq \sum_{l=1}^{m} w_l \cdot \left|ep^l_{ij} - p^l_{ij}\right| \leq \max_l \left|ep^l_{ij} - p^l_{ij}\right|\,.$$

Therefore, we have proved the following:

Proposition 3. *The error between a collective preference value and its estimated one is lower or equal to the maximum error between the individual preference values and their estimated ones.*

The following holds $|ep_{ik} - p_{ik}| = |ep_{ik} - cp_{ik}| + |cp_{ik} - p_{ik}|\ \forall i, k$; and consequently $\varepsilon p_{ik} \leq |ep_{ik} - p_{ik}|\ \forall i, k$. Thus, when all estimated values of each one of the individual preference relations of the set $\{P^1, \ldots, P^m\}$ are in $[0, 1]$ the consistency degree of a weighted mean collective preference relation will be greater or equal than the minimum of the individual consistency degrees, $\min_l cd^l$. When one or more individual estimated values are not in $[0, 1]$ then this limit is reduced by the quantity $\sum_{i,j;j \neq i} \frac{\max_l |ep^l_{ij} - cp^l_{ij}|}{n(n-1)}$. In a situation of high individual consistency degrees, the distance between $cp^l_{ij} = med\{0, 1, ep^l_{ik}\}$ and p^l_{ij} will be small (or zero) and as a consequence the distance between ep^l_{ij} and cp^l_{ij} will also be small (or zero). All this together can be used to claim that in GDM problems in which all experts provide highly consistent preferences the (weighted mean) collective preference will also be highly consistent.

3.2. *Consistency control*

We assume that before providing any preferences the group of experts agree on a threshold consistency degree value (β) for an expert to be considered as consistent. After providing preferences, experts' associated consistency degrees are obtained, $cd^i\ \forall i$. If all experts are consistent, i.e. $cd^i \geq \beta\ \forall i$, then the consensus reaching process is applied. Otherwise, a consistency advice system is applied (i) to identify the inconsistent experts, alternatives, and preference values; and (ii) to generate an alternative consistent value for each one of the inconsistent preference values.

3.3. *Consistency advice system*

This system will suggest experts changes on the preference values with a consistency degree below a specified threshold. To do so, the following three steps are carried out:

(1) To identify those experts (l) in the group with a global consistency degree (cd^l) lower than the minimum threshold consistency value (β).
(2) To identify for each one of these experts those alternatives (i) with a consistency degree (cd^l_i) lower than β.
(3) To identify for each one of these alternatives the preference values with a consistency degree (cd^l_{ij}) lower than β.
The set of preference values to be recommended for change will be:

$$\{(l, i, j) \mid \max\{cd^l, cd^l_i, cd^l_{ij}\} < \beta\}.$$

Based on Proposition 1, a preference value of the above set (p^l_{ij}) will be recommended to be changed to a value closer to its final estimated value (cp^l_{ij}). This change will bring the original individual preference relation (P^l) closer to its estimated one (CP^l) and therefore it will become more consistent globally. Thus, if $cd^l_{ij} < \beta$, in order to reach the minimum threshold value, p^l_{ij} will be recommended to be changed to

$$\bar{p}^l_{ij} = p^l_{ij} + sign(cp^l_{ij} - p^l_{ij}) \cdot (\beta - cd^l_{ij}),$$

where $sign(X)$ returns the sign of X. Finally, in order to maintain reciprocity, the value p^l_{ji} will be recommended to be changed to $\bar{p}^l_{ji} = 1 - \bar{p}^l_{ij}$.

Example 3. Suppose we have a set of four experts providing the following fuzzy preference relations on a set of four alternatives:

$$P^1 = \begin{pmatrix} - & 0.2 & 0.6 & 0.4 \\ 0.8 & - & 0.9 & 0.7 \\ 0.4 & 0.1 & - & 0.3 \\ 0.6 & 0.3 & 0.7 & - \end{pmatrix} \qquad P^2 = \begin{pmatrix} - & 0.7 & 0.9 & 0.5 \\ 0.3 & - & 0.6 & 0.7 \\ 0.1 & 0.4 & - & 0.8 \\ 0.5 & 0.3 & 0.2 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.3 & 0.5 & 0.7 \\ 0.7 & - & 0.1 & 0.3 \\ 0.5 & 0.9 & - & 0.25 \\ 0.3 & 0.7 & 0.75 & - \end{pmatrix} \qquad P^4 = \begin{pmatrix} - & 0.25 & 0.15 & 0.65 \\ 0.75 & - & 0.6 & 0.8 \\ 0.85 & 0.4 & - & 0.5 \\ 0.35 & 0.2 & 0.5 & - \end{pmatrix}$$

Let the threshold value be $\beta = 0.8$. We have the following global consistency values:

$$cd^1 = 1,\ cd^2 = 0.73,\ cd^3 = 0.63;\ cd^4 = 0.82.$$

This means that recommendations of change will be given to experts e_2 and e_3. Because

$$CD^2 = \begin{pmatrix} - & 0.85 & 0.6 & 0.5 \\ 0.85 & - & 0.95 & 0.9 \\ 0.6 & 0.95 & - & 0.55 \\ 0.5 & 0.9 & 0.55 & - \end{pmatrix} \qquad CD^3 = \begin{pmatrix} - & 0.4 & 0.93 & 0.48 \\ 0.4 & - & 0.48 & 0.93 \\ 0.93 & 0.48 & - & 0.55 \\ 0.48 & 0.93 & 0.55 & - \end{pmatrix}$$

$$cd^2_1 = 0.65,\ cd^2_2 = 0.9 \qquad cd^3_1 = 0.6,\ cd^3_2 = 0.6$$

$$cd^2_3 = 0.7,\ cd^2_4 = 0.65 \qquad cd^3_3 = 0.65,\ cd^3_4 = 0.65$$

the recommended new preference values would be:

$$\bar{p}^2_{13} = 0.7(\bar{p}^2_{31} = 0.3);\ \bar{p}^2_{14} = 0.8(\bar{p}^2_{41} = 0.2);\ \bar{p}^2_{34} = 0.55(\bar{p}^2_{43} = 0.45)$$

$$\bar{p}^3_{12} = 0.7(\bar{p}^3_{21} = 0.3);\ \bar{p}^3_{14} = 0.38(\bar{p}^3_{41} = 0.62);\ \bar{p}^3_{23} = 0.43(\bar{p}^3_{32} = 0.57);\ \bar{p}^3_{34} = 0.5(= \bar{p}^3_{43}).$$

If these recommended values were assumed by these experts, their new global consistency values would become $cd^2 = 0.94$ and $cd^3 = 0.92$, which represent a considerable improvement regarding their previous global consistency degrees. Afterwards the consensus and selection processes are carried out.

4. Adaptive Consensus Reaching Process

A consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator, who helps the experts to bring their opinions closer. In each step of this process, the moderator, by means of a consensus measure, knows the actual level of consensus (agreement) between the experts, which establishes the distance to the ideal state of consensus. If the consensus level is not acceptable, i.e., if it is lower than a specified threshold, then the moderator would urge the experts to discuss their opinions further in an effort to bring them closer. On the contrary, when the consensus level is acceptable, the moderator would apply the selection process in order to obtain the final consensus solution to the GDM problem. In this framework, consensus support systems (CSS) have been designed to model the moderator's actions with the aim to automate the consensus reaching process.[12,13]

Obviously, when the level of agreement between the experts is 'high', a few changes of opinions from some of the experts might lead to consensus in a few discussion rounds. On the contrary, when the level of agreement between the experts is 'low', a high number of changes of opinions and many group discussion rounds might be necessary for consensus to be achieved. In this second case, it seems reasonable that many experts' preferences should be changed if they try to achieve a common solution. As the level of agreement increases, less and less experts might need to change their opinions. In fact, in these cases it might be expected that only those experts whose preference values are furthest from the group ones should change them. In other words, the number of changes in different stages of a consensus process is clearly related to the actual level of agreement. Therefore, CSSs should adapt their behavior to the level of agreement between the experts. This can be done by modifying the policy for searching the preferences to be changed in each consensus round by taking into account the actual level of agreement. A first proposal in this direction was presented in Ref. 15 with the introduction of an adaptive search of preferences based on a broad classification of the global consensus level as 'low', 'medium' and 'high'. In this paper we refine this first proposal by introducing a new recommendation module. The different tasks to carry out in this new adaptive consensus reaching process are illustrated in Fig. 3, and are explained in detail in the following subsections.

4.1. *Computation of the level of agreement*

The computation of the level of agreement among the experts at the beginning of a discussion round is done by measuring the distance between their preference values. We use the function $s(p_{ij}^r, p_{ij}^t) = 1 - \left|p_{ij}^r - p_{ij}^t\right|$ to measure the similarity of the preference values of two experts, e_r and e_t, on a pair of alternatives, x_i and x_j. Indeed, reciprocity of preferences implies that $s(p_{ij}^r, p_{ij}^t) = s(p_{ji}^r, p_{ji}^t)$.

The above similarity function can be used for measuring consensus degrees and proximity measures. The first ones are calculated by fusing the similarity of the preference values of all the experts on each pair of alternatives. The second ones are calculated by measuring the similarity between the preferences of each expert in the group and the collective preferences, previously obtained by fusing all the individual experts' preferences.

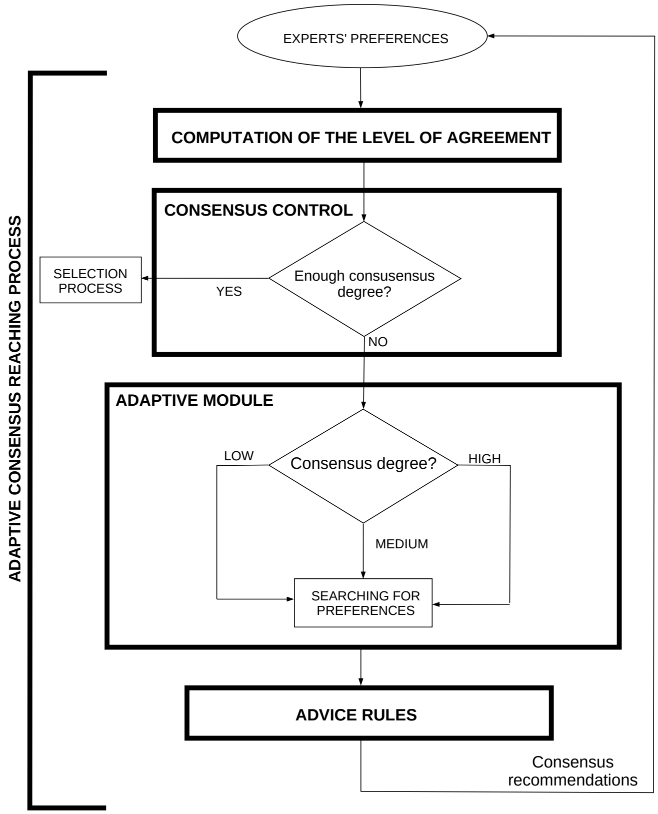

Fig. 3. Adaptive consensus reaching process.

4.1.1. *Consensus degrees*

The computation of consensus degrees is carried out as follows:

(1) For each pair of experts r and t ($r < t$), a *similarity matrix* is calculated

$$SM^{rt} = \left(sm_{ij}^{rt}\right)$$

with $sm_{ij}^{rt} = s(p_{ij}^r, p_{ij}^t),\ i,j = 1,\ldots,n\ \wedge\ i \neq j$.

(2) A *consensus matrix*, $CM = (cm_{ij})$, is obtained by aggregating all similarity matrices:

$$cm_{ij} = \phi(sm_{ij}^{rt});\ r,t = 1,\ldots,m\ ;\ i,j = 1,\ldots,n\ \wedge\ r < t\,,$$

We use the arithmetic mean as the aggregation function ϕ, although obviously other aggregation operators maintaining the reciprocity property could be used: $cm_{ij} = cm_{ji}$ $(\forall\ i,\ j)$.

(3) Consensus degrees are defined in each one of the three different levels of a relation:

Level 1. *Consensus on pairs of alternatives,* cp_{ij}. It measures the agreement among all experts on the pair of alternatives (x_i, x_j) :

$$cp_{ij} = cm_{ij} \,. \tag{12}$$

Level 2. *Consensus on alternatives,* ca_i. It measures the agreement among all experts on the alternative x_i, and it is defined as the average of the consensus degrees of all the pairs of alternatives involving it:

$$ca_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{cp_{ij}}{n-1} \,. \tag{13}$$

Level 3. *Consensus on the relation,* cr. It measures the global agreement among all experts, and it is defined as the average of the consensus degrees of all the alternatives, i.e. it is the average of all the consensus degrees at level of pairs of alternatives:

$$cr = \sum_{i=1}^{n} \frac{ca_i}{n} \,. \tag{14}$$

Example 4. (Example 3 continuation) The consensus degrees at the three levels associated to the previous set of four reciprocal fuzzy preference relations are:

Pairs of alternatives	Alternatives	Relation
$CM = \begin{pmatrix} - & 0.67 & 0.7 & 0.74 \\ 0.67 & - & 0.76 & 0.75 \\ 0.7 & 0.76 & - & 0.87 \\ 0.74 & 0.75 & 0.87 & - \end{pmatrix}$	$ca_1 = 0.71$; $ca_2 = 0.73$ $ca_3 = 0.78$; $ca_4 = 0.79$	$cr = 0.75$

4.1.2. *Proximity measures*

Proximity measures will be used to identify the experts which are furthest from the group. The first step here is therefore the computation of the collective fuzzy preference relation:

$$P^c = (p^c_{ij}); \; p^c_{ij} = \psi(p^1_{ij}, \ldots, p^m_{ij}).$$

As before, we use the arithmetic mean as the aggregation function ψ, although obviously other aggregation operators maintaining the reciprocity property could be used. For each expert, e_t, a *proximity matrix* is obtained:

$$PM^t = (pm^t_{ij}); \; pm^t_{ij} = s(p^t_{ij}, p^c_{ij}).$$

Proximity measures are computed in each one of the three different levels of a relation:

Level 1. *Proximity on pairs of alternatives, pp_{ij}^t*, which measures the proximity between the preference value of an expert and the corresponding collective one on a pair of alternatives (x_i, x_j):

$$pp_{ij}^t = pm_{ij}^t \,. \tag{15}$$

Level 2. *Proximity on alternatives, pa_i^t*, which measures the proximity between an expert's preference values of one alternative, x_i, over the rest of alternatives and the corresponding collective ones:

$$pa_i^t = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{pp_{ij}^t}{n-1} \,. \tag{16}$$

Level 3. *Proximity on the relation, pr^t*, which measures the global proximity between an expert and the group:

$$pr^t = \sum_{i=1}^{n} \frac{pa_i^t}{n} \,. \tag{17}$$

Example 5. (Example 4 continuation) The collective fuzzy preference relation is:

$$P^c = \begin{pmatrix} - & 0.4625 & 0.4875 & 0.5575 \\ 0.5375 & - & 0.6325 & 0.625 \\ 0.5125 & 0.3675 & - & 0.4625 \\ 0.4425 & 0.3750 & 0.5375 & - \end{pmatrix}$$

and the corresponding proximity measures are:
[Proximity on pairs of alternatives:]

$$PM^1 = \begin{pmatrix} - & 0.73 & 0.88 & 0.84 \\ 0.73 & - & 0.73 & 0.92 \\ 0.88 & 0.73 & - & 0.83 \\ 0.84 & 0.92 & 0.83 & - \end{pmatrix} \quad PM^2 = \begin{pmatrix} - & 0.76 & 0.78 & 0.75 \\ 0.76 & - & 0.96 & 0.92 \\ 0.78 & 0.96 & - & 0.91 \\ 0.75 & 0.92 & 0.91 & - \end{pmatrix}$$

$$PM^3 = \begin{pmatrix} - & 0.76 & 0.98 & 0.82 \\ 0.76 & - & 0.79 & 0.67 \\ 0.98 & 0.79 & - & 0.96 \\ 0.82 & 0.67 & 0.96 & - \end{pmatrix} \quad PM^4 = \begin{pmatrix} - & 0.78 & 0.66 & 0.9 \\ 0.78 & - & 0.96 & 0.82 \\ 0.66 & 0.96 & - & 0.96 \\ 0.9 & 0.82 & 0.96 & - \end{pmatrix}$$

[Proximity on alternatives:]

x_1	x_2	x_3	x_4
$pa_1^1 = 0.82$	$pa_2^1 = 0.79$	$pa_3^1 = 0.81$	$pa_4^1 = 0.86$
$pa_1^2 = 0.76$	$pa_2^2 = 0.88$	$pa_3^2 = 0.88$	$pa_4^2 = 0.86$
$pa_1^3 = 0.85$	$pa_2^3 = 0.74$	$pa_3^3 = 0.91$	$pa_4 = 0.82$
$pa_1^4 = 0.78$	$pa_2^4 = 0.86$	$pa_3^4 = 0.86$	$pa_4^4 = 0.89$

[Proximity on relations:]

$$pr^1 = 0.83, \;\; pr^2 = 0.85, \;\; pr^3 = 0.83, \;\; pr^4 = 0.85\,.$$

4.2. *Consensus control*

We assume that before providing any preferences the group of experts agree on a consensus threshold $\gamma \in [0,1]$ such that when $cr \geq \gamma$ the consensus process will stop and the selection process will be applied to obtain the solution of consensus. Otherwise, the consensus process continues and a (new) discussion round would be necessary for the experts to change preferences in an attempt to increase their global consensus level. The value γ depends on the particular problem dealt with. When the consequences of the decision making are of a significant importance, the minimum level of agreement required should be set as very high. On the contrary, or when it is urgent to obtain a solution of consensus, this value might not be set very high.

4.3. *Adaptive module*

The CSS adapts the search of preferences to be changed by the experts according to three different rules, which are based on the classification of the actual global consensus cr as 'low', 'medium' or 'high'. To do this, two new parameters are introduced, $\theta_1 < \theta_2 < \gamma$, and the following algorithm is applied at the beginning of each discussion round:

Table 1. Adaptive preference search algorithm.

```
IPUTS
    cr, γ, θ1, θ2
BEGIN
    IF cr ≥ γ
    THEN
        Execute Selection Process
    ELSE
        IF cr ≤ θ1
        THEN
            Execute Low Consensus Preference Search
        ELSE
            IFcr ≤ θ2
            THEN
                Execute Medium Consensus Preference Search
            ELSE
                Execute High Consensus Preference Search
            END-IF
        END-IF
    END-IF
END
```

4.3.1. *Low consensus preference search procedure:* PSP_L

This procedure identifies the pairs of alternatives (x_i, x_j) with consensus degree below the global consensus level

$$P = \{(i,j) \mid cp_{ij} < cr\}$$

and recommends that all experts change their preferences on such pairs of alternatives. Therefore, the set of preferences that each expert e_t should change is

$$PREFECH_L^t = \{(i,j) \mid (i,j) \in P\}\,.$$

4.3.2. *Medium consensus preference search procedure:* PSP_M

This procedure identifies first the alternatives with consensus degree below the global consensus level:

$$X_{ch} = \{i \mid ca_i < cr\}$$

and for each one of these alternatives, those preference values with consensus degree also below the global consensus level

$$P = \{(i,j),\ (j,i) \mid i \in X_{ch}\ \wedge\ cp_{ij} = cp_{ji} < cr\}\,.$$

Additionally, the number of experts required to make changes on these identified preferences is reduced to just those furthest on the identified alternatives. To do this, the average of all proximity values on the particular identified alternative will be used as the threshold value to select the experts that will be asked to change preferences. Therefore, the set of preferences that each expert e_t should change is

$$PREFECH_M^t = \left\{(i,j) \mid (i,j) \in P \wedge pa_i^t < \sum_r \frac{pa_i^r}{m}\right\}.$$

Clearly, the new restriction reduces the number of preferences and experts required to make changes. Consequently, we have that:

$$\#(\bigcup_t PREFECH_M^t) \leq \#(\bigcup_t PREFECH_L^t)\,.$$

4.3.3. *High consensus preference search procedure:* PSP_H

This procedure identifies the alternatives and their preference values with consensus degrees below the global consensus level. The experts required to make changes in these identified preferences will be those furthest on the alternatives and on the preference values to be changed. Again, the threshold values used to identify these experts will be the corresponding averages of all proximity values. Therefore, the set of preferences that each expert e_t should change is

$$PREFECH_H^t = \left\{(i,j) \mid (i,j) \in P \wedge pa_i^t < \sum_r \frac{pa_i^r}{m} \wedge pp_{ij}^t < \sum_r \frac{pp_{ij}^r}{m}\right\}.$$

Clearly, we have that $\#(\bigcup_t PREFECH_H^t) \leq \#(\bigcup_t PREFECH_L^t)$.

Example 6. (Example 5 continuation) Let us suppose $\gamma = 0.85$, $\theta_1 = 0.7$, $\theta_2 = 0.8$. The global consensus of the initial preferences is $cr = 0.75$, which means that the medium consensus preference search procedure is executed. We have:

$$\begin{aligned}
&X_{ch} = \{x_1, x_2\}\\
&P = \{(1,2),(2,1),(1,3),(3,1),(1,4),(4,1),(2,4),(4,2)\}\\
&PREFECH_M^1 = \{(2,1),(1,2),(2,4),(4,2)\}\\
&PREFECH_M^2 = \{(1,2),(2,1),(1,3),(3,1),(1,4),(4,1)\}\\
&PREFECH_M^3 = \{(2,1),(1,2),(2,4),(4,2)\}\\
&PREFECH_M^4 = \{(1,2),(2,1),(1,3),(3,1),(1,4),(4,1)\}\,.
\end{aligned}$$

4.4. *Advice rules*

Once the preferences to be changed have been identified, the model suggests the experts the right direction of the changes if the global consensus is to be increased. This is done via simple "advice rules" based on a broad comparison between the individual and collective preferences:

DR.1. If $p_{ij}^t - p_{ij}^c < 0$, expert e_t will be recommended to increase p_{ij}^t and decrease p_{ji}^t in the same quantity.

DR.2. If $p_{ij}^t - p_{ij}^c > 0$, expert e_t will be recommended to decrease p_{ij}^t and increase p_{ji}^t in the same quantity.

DR.3. If $p_{ij}^t - p_{ij}^c = 0$, expert e_t will not receive a recommendation of change for p_{ij}^t and p_{ji}^t.

Example 7. (Example 6 continuation) Let us assume that the experts follow the recommendations given by the CSS and that they provide the following new preference values:

$$\begin{aligned}
&e_1:\; p_{21}^1 = 0.67\; p_{12}^1 = 0.33;\quad p_{24}^1 = 0.67\; p_{42}^1 = 0.33\\
&e_2:\; p_{12}^2 = 0.58\; p_{21}^2 = 0.42;\quad p_{13}^2 = 0.59\; p_{31}^2 = 0.41;\; p_{14}^2 = 0.68\; p_{41}^2 = 0.32\\
&e_3:\; p_{21}^3 = 0.42\; (p_{12}^3 = 0.58);\; p_{24}^3 = 0.46\; (p_{42}^3 = 0.54)\\
&e_4:\; p_{12}^4 = 0.36\; (p_{21}^2 = 0.64);\; p_{13}^4 = 0.32\; (p_{31}^4 = 0.68);\; p_{14}^4 = 0.6\; (p_{41}^4 = 0.4)\,.
\end{aligned}$$

The new global consensus degree increases from the previous 0.75 to 0.82. Because the consensus threshold is not reached, the high consensus preference search procedure is applied. This result in experts e_1 and e_3 being recommended to change the following preference values:

$$PREFECH_H^1 = \{(2,3),(3,2)\};\quad PREFECH_H^3 = \{(2,3),(3,2),(2,4),(4,2)\}\,.$$

Assuming that the new preference values are:

$$\begin{aligned}
&e_1:\; p_{23}^1 = 0.77\; p_{32}^1 = 0.33\\
&e_3:\; p_{23}^3 = 0.53\; p_{32}^3 = 0.47;\; p_{24}^3 = 0.56\; p_{42}^3 = 0.44
\end{aligned}$$

we have a new global consensus degree $cr = 0.86$, above the consensus threshold $\gamma = 0.85$, which will result in the execution of the selection process.

5. Conclusions

In any rational GDM process, both consensus and consistency should be sought after. In this paper we have addressed the issues of measuring consistency and of achieving a high level of consistency within an adaptive consensus reaching process. For doing that, we have developed a consistency advice module, based on theoretical results, for recommending 'consistent' changes to experts for the most/more inconsistent preference values. We have argued that consistency is needed to be checked just once before the application of the consensus process, because

(a) when all individual experts provide highly consistent preferences the (weighted mean) collective preference will also be highly consistent, and
(b) the consensus process tends to make the individual opinions closer to the collective ones.

Also, if we were to secure consensus and only thereafter consistency, we could destroy the consensus in favor of the individual consistency and the final solution could not be acceptable for the group of experts.

Because the number of changes in each discussion round of a consensus process is clearly related to the actual level of agreement, we have argued that CSSs should adapt their behavior to the level of agreement between the experts. One way of doing this is by modifying the policy for searching the preferences to be changed in each consensus round. In this paper, we have refine the first proposal in this direction,[15] with the introduction of an adaptive recommendation module based on a broad classification of the global consensus level as 'low', 'medium' and 'high'.

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