Additive Consistency of Fuzzy Preference Relations: Characterization and Construction

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Extended Abstract

As it is known, most decision processes are based on preference relations, in the sense that processes are linked to some degree of preference of any alternative over another. The use of preference relations is usual in decision making [1, 5, 10, 14]. Therefore, to establish properties to be verified by such preference relations is very important for designing good decision making models.

One of these properties is the so called *consistency property*. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, if not crucial, to study conditions under which consistency is satisfied [5, 10]. On the other hand, perfect consistency is difficult to obtain in practice, specially when measuring preferences on a set with a large number of alternatives.

Clearly, the problem of consistency itself includes two problems [3, 7]:

(i) when an expert, considered individually, is said to be consistent and,

(ii) when a whole group of experts are considered consistent.

In this contribution we will focus on the first problem, assuming that expert's preferences are expressed by means of a preference relation defined over a finite and fixed set of alternatives.

In a crisp model, where an expert provides his/her opinion on the set of alternatives, $X = \{x_1, x_2, \ldots, x_n; n \geq 2\}$, by means of a binary preference relation, R, the concept of consistency has traditionally been defined in terms of acyclicity [12], that is the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ with $x_j R\, x_{j+1} \forall j = 1, \ldots, k$.

In a fuzzy context, where an expert expresses his/her opinions using fuzzy preference relations, a traditional requirement to characterize consistency is using transitivity, in the sense that if an alternative x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k. Stronger conditions have been given to define consistency, for example *max-min transitivity property* or *additive transitivity property*

[5, 13, 14, 15]. However, the problem is the difficulty to check and to guarantee such consistency conditions in the decision making processes.

In this contribution, we present some issues to study and to guarantee consistency in the decision making problems under fuzzy preference relations, taking in consideration 2 aspects:

1. We present a characterization of fuzzy consistency based on the additive transitivity property which facilitates the verification of consistency in the case of fuzzy preference relations.

2. Using this new characterization we present a method to construct consistent fuzzy preference relations from $n-1$ given preference values.

The following sections of this extended abstract present the following study: Section 1 presents the use of the preference relations in decision making. Section 2 studies the different characterizations of consistency of fuzzy preference relations. Section 3 defines a new characterization of consistency and the constructing method of consistent fuzzy preference relations. Finally, some future studies are briefly introduced.

1 The use of the preference relations

Preference relation is the most common representation of information used in decision making problems because it is a useful tool in modeling decision processes, above all when we want to aggregate experts' preferences into group preferences [5, 10, 11, 13]. In a preference relation an expert associates to every pair of alternatives a value that reflects some degree of preference of the first alternative over the second one. Many important decision models have been developed using mainly two kinds of preference relations:

Multiplicative preference relations [10, 11]: A multiplicative preference relation A on a set of alternatives X is represented by a matrix $A \subset X\mathrm{x}X$, $A = (a_{ij})$, being a_{ij} interpreted as the ratio of the preference intensity of alternative x_i to that of x_j, i.e., it is interpreted as x_i is a_{ij} times as good as x_j. Saaty suggests measuring a_{ij} using a ratio scale, and precisely the 1 to 9 scale [10, 11]: $a_{ij} = 1$ indicates indifference between x_i and x_j, $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j, and $a_{ij} \in \{1, \ldots, 9\}$ indicates intermediate preference evaluations. In this case, the preference relation, A, is usually assumed multiplicative reciprocal, i.e.,

$$a_{ij} \cdot a_{ji} = 1 \ \ \forall i, j \in \{1, \ldots, n\}.$$

Saaty means by consistency what he calls *cardinal transitivity* in the strength of preferences which is a stronger condition than the traditional requirement of the transitivity of preferences. Thereby, the definition of consistency proposed by Saaty is the following [10, 11]

Definition 1. A reciprocal multiplicative preference relation $A = (a_{ij})$ is consistent if

$$a_{ij} \cdot a_{jk} = a_{ik} \ \ \forall i, j, k = 1, \ldots, n.$$

Fuzzy preference relations [1, 5, 14]: A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function

$$\mu_P : X \times X \longrightarrow [0, 1].$$

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ij})$ being $p_{ij} = \mu_P(x_i, x_j)$ $\forall i, j \in \{1, \dots, n\}$. p_{ij} is interpreted as the preference degree of the alternative x_i over x_j: $p_{ij} = 1/2$ indicates indifference between x_i and x_j ($x_i \sim x_j$), $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, and $p_{ij} > 1/2$ indicates that x_i is preferred to x_j ($x_i \succ x_j$). In this case, the preference matrix, P, is usually assumed additive reciprocal, i.e.,

$$p_{ij} + p_{ji} = 1 \quad \forall i, j \in \{1, \dots, n\}.$$

In [2] we studied the transformation function between reciprocal multiplicative preference relations with values in the interval scale $[1/9, 9]$ and reciprocal fuzzy preference relations with values in $[0, 1]$. This study can be summarized in the following proposition.

Proposition 1. Suppose that we have a set of alternatives, $X = \{x_1, \dots, x_n\}$, and associated with it a reciprocal multiplicative preference relation $A = (a_{ij})$ with $a_{ij} \in [1/9, 9]$. Then, the corresponding reciprocal fuzzy preference relation, $P = (p_{ij})$ with $p_{ij} \in [0, 1]$, associated with A is given as follows:

$$p_{ij} = g(a_{ij}) = \frac{1}{2} \cdot (1 + \log_9 a_{ij}).$$

With such a transformation function g we can relate the research issues obtained for both kinds of preference relations.

In the following section, we study briefly the different proposals to characterize consistency of the fuzzy preference relations existing in the literature.

2 On consistency of the fuzzy preference relations

For making a consistent choice when assuming fuzzy preference relations a set of consistency properties to be satisfied by such relations have been suggested. Transitivity is one of the most important properties concerning preferences, and it represents the idea that the preference value obtained by comparing directly two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives [4, 9, 14]. Some of the suggested properties are given:

1. *Triangle condition [9]:* $p_{ij} + p_{jk} \geq p_{ik}$ $\forall i, j, k$.
2. *Weak transitivity [14]:* $p_{ij} \geq 0.5$, $p_{jk} \geq 0.5 \Rightarrow p_{ik} \geq 0.5$ $\forall i, j, k$.
3. *Max-min transitivity [4, 15]:* $p_{ik} \geq \min(p_{ij}, p_{jk})$ $\forall i, j, k$.
4. *Max-max transitivity [4, 15]:* $p_{ik} \geq \max(p_{ij}, p_{jk})$ $\forall i, j, k$.

5. *Restricted max-min transitivity [14]:* $p_{ij} \geq 0.5,\ p_{jk} \geq 0.5 \Rightarrow p_{ik} \geq \min(p_{ij}, p_{jk})\ \ \forall i, j, k$.

6. *Restricted max-max transitivity [14]:* $p_{ij} \geq 0.5,\ p_{jk} \geq 0.5 \Rightarrow p_{ik} \geq \max(p_{ij}, p_{jk})\ \ \forall i, j, k$.

7. *Multiplicative transitivity [14]:* $\frac{p_{ji}}{p_{ij}} \cdot \frac{p_{kj}}{p_{jk}} = \frac{p_{ki}}{p_{ik}}\ \ \forall i, j, k$.

8. *Additive transitivity [13, 14]:* $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5)\ \ \forall i, j, k$, or equivalently $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2}\ \ \forall i, j, k$.

Remark. This type of transitivity has the following interpretation: suppose we do want to establish a ranking between three alternatives x_i, x_j, and x_k. If we do not have any information about these alternatives it is natural to start assuming that we are in an indifference situation, that is, $x_i \sim x_j \sim x_k$, and therefore when giving preferences this situation is represented by $p_{ij} = p_{jk} = p_{ik} = 0.5$. Suppose now that we have a piece of information that says alternative $x_i \prec x_j$, that is $p_{ij} < 0.5$. It is clear then that p_{jk} or p_{ki} have to change otherwise there would be a contradiction because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must conclude then that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ji} = p_{ki}$, and so $p_{ij}+p_{jk}+p_{ki} = p_{ij}+p_{jk}+p_{ji} = 1+0.5 = 1.5$. We have the same conclusion if $p_{ki} = 0.5$. In the case of being $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ki} has to be equal to or greater than p_{ji}, being equal only in the case of $p_{jk} = 0.5$ as we have seen. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seem reasonable to suppose that the intensity of preference of x_k over x_i should be equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is $p_{ki} - 0.5 = (p_{kj} - 0.5) + (p_{ji} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$. The reciprocal fuzzy preference relation P, given above, verifies additive transitivity. It is easy to prove that additive transitivity is a stronger concept than restricted max-max transitivity [13, 14].

¿From the above list, the additive transitivity seems to be an acceptable property to characterize consistency in the case of fuzzy preference relations given that:

- The weak transitivity is the minimum requirement condition that a consistent fuzzy preference relation should verify.

- The max-min transitivity has been the traditional requirement to characterize consistency in the case of fuzzy preference relations.

- The max-max transitivity is a stronger concept than max-min transitivity.

  Both transitivity concepts are too strong in the sense that they could not be verified even when a fuzzy preference relation is considered perfectly consistent from a practical point of view (as was shown in the above example).

- Restricted max-min and restricted max-max transitivity concepts seem good alternatives to them, being restricted max-max transitivity even more adequate from a practical point of view than restricted max-min transitivity; moreover, restricted max-max transitivity implies restricted max-min transitivity.

- The multiplicative transitivity concept is valid only in the case of being $p_{ij} > 0 \; \forall i, j$.
- The additive transitivity is a stronger concept than restricted max-max transitivity and it implies restricted max-max transitivity. If we want to include a some kind of measure of strength of preference in the concept of transitivity then additive transitivity includes this idea of ordinal strength of preferences. Furthermore, as it is shown in the next result, the consistency definition in the case of the multiplicative preference relations via the above transformation function g (given in proposition 1) is equivalent to the additive transitivity property.

Proposition 2. Let $A = (a_{ij})$ be a consistent multiplicative preference relation, then the corresponding reciprocal fuzzy preference relation, $P = g(A)$ verifies additive transitivity property.

In such a way, in this contribution we consider the following definition of the consistent fuzzy preference relation.

Definition 2. A reciprocal fuzzy preference relation $P = (p_{ij})$ is consistent if

$$p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \;\; \forall \; i, j, k = 1, \ldots, n.$$

In what follows, we will use the term *additive consistency* to refer to consistency for fuzzy preference relations based on the additive transitivity property.

3 Additive consistency

In this section we present a new characterization of the additive consistency condition, which states that for checking additive consistency of a fuzzy preference relation P, it is only necessary to check those triplets of values $(i, \; j, \; k)$ verifying $i \leq j \leq k$. As a consequence of this equivalent condition, we design a method to construct consistent fuzzy preference relations from a set of n-1 preference values which guarantees consistency of the fuzzy preference relations provided by the experts. We conclude this section by exporting the above research issues on additive consistency to the multiplicative decision models [10, 11], i.e., decision models based on multiplicative preference relations.

3.1 Characterization of additive consistency

Proposition 3. For a reciprocal fuzzy preference relation $P = (p_{ij})$, the following statements are equivalent:

1. $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \;\; \forall \; i, j, k$.
2. $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \;\; \forall \; i < j < k$.

Proposition 3 can be rewritten as follows.

Proposition 4. A fuzzy preference relation $P = (p_{ij})$ is consistent if and only if

$$p_{ij} + p_{jk} + p_{ik} = \frac{3}{2}, \quad \forall\, i \leq j \leq k.$$

The following result characterizes additive consistency.

Proposition 5. For a reciprocal fuzzy preference relation $P = (p_{ij})$, the following statements are equivalent:

1. $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \;\; \forall\, i < j < k$,
2. $p_{i(i+1)} + p_{(i+1)(i+2)} + \ldots + p_{(j-1)j} + p_{ji} = \frac{j-i+1}{2} \;\; \forall\, i < j$.

3.2 A method to construct consistent fuzzy preference relations

The result presented in proposition 5 is very important because it can be used to construct a consistent fuzzy preference relation from the set of n-1 values $\{p_{12}, p_{23}, \ldots, p_{n-1n}\}$. In such a way, we can facilitate experts the expression of consistent preferences in the decision processes.

Example. Suppose that we have a set of four alternatives $\{x_1, x_2, x_3, x_4\}$ where we have certain knowledge to assure that alternative x_1 is weakly more important than alternative x_2, alternative x_2 is more important than x_3 and finally alternative x_3 is strongly more important than alternative x_4. Suppose that this situation is modeled by the preference values $\{p_{12} = 0.55, p_{23} = 0.65, p_{34} = 0.75\}$. Applying Proposition 5, we obtain:

$$p_{31} = 1.5 - p_{12} - p_{23} = 1.5 - 0.55 - 0.65 = 0.3,$$
$$p_{41} = 2 - p_{12} - p_{23} - p_{34} = 2 - 0.55 - 0.65 - 0.75 = 0.05,$$
$$p_{42} = 1.5 - p_{23} - p_{34} = 1.5 - 0.65 - 0.75 = 0.1,$$
$$p_{21} = 1 - p_{12} = 0.45, p_{13} = 1 - p_{31} = 0.7, p_{14} = 1 - p_{41} = 0.95,$$
$$p_{32} = 1 - p_{23} = 0.35, p_{24} = 1 - p_{42} = 0.9, p_{43} = 1 - p_{34} = 0.25.$$

We make note that, if the primary values are different then we would have obtained a matrix P with entries not in the interval $[0, 1]$, but in an interval $[-a, 1+a]$, being $a > 0$. In such a case, we would need to transform the obtained values using a transformation function which preserves reciprocity and additive consistency, that is a function $f : [-a, 1+a] \longrightarrow [0, 1]$, verifying

1. $f(-a) = 0$.
2. $f(1+a) = 1$.
3. $f(x) + f(1-x) = 1, \; \forall x \in [-a, 1+a]$.

4. $f(x)+f(y)+f(z)=\frac{3}{2},\ \forall x,y,z\in[-a,1+a]$ such that $x+y+z=\frac{3}{2}$.

The linear solution verifying 1 and 2 takes the form $f(x)=\varphi\cdot x+\beta$, being $\varphi,\ \beta\in R$. This function is

$$f(x)=\frac{1}{1+2a}\cdot x+\frac{a}{1+2a}=\frac{x+a}{1+2a}$$

which verifies 3

$$f(x)+f(1-x)=\frac{x+a}{1+2a}+\frac{1-x+a}{1+2a}=\frac{x+a+1-x+a}{1+2a}=1$$

and when $x+y+z=\frac{3}{2}$

$$f(x)+f(y)+f(z)=\frac{x+a}{1+2a}+\frac{y+a}{1+2a}+\frac{z+a}{1+2a}=\frac{x+y+z+3a}{1+2a}=\frac{3/2+3a}{1+2a}=\frac{3}{2}$$

verifies 4.

Summarizing: The method to construct a consistent reciprocal fuzzy preference relation P' on $X=\{x_1,\ldots,x_n,n\geq 2\}$ from n-1 preference values $\{p_{12},p_{23},\ldots,p_{n-1n}\}$ presents the following steps:

1. Compute the set of preference values B as

$$B=\{p_{ij},\ i<j\wedge p_{ij}\notin\{p_{12},p_{23},\ldots,p_{n-1n}\}\}$$

$$p_{ij}=\frac{j-i+1}{2}-p_{ii+1}-p_{i+1i+2}\cdots-p_{j-1j}.$$

2. $a=|min\{B\cup\{p_{12},p_{23},\ldots,p_{n-1n}\}\}|$

3. $P=\{p_{12},p_{23},\ldots,p_{n-1n}\}\cup B\cup\{1-p_{12},1-p_{23},\ldots,1-p_{n-1n}\}\cup\neg B.$

4. The consistent fuzzy preference relation P' is obtained as $P'=f(P)$ such that

$$f:[-a,1+a]\longrightarrow[1,0]$$

$$f(x)=\frac{x+a}{1+2a}.$$

Future research

Future studies will be focused on the second problem of consistency in decision making, i.e., when a whole group of experts are considered consistent and the aggregation of consistent preferences.

Remarks: The proofs of the propositions introduced in this paper can be found in [8].

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