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Linguistic Measures Based on Fuzzy Coincidence for Reaching Consensus in Group Decision Making

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ABSTRACT

Assuming a linguistic framework, a model for the consensus reaching problem in heterogeneous group decision making is proposed. This model contains two types of linguistic consensus measures: linguistic consensus degrees *and* linguistic proximities *to guide the consensus reaching process. These measures evaluate the current consensus state on three levels of action:* level of the pairs of alternatives, level of the alternatives, *and* level of the relation. *They are based on a fuzzy characterization of the concept of coincidence, and they are obtained by means of several conjunction functions for handling linguistic weighted information, the LOWA operator for aggregating linguistic information, and linguistic quantifiers representing the concept of fuzzy majority.* © *1997 Elsevier Science Inc.*

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1. INTRODUCTION

Consensus or *synthesis* consists in combining a data set provided by different information sources with a view to obtaining more elaborate

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information [31, 32]. When the information sources provide imprecise information, the use of *fuzzy set theory* to deal with this type of information is most advisable. A usual situation, in the real world, which presents the appropriate characteristics to apply consensus theory and fuzzy set theory together, is the *group decision making* (GDM) situation.

In a classical GDM situation there is a problem to solve, a set of possible solution alternatives, and a group of two or more experts, who express their opinions about the set of solution alternatives and attempt to reach a collective decision with the maximum possible consensus on this question: what is/are the best solution alternative(s) to the problem?. Many papers on consensus theory applied to GDM make use of Arrow's work [1] as a starting point and a basic guide. Arrow proposed a qualitative setting composed by a set of axioms, which any acceptable consensus tool for GDM should satisfy. *Arrow's impossibility theorem* was an important result thereof. According to this theorem, it is impossible to aggregate individual preferences into group preference in a completely rational way. This is a problem that disappears in a cardinal setting in a fuzzy context, on introducing preference intensities, which provide additional degrees of freedom to any aggregation model [13, 9].

In a fuzzy context, the application of consensus theory to GDM problems presents two ways to relate to different decision schemata [6]. The first way, called *algebraic consensus*, consists in establishing a *group choice process* which obtains a decision scheme as a solution to the GDM problem. The second way, called *topologic consensus*, consists in establishing a *group consensus reaching process*, which, guided by means of a measure of closeness among different decision schemata, called the *consensus measure*, attempts to achieve the maximum possible degree of consensus on solution alternative(s). Both consensus types may be combined in a resolution scheme (see Figure 1). Given that the set of experts initially have diverging opinions, firstly, topologic consensus is applied, and in each step, the degree of existing consensus among experts' opinions is measured. If the moderator thinks that the consensus degree is satisfactory, then algebraic consensus is applied in order to obtain a solution; otherwise, the experts are persuaded to update their opinions. In this way, a GDM process may be defined as a dynamic and iterative process, in which the experts, via the exchange of information and rational arguments, agree to update their opinions until they become sufficiently similar, and then the solution alternative(s) is/are obtained. Here, we shall focus our research on the topologic consensus.

As was mentioned earlier, the topologic consensus is guided by means of a consensus measure. Assuming numerical preference relations for providing the experts' opinions, several authors introduced *hard consensus measures* varying between 0 (no consensus or partial agreement) and 1 (full

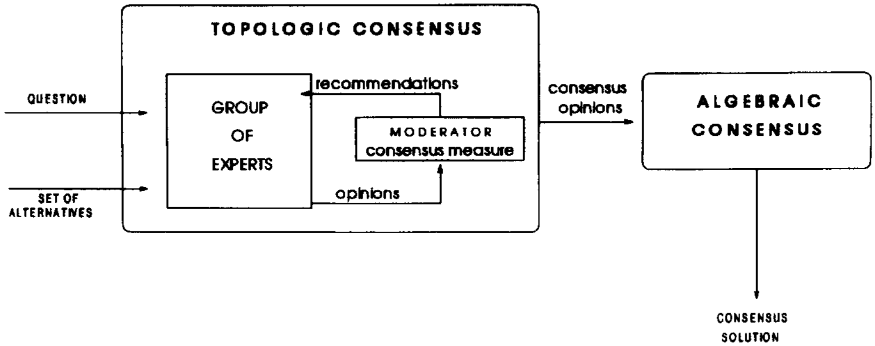

Figure 1. Group decision making process.

consensus or complete agreement) [2, 3, 28, 29]. However, consensus as a full and unanimous agreement is far from being achieved in real situations, and even if it is, in such a situation, the consensus reaching process could be unacceptably costly. So, in practice, a more realistic approach is to use "softer consensus measures" [24], which assess the consensus degree in a more flexible way, and therefore reflect the large spectrum of possible partial agreements, and guide topologic consensus until widespread agreement (not always full) is achieved among experts.

Along this line of reasoning, but in different fuzzy GDM contexts, several alternative consensus measures have been proposed: in a numerical context, i.e., with numerical assessments on the unit interval $[0, 1]$, by Kacprzyk [24], Kacprzyk and Fedrizzi [25, 26], and Fedrizzi, Kacprzyk, and Nurmi [15]; and in a linguistic context, i.e., with linguistic assessments on a preestablished label set S, by Fedrizzi and Mich [14], Mich, Gaio, and Fedrizzi [27], Herrera, Herrera-Viedma, and Verdegay [21, 23], and Bordogna, Fedrizzi, and Pasi [5]. In all these cases, the authors have based their consensus measures on the concept of *coincidence*, i.e., observing the existing coincidence among experts' opinions. Different coincidence meanings have been considered, some based on *strict coincidence*, i.e., accepting only the total coincidence or null coincidence cases [24–26, 15, 27, 21, 23], and others on *less strict coincidence*, i.e., accepting different coincidence degrees [24–26, 15, 14, 5].

Here, in a linguistic context, we propose to use a more flexible idea of the concept of coincidence, i.e., using it as a fuzzy concept. We present *fuzzy coincidence* as a fuzzy set defined on the set of expert pairs and characterized by closeness observed among their respective opinions. In particular, we assume a heterogeneous linguistic context to introduce the new fuzzy coincidence concept, i.e., we define the gradation of the coincidence degree existing among two experts from a label set, S, used to express the experts' opinions, to a new and more appropriate (preestablished) label set, in order to express the coincidence degrees, G. In this way, we present various ways to measure the closeness observed among experts' opinions. Moreover, we study the fuzzy coincidence among experts on three levels of action: the *level of the pairs of alternatives*, the *level of the alternatives*, and the *level of the relation*. Then, using this new idea of fuzzy coincidence, we further advance our previous consensus models [21, 23] for deriving some new softer linguistic consensus measures, which are applied on the three coincidence levels. All consensus measures are obtained using different *conjunction functions* to manipulate weighted linguistic information [16], the *linguistic ordered weighting averaging* (LOWA) operator [18, 22] to aggregate linguistic information, and the *linguistic quantifiers* [42] representing the fuzzy majority concept.

In order to do so, in the next section we present some prior considerations on some consensus measures proposed in the literature with a view to clarifying the contributions in this paper. In Section 3, we present briefly the linguistic setting of the GDM problem considered. In Section 4, we present the new linguistic consensus measures. In Section 5, 6, and 7, we show the derivation model of the consensus measures, and finally, some conclusions are pointed out.

2. BACKGROUND ON CONSENSUS MEASURES

As we said at the beginning, in a fuzzy context, several alternative softer consensus measures have been proposed. In this section, we briefly analyze these measures with a view to better clarifying the new developments proposed in this paper.

In a numerical context, Kacprzyk [24] presented three numerical consensus measures, which are:

- assessed on unit interval, [0, 1];
- developed in a simple GDM context with a homogeneous group of experts (all experts' opinions have the same importance degree) and a homogeneous set of alternatives (all the alternatives have the same relevance degree);
- calculated across the global set of the alternatives in a hierarchical pooling process from the experts' opinions, provided by means of the numerical preference relations, and using the fuzzy majority concept represented by a linguistic quantifier [42]; and finally
- obtained: (1) the first measure, using a strict idea of the concept of coincidence, that is, establishing a particular pair of alternatives: if the opinions of two experts are equal then they are in agreement (value

1), and otherwise they are in disagreement (value 0); (2) the second one, using a less strict idea of the concept of coincidence, that is, establishing a particular alternative pair: if the opinions of two experts are more or less equal according to a degree α (preestablished), then they are in agreement (value 1), and otherwise, they are in disagreement (value 0); and (3) the third one, using another less strict idea of the concept of coincidence represented by a function, $s:[0,1] \rightarrow [0,1]$ defined on the closeness between experts' opinions.

Kacprzyk and Fedrizzi [25, 26] extended Kacprzyk's measures of GDM contexts with a heterogeneous set of alternatives and a heterogeneous group of experts, respectively. Fedrizzi, Kacprzyk, and Nurmi [15] modified the definition of Kacprzyk and Fedrizzi's measures and calculated them using the *ordered weighted averaging* (OWA) operator [34].

On the other hand, in a linguistic context, Fedrizzi and Mich [14] presented a new numerical consensus measure, which is:

- developed in a homogeneous GDM context with multiple criteria;
- calculated for each alternative, independently, from the experts' opinions provided by linguistic labels (not preference relations) by means of computation on a fuzzy representation of linguistic labels (trapezoidal membership functions); and
- obtained using a less strict coincidence concept represented by means of a euclidean distance d, which implements the linguistic approximation [30].

Mich, Gaio, and Fedrizzi [27] modified this measure and obtained it by applying a strict coincidence concept, which divided the expert group into subsets according to their evaluations. Herrera, Herrera-Viedma, and Verdegay [21] presented two types of linguistic consensus measures, one to measure the consensus degree and another to measure the closeness between experts opinions. Both are:

- assessed on the same label set, S, used to express experts' opinions;
- developed in a GDM context with a heterogeneous group of experts and a heterogeneous set of the alternatives with importance and relevance degrees assessed on $[0,1]$;
- calculated from the experts' opinions, provided by linguistic preference relations, using linguistic quantifiers and a linguistic aggregation operator by direct computation on the labels (the LOWA operator [18, 22]) on three levels of action: preference on the pairs of alternatives, preference on the individual alternatives, and preference on the global set of the alternatives; and
- obtained by applying a strict coincidence concept, similar to Mich, Gaio, and Fedrizzi's concept, but according to an average consensus policy, that is, using every subset of experts with over two experts.

In their second paper [23], Herrera, Herrera-Viedma, and Verdegay modified their measures to work in a GDM context with heterogeneous groups of experts with importance degrees assessed on S, and homogeneous sets of alternatives. The consensus measures were obtained according to a strict coincidence concept but by means of a strict consensus policy, that is, considering only the subset of experts with maximum cardinality. This consensus model incorporated a new development: it integrated two types of *linguistic rationality measures* to achieve less distorted consensus solutions. Finally, Bordogna, Fedrizzi, and Pasi [5] presented a linguistic consensus measure, which is:

- assessed on the same label set, S, used to express the experts' opinions;
- developed in a linguistic GDM context, similar to Herrera, Herrera-Viedma, and Verdegay's, but with a heterogeneous set of criteria and having linguistic importance degrees assessed on S;
- calculated for each alternative independently, from the experts' opinions, provided by linguistic labels, by means of the linguistic version of the OWA operator [35] and considering linguistic quantifiers; and
- obtained using a less strict coincidence concept represented by means of a usual distance function d defined directly on S and proposed initially by Herrera, Herrera-Viedma, and Verdegay [21].

Now, we present a consensus model with a structure similar to [21, 23], i.e., with two types of linguistic consensus measures calculated on three levels of action, but with the following peculiarities:

- it is designed for GDM situations with heterogenous groups of experts and heterogeneous sets of alternatives using linguistic weighting degrees;
- it is developed in a heterogeneous linguistic context, i.e., using different linguistic domains to express the opinions, the importance, and relevance degrees, as well as the consensus measures;
- its consensus measures are obtained using a fuzzy formulation of the concept of coincidence.

3. LINGUISTIC SETTING OF THE GDM PROBLEM

As was mentioned earlier, we assume a GDM problem developed in a linguistic context, i.e., the experts use linguistic terms instead of numerical values to express their preferences [10, 12, 18, 22, 30, 35, 40]. We consider finite and totally ordered term sets on [0, 1], $S = \{s_i\}$, $i \in H = \{0, \ldots, T\}$, with an odd cardinal, in which the middle label represents an uncertainty of "approximately 0.5" and the remaining terms are placed around it symmetrically, as in [4]. Moreover, the term set must have the following characteristics:

1. The set is ordered: $s_i \geq s_j$ if $i \geq j$.
2. There is the negative operator: $\mathrm{Neg}(s_i) = s_j$ such that $j = T - i$.
3. Maximization operator: $\mathrm{Max}(s_i, s_j) = s_i$ if $s_i \geq s_j$.
4. Minimization operator: $\mathrm{Min}(s_i, s_j) = s_i$ if $s_i \leq s_j$.

We consider that the semantic of the elements in the term set is given by fuzzy numbers defined on the interval $[0, 1]$, which are described by linear trapezoidal membership functions. This representation is achieved by the 4-tuple $(a_i, b_i, \alpha_i, \beta_i)$. The first two parameters indicate the interval in which the membership value is 1; the third and fourth parameters indicate the left and right width.

Example 3.1. The following seven label set, S, verifies the aforementioned properties:

MA	*Maximum*	$(1, 1, .25, 0)$
VM	*Very_Much*	$(.75, .75, .15, .25)$
Mu	*Much*	$(.6, .6, .1, .15)$
M	*Medium*	$(.5, .5, .1, .1)$
L	*Little*	$(.4, .4, .15, .1)$
VL	*Very_Little*	$(.25, .25, .25, .15)$
MI	*Minimum*	$(0, 0, 0, .25)$

In this linguistic context, the mathematical model of the GDM problem considered is the following. Let $X = \{x_1, \dots, x_n\}$ be a heterogeneous, nonempty, and finite set of alternatives to be analyzed by a heterogeneous, nonempty, and finite set of experts $E = \{e_1, \dots, e_m\}$. Assuming a label set, $V = \{v_i\}$, $i \in I = \{0, \dots, M\}$, to express importance and relevance degrees, for each alternative, $x_i \in X$, we suppose that a *linguistic relevance degree* is defined, $\mu_R(i) \in V$, from v_0 standing for "definitely irrelevant" to v_M standing for "definitely relevant," across all the intermediate values. Similarly, for each expert $e_k \in E$, we assume that a *linguistic importance degree* is known, $\mu_E(k) \in V$, assigned by a distinguished person, called the moderator, to each expert e_k. Then, each expert e_k provides his/her opinions on X as a linguistic preference relation, $P^k \subset X \times X$, with membership function $\mu_{P^k} : X \times X \to S$, where $\mu_{P^k}(x_i, x_j) = p_{ij}^k$ denotes the linguistic preference degree of the alternative x_i over x_j. We assume, without loss of generality, that P^k is reciprocal in the sense that $p_{ij}^k = \mathrm{Neg}(p_{ji}^k)$, and by definition $p_{ii}^k = s_0$ (the minimum label in S).

Given an expert e_k, his importance degree, $\mu_E(k)$, is interpreted as the degree to which the expert is really a decision maker in relation to the decision problem. And given an alternative x_i, its relevance degree, $\mu_R(i)$, is interpreted as the degree to which the alternative is really an option in relation to the problem domain.

EXAMPLE 3.2 Assume the following nine label set V to express the importance and relevance degrees:

T	*Total*	$(1, 1, 0, 0)$
EH	*Extremely_High*	$(.98, .99, .05, .01)$
VH	*Very_High*	$(.78, .92, .06, .05)$
H	*High*	$(.63, .80, .05, .06)$
M	*Medium*	$(.41, .58, .09, .07)$
L	*Low*	$(.22, .36, .05, .06)$
VL	*Very_Low*	$(.1, .18, .06, .05)$
EL	*Extremely_Low*	$(.01, .02, .01, .05)$
N	*Null*	$(0, 0, 0, 0)$

Let $X = \{x_1, x_2, x_3, x_4\}$ be a heterogeneous set of four alternatives, for which the respective linguistic relevance degrees are

$$\mu_R(1) = EH, \qquad \mu_R(2) = M, \qquad \mu_R(3) = VH, \qquad \mu_R(4) = VL.$$

Let $E = \{e_1, e_2, e_3, e_4\}$ be a heterogeneous group of four experts, for which the respective linguistic importance degrees are

$$\mu_E(1) = M, \qquad \mu_E(2) = VH, \qquad \mu_E(3) = M, \qquad \mu_E(4) = L.$$

Then, following Example 3.1, linguistic preference relations over X, in this linguistic context, may be considered as:

$$P^1 = \begin{bmatrix} - & VL & VM & VL \\ VM & - & M & M \\ VL & L & - & VL \\ VM & L & VM & - \end{bmatrix}, \qquad P^2 = \begin{bmatrix} - & L & M & VL \\ M & - & VM & L \\ L & VL & - & VL \\ VM & M & VM & - \end{bmatrix},$$

$$P^3 = \begin{bmatrix} - & M & VM & MI \\ M & - & VM & L \\ VL & VL & - & VL \\ MA & M & VM & - \end{bmatrix}, \qquad P^4 = \begin{bmatrix} - & L & VM & VL \\ M & - & L & VL \\ VL & M & - & VL \\ VM & VM & VM & - \end{bmatrix}.$$

In the GDM problem, in order to aggregate linguistic labels, we use the LOWA operator [18, 22], which allows us to represent the concept of fuzzy majority in the aggregation processes. The LOWA operator is based on the *ordered weighted averaging* (OWA) operator defined by Yager [34], and on the *convex combination of linguistic labels* defined by Delgado et al. [11].

DEFINITION 3.1 *Let $A = \{a_1, \ldots, a_m\}$ be a set of labels to be aggregated. Then the LOWA operator ϕ is defined as*

$$\phi(a_1, \ldots, a_m) = WB^T = \mathscr{C}^m\{w_k, b_k, k = 1, \ldots, m\}$$
$$= w_1 \odot b_1 \oplus (1 - w_1) \odot \mathscr{C}^{m-1}\{\beta_h, b_h, h = 2, \ldots, m\}$$

where $W = [w_1, \ldots, w_m]$, is a weighting vector such that (i) $w_i \in [0,1]$ *and* (ii) $\Sigma_i = 1$, $\beta_h = w_h / \Sigma_2^m w_k$, $h = 2, \ldots, m$, *and $B = \{b_1, \ldots, b_m\}$ is a vector associated with A such that $B = \sigma(A) = \{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\}$, where $a_{\sigma(j)} \leq a_{\sigma(i)}$ $\forall i \leq j$, with σ being a permutation over the set of labels A. $\mathscr{C}^m$ is the convex combination operator of m labels, and if $m = 2$, then it is defined as*

$$\mathscr{C}^2\{w_i, b_i, i = 1,2\} = w_1 \odot s_j \oplus (1 - w_1) \odot s_i = s_k, \qquad s_j, s_i \in S \quad (j \geq i)$$

such that, $k = \mathrm{MIN}\{T, i + \mathrm{round}(w_1 \cdot (j - i))\}$, where round *is the usual rounding operation, and $b_1 = s_j$, $b_2 = s_i$. If $w_j = 1$ and $w_i = 0$ with $i \neq j$ $\forall i$, then the convex combination is defined as $\mathscr{C}^m$ $\{w_i, b_i, i = 1, \ldots, m\} = b_j$.*

Other approaches to aggregation of linguistic labels may be found in [4, 11, 30, 35, 36, 38–40].

How to calculate the weighting vector of the LOWA operator, W, is a basic question to decide. Yager proposed in [34, 37] an interesting way to compute the weights of the OWA aggregation operator using linguistic quantifiers [42], representing the concept of *fuzzy majority*. In our case, we use two types of fuzzy majority:

- *Fuzzy majority of alternatives*, used to quantify the different fuzzy coincidence degrees according to one pair of experts' opinions.
- *Fuzzy majority of experts*, used to quantify the different consensus measures according to every pair of experts' opinions.

According to Yager [34, 37] the weights can be obtained by means of the following expression:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \qquad i = 1, \ldots, n,$$

where Q is a nondecreasing proportional quantifier represented by the following membership function:

$$Q(r) = \begin{cases} 0 & \text{if } r < a, \\ \dfrac{r-a}{b-a} & \text{if } a \leq r < b, \\ 1 & \text{if } r \geq b \end{cases}$$

with $a, b, r \in [0,1]$.

EXAMPLE 3.3 Some proportional quantifiers are shown in Figure 2, in which the parameters (a, b) are $(0.3, 0.8)$, $(0, 0.5)$, and $(0.5, 1)$, respectively.

When a fuzzy linguistic quantifier Q is used to compute the weights of the LOWA operator ϕ, it is symbolized by ϕ_Q.

4. LINGUISTIC MEASURES BASED ON FUZZY COINCIDENCE FOR REACHING CONSENSUS

Assuming the aforementioned linguistic GDM problem setting, we present new linguistic consensus measures based on a fuzzy characterization of the concept of coincidence. They allow us to know the current state of consensus from different viewpoints, and therefore, to guide more correctly the consensus reaching processes. As in [21, 23], we present two types of *consensus measures*, one to measure the concensus degree among experts' opinions, called *consensus degrees*, and another to measure the closeness among experts' opinions and group opinion, called *linguistic proximities*. These measures evaluate the consensus state on three levels of action: the *level of the alternative pairs*, the *level of the alternatives*, and the *level of the relation.*

Therefore, the *consensus degrees* are:

1. *Pair linguistic consensus degree:* measuring the social consensus degree considering the experts' opinions expressed on a single pair of alternatives (x_i, x_j).
2. *Alternative linguistic consensus degree:* measuring the social consensus degree considering the experts' opinions expressed on the subset of pairs of alternatives determined by a single alternative, x_i.
3. *Relation linguistic consensus degree:* measuring the social consensus degree considering the total set of experts' opinions, i.e., the relation P^k.

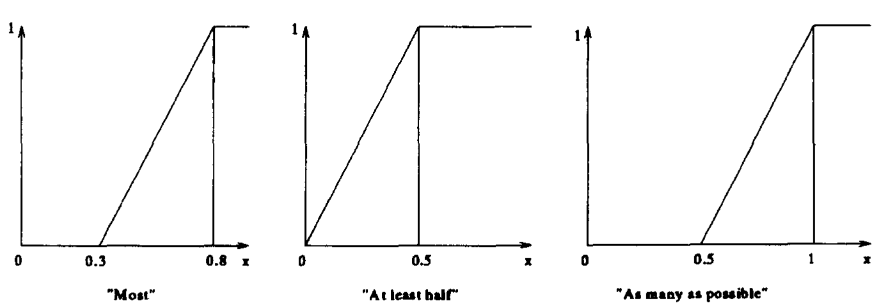

Figure 2. Proportional fuzzy linguistic quantifiers.

And the *linguistic proximities* are:

1. *Pair linguistic proximity:* measuring the closeness degree among opinions given by a single expert, e_k, and the remaining ones, on a single pair of the alternatives (x_i, x_j).
2. *Alternative linguistic proximity:* measuring the closeness degree among opinions given by a single expert, e_k, and the remaining ones, on the subset of pairs of alternatives determined by a single alternative, x_i.
3. *Relation linguistic proximity:* measuring the closeness degree among opinions given by a single expert, e_k, and the remaining ones, on the total set of the pairs of the alternatives, i.e., on the relation, P^k.

Each measure is defined on its own level, and helps the moderator to decide about the need to continue the consensus reaching process and to make his recommendations.

The derivation model of these linguistic consensus measures may be viewed as a *hierarchical amalgamation model* shown in Figure 3, developed from the lower level, the level of the pairs of alternatives, to the upper level, the level of the relation. It is formed by two processes:

1. *Coincidence process*, which evaluates the fuzzy coincidence between every different pair of experts in each level of action.
2. *Computing process*, which evaluates the consensus measures among and for all the experts in each level of action.

Each process is formed by three phases: (1) *working on the pairs of alternatives*, (2) *working on the alternatives*, and (3) *working on the relation*. These phases are applied consecutively, beginning with phase 1 of the coincidence process. Briefly, the derivation model is developed as follows:

1. *Working on the pairs of alternatives.* First, working on the level of the alternative pairs, for each pair of the alternatives, (x_i, x_j), the fuzzy coincidence degrees on the pair are found out according to every pair of experts, considering the closeness among their preferences assigned to this pair. Then, linguistic consensus measures of this level of action are obtained.
2. *Working on the alternatives.* Second, working on the level of the alternative, for each alternative x_i, the fuzzy coincidence degrees on the alternative are found out according to every pair of experts, considering the fuzzy coincidence degrees obtained in the previous phase for all pairs in which the alternative x_i appears. These fuzzy coincidence degrees are calculated by amalgamation by means of the LOWA operator, some linguistic conjunction functions [16], and an appropriate linguistic quantifier, Q^2, which expresses the fuzzy majority of the alternative. The, linguistic consensus measures of this level of action are obtained.

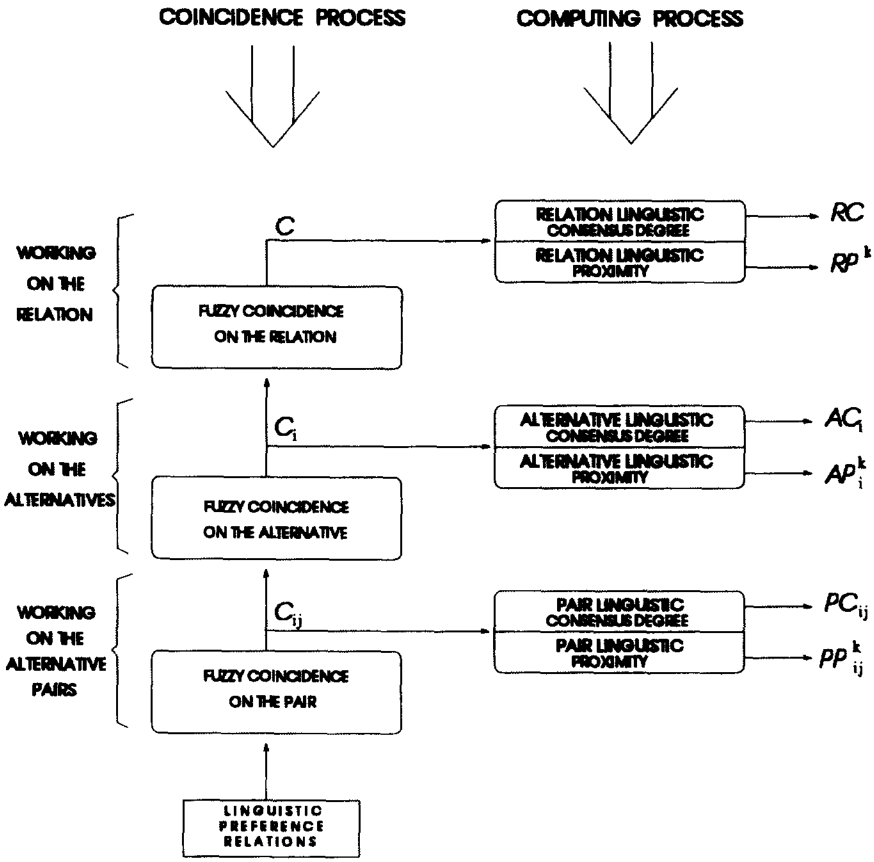

Figure 3. Hierarchical amalgamation model.

3. *Working on the relation.* And finally, working on the level of the relation, the fuzzy coincidence degrees on the overall set of the alternatives, X, are found out according to every pair of experts, considering the fuzzy coincidence degrees obtained in the previous phase for all alternatives. Similarly, they are calculated as in the previous phase. Then, linguistic consensus measures of this level of action are obtained.

In all these phases, linguistic consensus measures are derived by amalgamation of these respective fuzzy coincidence degrees together with the importance degrees by means of the LOWA operator, some linguistic conjunction functions, and an appropriate linguistic quantifier Q^1, which expresses the fuzzy majority of experts. In this regard,

- the *pair linguistic consensus degree* expresses the consensus state of the Q^1 pairs of experts according to their preferences on a pair of alternatives,
- the *alternative linguistic consensus degree* expresses the consensus state of the Q^1 pairs of experts according to their preferences on Q^2 pairs of alternatives in which one given alternative appears,
- the *relation linguistic consensus degree* expresses the consensus state of the Q^1 pairs of experts according to their preferences on Q^2 pairs of alternatives,
- the *pair linguistic proximity* expresses the state of agreement of an expert, e_k, with the Q^1 remaining experts according to their preferences on a pair of alternatives,
- the *alternative linguistic proximity* expresses the state of agreement of an expert, e_k, with the Q^1 remaining experts according to their preferences on Q^2 pairs of alternatives in which one given alternative appears, and
- the *relation linguistic proximity* expresses the state of agreement of an expert, e_k, with the Q^1 remaining experts according to their preferences on Q^2 pairs of alternatives.

Next, we develop the derivation model in detail, studying its phases with their coincidence and computing processes.

5. PHASE 1: WORKING ON THE PAIRS OF ALTERNATIVES

5.1. Coincidence Process

In this step the fuzzy coincidence concept among experts' opinions is defined, working on the level of the pair of alternatives.

As we said at the beginning, we assume a label set to be established, $G = \{g_i\}$, $i \in J = \{0, \ldots, U\}$, to express the new linguistic consensus measures, with an appropriate meaning to express the fuzzy coincidence degrees, different from that used to provide the experts' opinions. Here, we do not deal with the way to obtain it. A possible option is to consider that experts decide about it before beginning the GDM process. In the following we present the definition of the fuzzy coincidence on a pair of alternatives according to the ideas previously mentioned.

DEFINITION 5.1 *The fuzzy coincidence on a pair of alternatives*, (x_i, x_j), $i \neq j$, *is defined as a fuzzy set*, C_{ij}, *in the nonfuzzy set of pairs of experts*,

$E^2 = \{e_{kl} = (e_k, e_l),\ k = 1, \ldots, m-1,\ l = k+1, \ldots, m\}$, namely $C_{ij} = \{(e_{kl}, \mu_{C_{ij}}(e_{kl}))\}$, characterized by a membership function, $\mu_{C_{ij}} : E^2 \to G$, with $\mu_{C_{ij}}(e_{kl}) = \mathrm{Neg}(d(p_{ij}^k, p_{ij}^l))$, indicating the coincidence degree between experts e_k and e_l's opinions on a pair of alternatives, where d stands for a closeness measure among opinions assessed linguistically on G, i.e., $d : S \times S \to G$.

In order to define the closeness measure d among linguistic opinions (labels), two main approaches may be considered: the first one by agreement among experts using the definition table, and the second one by linguistic approximation using associated membership functions [4, 30, 14]. Most of the available techniques belong to the latter kind. However, its difficulty is that it obtains fuzzy sets which do not correspond to any label in the original term set, and if one finally wants to have a label, then a "linguistic approximation" is needed [4, 30, 14]. Therefore, since, in our context, we consider an environment where experts can discriminate perfectly the same label set under a similar conception, and since we use linguistic aggregation operators by direct computation on labels, in this paper we assume the first approach.

This approach consists in establishing an *ad hoc closeness table*, $\Omega : S \times S \to G$, according to the experts' feeling, in such a way that if $p_{ij}^k = s_t$ and $p_{ij}^l = s_v$ then $d(p_{ij}^k, p_{ij}^l) = \Omega(s_t, s_v)$, $\Omega(s_t, s_v) \in G$, $t, v \in \{0, \ldots, T\}$. This, together with the determination of the label set G, may be done in a state prior to the GDM process.

EXAMPLE 5.1 Assume the label set given in Example 3.1 to express the opinions; and as the label set to express linguistic consensus measures, G, assume the set V that was given in Example 3.2 to express importance and relevance degrees, i.e., $G = V$. Then the table may be defined as shown in Figure 4.

Let us point out that we do not deal here with how the table is obtained, i.e., it is not built according to the axiom set, but it should be the result of the particular way of evaluation of the group of experts. Therefore, we may find curious situations, such as for example in Figure 4, where $\Omega(MI, L) = VL$ and $\Omega(MI, VL) = VL$.

EXAMPLE 5.2 Let d be the closeness table Ω considered in Example 5.1. Then, in the GDM context given in Example 3.2, for each pair of alternatives (x_i, x_j), its set of fuzzy coincidences, C_{ij} in E^2 is obtained, resulting in

$$C_{12} = \{(e_{12}, VH), (e_{13}, M), (e_{14}, VH), (e_{23}, M), (e_{24}, T), (e_{34}, M)\}.$$

$$C_{13} = \{(e_{12}, M), (e_{13}, T), (e_{14}, T), (e_{23}, M), (e_{24}, M), (e_{34}, T)\},$$

S={MI, VL, L, M, MU, VM, MA}
G={N, EL, VL, L, M, H, VH, EH, T}

$\Omega : S\times S \longrightarrow G$

Ω	MI	VL	L	M	MU	VM	MA
MA	T	EH	VH	H	M	L	N
VM	EH	H	H	M	M	N	L
MU	VH	H	H	M	N	M	M
M	H	M	M	N	M	M	H
L	VL	VL	N	M	H	H	VH
VL	VL	N	VL	M	H	H	EH
MI	N	VL	VL	H	VH	EH	T

Figure 4. Ad hoc closeness table.

$$C_{14} = \{(e_{12},T),(e_{13},VH),(e_{14},T),(e_{23},VH),(e_{24},T),(e_{34},VH)\},$$

$$C_{21} = \{(e_{12},M),(e_{13},M),(e_{14},M),(e_{23},T),(e_{24},T),(e_{34},T)\},$$

$$C_{23} = \{(e_{12},M),(e_{13},M),(e_{14},M),(e_{23},T),(e_{24},L),(e_{34},L)\},$$

$$C_{24} = \{(e_{12},M),(e_{13},M),(e_{14},M),(e_{23},T),(e_{24},VH),(e_{34},VH)\},$$

$$C_{31} = \{(e_{12},VH),(e_{13},T),(e_{14},T),(e_{23},VH),(e_{24},VH),(e_{34},T)\},$$

$$C_{32} = \{(e_{12},VH),(e_{13},VH),(e_{14},M),(e_{23},T),(e_{24},M),(e_{34},T)\},$$

$$C_{34} = \{(e_{12},T),(e_{13},T),(e_{14},T),(e_{23},T),(e_{24},T),(e_{34},T)\},$$

$$C_{41} = \{(e_{12},T),(e_{13},H),(e_{14},N),(e_{23},H),(e_{24},T),(e_{34},T)\},$$

$$C_{42} = \{(e_{12},M),(e_{13},M),(e_{14},L),(e_{23},T),(e_{24},M),(e_{34},M)\},$$

$$C_{43} = \{(e_{12},T),(e_{13},T),(e_{14},T),(e_{23},T),(e_{24},T),(e_{34},T)\}.$$

For example, $\mu_{C_{42}}(e_{12})$ is obtained as

$$\begin{aligned}\mu_{C_{42}}(e_{12}) &= \mathrm{Neg}\big(d(p_{42}^1, p_{42}^2)\big) = \mathrm{Neg}(d(L,M)) \\ &= \mathrm{Neg}(\Omega(s_2,s_3)) = \mathrm{Neg}(M) = M,\end{aligned}$$

since $L = s_2$ and $M = s_3$.

5.2. Computing Process

In this first step of the computing process, for each pair of alternatives, (x_i, x_j), the different *pair linguistic consensus measures* are calculated according to the following definitions:

DEFINITION 5.2 *The pair linguistic consensus degree, PC_{ij}, is defined according to this expression*:

$$PC_{ij} = \phi_{Q^1}\Big(LC^{\rightarrow}\left(\mu_{C_{ij}}(e_{kl}), r_{kl}\right), k = 1,\ldots,m-1, l = k+1,\ldots,m\Big),$$

and $r_{kl} = \phi(\mu_E(k), \mu_E(l))$, with weighting vector of the LOWA operator, $W = [0.5, 0.5]$.

r_{kl} is an *averaging importance degree*, which represents the importance degree of the coincidence degree of the pair of experts, e_{kl}. It is obtained by means of the LOWA operator ϕ with that weighting vector in order to achieve a mean aggregation of the importance degrees. $LC^{\rightarrow}$ represents a family of connectives, i.e., *linguistic conjunction functions* [17]. We shall use as linguistic conjunction functions the following t-norms, which are monotonically nonincreasing in the weights w, and satisfy the properties required for any transformation function of the weighted information (a, w) [16, 17]:

1. The classical Min operator:

$$LC_1^{\rightarrow}(a,w) = \mathrm{Min}(a,w).$$

2. The nilpotent Min operator:

$$LC_2^{\rightarrow}(a,w) = \begin{cases} \mathrm{Min}(a,w) & \text{if} \quad w > \mathrm{Neg}(a), \\ g_0 & \text{otherwise}. \end{cases}$$

3. The weakest conjunction:

$$LC_3^{\rightarrow}(a,w) = \begin{cases} \mathrm{Min}(a,w) & \text{if} \quad \mathrm{Max}(a,w) = g_M, \\ g_0 & \text{otherwise}. \end{cases}$$

And ϕ_{Q^1} is the LOWA operator for which the weighting vector is obtained by means of the linguistic quantifier, Q^1, used to represent the concept of fuzzy majority of experts.

REMARK 5.1 Note that Definition 5.2 explicitly requires this restriction, $G = V$, i.e., that the linguistic domain used to express consensus measures is the same one used to express importance and relevance degrees. This limitation may be bridged if we use a method to transform labels among different linguistic domains, but this is not our goal in this paper.

EXAMPLE 5.3 Continuing with the GDM context given in Example 3.2, from importance degrees of experts, for each pair of experts, e_{kl}, the averaging importance degrees, $r_{kl} \in V$, are calculated,

$$\{r_{12} = H, r_{13} = M, r_{14} = M, r_{23} = H, r_{24} = H, r_{34} = M\},$$

in which, for example, as $\mu_E(1) = M = v_4$ and $\mu_E(2) = VH = v_6$, then $r_{12} = H = v_5$, since $5 = \mathrm{MIN}\{8, 4 + \mathrm{round}((6-4) \times 0.5)\}$.

Here and in the next examples, we assume the *nilpotent* Min *operator*, $LC_2^{\rightarrow}$, to manipulate linguistic weighted information, and as the linguistic quantifier Q^1 the quantifier given in Example 3.3, "As many as possible," with the pair (0.5, 1).

Then, in this context, from the fuzzy coincidence sets obtained in Example 5.2 and from the previous averaging importance degrees, on each pair of alternatives, (x_i, x_j), $i \neq j$, we calculate the *pair linguistic consensus degree*, PC_{ij}, by means of the conjunction function, $LC_2^{\rightarrow}$, and of the LOWA operator, ϕ_{Q^1}, with the weighting vector, $W = [0, 0, 0, 0.32, 0.35, 0.33]$:

$$\{PC_{12} = EL, PC_{13} = M, PC_{14} = M,\},$$
$$\{PC_{21} = EL, PC_{23} = N, PC_{24} = EL,\},$$
$$\{PC_{31} = M, PC_{32} = EL, PC_{34} = M,\},$$
$$\{PC_{41} = L, PC_{42} = N, PC_{43} = M,\},$$

in which, for example, PC_{41} is obtained as

$$PC_{41} = \phi_{Q^1}(LC_2^{\rightarrow}(T, H), LC_2^{\rightarrow}(H, M), LC_2^{\rightarrow}(N, M),$$
$$LC_2^{\rightarrow}(H, H), LC_2^{\rightarrow}(T, H), LC_2^{\rightarrow}(T, M)) = L.$$

COMMENT 5.1 In general, the consensus degrees are low. For example, on the pairs of alternatives (x_2, x_3) and (x_4, x_2), there is no consensus among the experts' opinions, and on the set of pairs of alternatives $\{(x_1, x_2), (x_2, x_1), (x_2, x_4), (x_3, x_2)\}$ the consensus is too low. Only a maximum consensus degree with a value M is achieved on some pairs of alternatives. However, if we observe the sets of fuzzy coincidences obtained in Example 5.2, their membership functions present, in general, values above the middle value, M, which should result in high consensus degrees. So, from this viewpoint, apparently, there is a contradiction. However, we must not forget that we are working implicitly in a heterogeneous GDM context with different meanings of fuzzy majority. So, this situation sometimes is due to the influence of the chosen conjunction function, and in others, it is due to the influence of the chosen linguistic quantifier. In our case, both the conjunction function, $LC_2^{\rightarrow}$ and the linguistic quantifier "As many as possible" induce a pessimistic influence of the consensus state. So, for example, if $LC_1^{\rightarrow}$ is chosen as a conjunction function, then $PC_{12} = M$, and similarly, other consensus degrees will be higher. In the same way, if "At least half" is chosen as a linguistic quantifier, maintaining $LC_2^{\rightarrow}$, then $PC_{12} = H$. Therefore, in short, we must choose both appropriate conjunction functions and linguistic quantifiers in tune with our consensus idea.

Now, similarly, from the fuzzy coincidence on the pairs of alternatives, we define another pair linguistic consensus measure.

DEFINITION 5.3 *The pair linguistic proximity PP_{ij}^k of an expert e_k is defined according to this expression:*

$$PP_{ij}^k = \phi_{Q^1}\Big(LC^{\rightarrow}\left(\mu_{C_{ij}}(e_{kl}), \mu_E(l)\right), l = 1, \ldots, m, l \neq k\Big),$$

knowing that when $\mu_{C_{ij}}(e_{kl}) \notin C_{ij}$ then $\mu_{C_{ij}}(e_{kl}) = \mu_{C_{ij}}(e_{lk})$.

REMARK 5.2 Note that in this definition, as in Definition 5.2, the consensus measure is defined by means of the LOWA operator ϕ_{Q^1} and the conjunction function $LC^{\rightarrow}$, and using the sets of fuzzy coincidence of Definition 5.1, but in this case considering only the importance degrees of the remaining experts and not the averaging importance degrees. Therefore, in this sense, $\mu_E(l)$ is used as the importance degree given to the coincidence degree observed between the expert analyzed and another expert e_l in the group.

EXAMPLE 5.4 As in Example 5.3, but this time assuming the importance degrees given in Example 3.2 instead of averaging importance degrees, on each pair of alternatives, (x_i, x_j), and for each expert e_k, we calculate *pair linguistic proximities PP_{ij}^k* by means of the conjunction function $LC_2^{\rightarrow}$, and the LOWA operator ϕ_{Q^1} with the weighting vector $W = [0, 0.32, 0.68]$:

1. *Expert e_1*:

$$\{PP_{12}^1 = L, PP_{13}^1 = M, PP_{14}^1 = M, PP_{21}^1 = N, PP_{23}^1 = N, PP_{24}^1 = N, PP_{31}^1 = M, PP_{32}^1 = M, PP_{34}^1 = M, PP_{41}^1 = M, PP_{42}^1 = N, PP_{43}^1 = M\}$$

2. *Expert e_2*:

$$\{PP_{12}^2 = L, PP_{13}^2 = N, PP_{14}^2 = M, PP_{21}^2 = L, PP_{23}^2 = N, PP_{24}^2 = L, PP_{31}^2 = M, PP_{32}^2 = M, PP_{34}^2 = M, PP_{41}^2 = M, PP_{42}^2 = N, PP_{43}^2 = M\}$$

3. *Expert e_3*:

$$\{PP_{12}^3 = L, PP_{13}^3 = M, PP_{14}^3 = M, PP_{21}^3 = L, PP_{23}^3 = N, PP_{24}^3 = L, PP_{31}^3 = M, PP_{32}^3 = M, PP_{34}^3 = M, PP_{41}^3 = M, PP_{42}^3 = N, PP_{43}^3 = M\}$$

4. *Expert* e_4:

$$\{PP^4_{12} = M,\ PP^4_{13} = M,\ PP^4_{14} = M,\ PP^4_{21} = M,\ PP^4_{23} = N,\ PP^4_{24} = M,$$
$$PP^4_{31} = M,\ PP^4_{32} = N,\ PP^4_{34} = M,\ PP^4_{41} = M,\ PP^4_{42} = N,\ PP^4_{43} = M\}$$

For example, PP^4_{42} is obtained as

$$PP^4_{42} = \phi_{Q^2}(LC_2^{\rightarrow}(L, M), LC_2^{\rightarrow}(M, VH), LC_2^{\rightarrow}(M, M)) = N.$$

COMMENT 5.2 Logically, as in Example 5.3 and for the same reasons, here, all the experts present very low proximities—in any case, never higher than the middle value M.

6. PHASE 2: WORKING ON THE ALTERNATIVES

6.1. Coincidence Process

In this step the concept of fuzzy coincidence among experts' opinions is defined, working on the level of the alternatives.

DEFINITION 6.1 *The fuzzy coincidence on an alternative, x_i, is defined as a fuzzy set C_i in the nonfuzzy set of pairs of experts, E^2, namely $C_i = \{(e_{kl}, \mu_{C_i}(e_{kl}))\}$, characterized by a membership function $\mu_{C_i} : E^2 \rightarrow G$ indicating the coincidence degree between experts e_k and e_l's opinions on pairs of alternatives in which the alternative x_i appears:*

$$\mu_{C_i}(e_{kl}) = \phi_{Q^2}\Big(LC^{\rightarrow}\left(\mu_{C_{ij}}(e_{kl}), r'_{ij}\right), LC^{\rightarrow}\left(\mu_{C_{ji}}(e_{kl}), r'_{ij}\right),$$
$$j \neq i, j = 1, \ldots, n\Big),$$

and $r'_{ij} = \phi(\mu_R(i), \mu_R(j))$, with the weighting vector of the LOWA operator given by $W = [0.5, 0.5]$.

r'_{ij} is an *averaging relevance degree*, which represents the relevance degree of the coincidence degree achieved on the pair of alternatives (x_i, x_j). It is obtained in the same way as r_{kl} in Definition 5.2. In this case, the coincidence degree $\mu_{C_i}(e_{kl})$ is obtained by means of a LOWA operator for which the weighting vector is calculated by means of the linguistic quantifier Q^2 used to represent the concept of fuzzy majority of alternatives. The restriction pointed out in Remark 5.1 is applied here too and in the next definitions.

EXAMPLE 6.1 Continuing with the GDM context given in Example 3.2, from the relevance degrees of the alternatives, for each pair of alternatives, (x_i, x_j), the averaging relevance degrees $r'_{ij} \in V$ are

$$\{r'_{12} = VH, r'_{13} = EH, r'_{14} = H, r'_{23} = H, r'_{24} = L, r'_{34} = M\}.$$

in which, for example, as $\mu_R(1) = EH = v_7$ and $\mu_R(2) = VH = v_6$, we have $r'_{12} = EH = v_7$, since $7 = \text{MIN}\{8, 6 + \text{round}((7-6) \times 0.5)\}$.

Assuming, like Q^2, the linguistic quantifier given in Example 3.3, i.e., "At least half," with the pair $(0, 0.5)$, then, from the fuzzy coincidence sets calculated in Example 5.2 and from the above averaging relevance degrees, by means of the LOWA operator ϕ_{Q^2} with $W = [0.33, 0.35, 0.32, 0, 0, 0]$, and the conjunction function $LC_2^{\rightarrow}$ for each alternative x_i its fuzzy coincidence set C_i in E^2, is obtained, resulting in

$$C_1 = \{(e_{12}, VH), (e_{13}, VH), (e_{14}, EH), (e_{23}, VH), (e_{24}, VH), (e_{34}, EH)\},$$

$$C_2 = \{(e_{12}, H), (e_{13}, M), (e_{14}, H), (e_{23}, H), (e_{24}, H), (e_{34}, H)\},$$

$$C_3 = \{(e_{12}, H), (e_{13}, VH), (e_{14}, H), (e_{23}, H), (e_{24}, M), (e_{34}, H)\},$$

$$C_4 = \{(e_{12}, H), (e_{13}, H), (e_{14}, M), (e_{23}, H), (e_{24}, H), (e_{34}, H)\}.$$

For example, $\mu_{C_4}(e_{14})$ is obtained as

$$\mu_{C_4}(e_{14}) = \phi_{Q^2}(LC_2^{\rightarrow}(N, H), LC_2^{\rightarrow}(M, L), LC_2^{\rightarrow}(M, M),$$
$$LC_2^{\rightarrow}(H, H), LC_2^{\rightarrow}(N, L), LC_2^{\rightarrow}(N, M)) = M.$$

6.2. Computing Process

In this second step, on each alternative, x_i, the different *alternative linguistic consensus measures* are calculated according to the following definitions:

DEFINITION 6.2 *The alternative linguistic consensus degree, AC_i, is defined according to this expression*:

$$AC_i = \phi_{Q^1}\left(LC^{\rightarrow}\left(\mu_{C_i}(e_{kl}), r_{kl}\right); k = 1, \dots, m-1, l = k+1, \dots, m\right).$$

EXAMPLE 6.2 In the same way as we did in Example 5.3, on each alternative, x_i, we calculate the alternative linguistic consensus degree, AC_i, but this time, considering the aforementioned fuzzy coincidence sets

$$\{AC_1 = M, AC_2 = M, AC_3 = M, AC_4 = M\},$$

in which, for example, AC_4 is obtained as

$$AC_4 = \phi_{Q^1}(LC^{\rightarrow}(H,H), LC^{\rightarrow}(H,M), LC^{\rightarrow}(M,M),$$
$$LC^{\rightarrow}(H,H), LC^{\rightarrow}(H,H), LC^{\rightarrow}(H,M)) = M$$

COMMENT 6.1 On this level, the effect pointed out in Comment 5.1 also appears, since there is a maximum consensus degree with value M on any alternative, in spite of the fact that some observed coincidence degrees are high. Besides, here the effect of the averaging relevance degrees, used to calculate the coincidence degrees, is included.

DEFINITION 6.3 *The alternative linguistic proximity AP_i^k of an expert e_k is defined according to this expression*:

$$AP_i^k = \phi_{Q^1}\Big(LC^{\rightarrow}\big(\mu_{C_i}(e_{kl}), \mu_E(l)\big), l = 1, \ldots, m, l \neq k\Big),$$

knowing that when $\mu_{C_i}(e_{kl}) \notin C_i$, then $\mu_{C_i}(e_{kl}) = \mu_{C_i}(e_{lk})$.

EXAMPLE 6.3 In the same way we did in Example 5.4, here, on each alternative x_i for each expert e_k, we calculate the alternative linguistic proximity AP_i^k, but this time, considering the aforementioned fuzzy coincidence sets

$$\{AP_1^1 = L, AP_2^1 = N, AP_3^1 = EL, AP_4^1 = EL\},$$
$$\{AP_1^2 = L, AP_2^2 = EL, AP_3^2 = EL, AP_4^2 = EL\},$$
$$\{AP_1^3 = L, AP_2^3 = N, AP_3^3 = EL, AP_4^3 = EL\},$$
$$\{AP_1^4 = M, AP_2^4 = M, AP_3^4 = M, AP_4^4 = EL\}.$$

Here, for example, AP_1^4 is obtained as

$$AP_1^4 = \phi_{Q^1}(LC^{\rightarrow}(EH,M), LC^{\rightarrow}(VH,VH), LC^{\rightarrow}(EH,M)) = M.$$

7. PHASE 3: WORKING ON THE RELATION

7.1. Coincidence Process

In this last phase, the concept of fuzzy coincidence among experts' opinions is defined, working on the level of the relation.

DEFINITION 7.1 *The fuzzy coincidence on the relation is defined as a fuzzy set $C = \{(e_{kl}, \mu_C(e_{kl}))\}$ in the nonfuzzy set of pairs of experts, E^2, characterized by a membership function, $\mu_C : E^2 \rightarrow G$, indicating the coincidence* *degree between experts e_k and e_l's opinions on all the pairs of alternatives*:

$$\mu_C(e_{kl}) = \phi_{Q^2}\Big(LC^{\rightarrow}\left(\mu_{C_i}(e_{kl}), \mu_R(i)\right), i = 1, \dots, n\Big).$$

REMARK 7.1 In this definition, the coincidence degree, $\mu_C(e_{kl})$, is obtained as in Definition 6.1, i.e., by means of the LOWA operator ϕ_{Q^2} and of the conjunction function $LC^{\rightarrow}$, but in this case, using relevance degrees $\mu_R(i)$ instead of averaging relevance degrees r'_{ij} representing the relevance degree of the coincidence degree achieved on each alternative x_i.

EXAMPLE 7.1 Assuming the relevance degrees given in Example 3.2, as was done in Example 6.1, for overall alternatives X, its set of fuzzy coincidence, C, in E^2 is obtained from relevance degrees and the fuzzy coincidence sets obtained in Example 6.1, by means of the LOWA operator ϕ_{Q^2} with the weighting vector $W = [0.5, 0.5, 0, 0]$ and the same conjunction function, $LC_2^{\rightarrow}$, resulting in

$$C = \{(e_{12}, VH), (e_{13}, VH), (e_{14}, VH), (e_{23}, VH), (e_{24}, H), (e_{34}, VH)\}.$$

For example, $\mu_C(e_{14})$ is obtained as

$$\mu_C(e_{14}) = \phi_{Q^2}(LC^{\rightarrow}(EH, EH), LC^{\rightarrow}(H, M), LC^{\rightarrow}(H, VH), LC^{\rightarrow}(M, VL)) = VH.$$

7.2. Computing Process

In this last computing process, on overall opinions, i.e., on the relation, the *relation linguistic consensus measures* are calculated according to the following definitions:

DEFINITION 7.2 *The relation of linguistic consensus degree, RC, is defined according to this expression*:

$$RC = \phi_{Q^1}(LC^{\rightarrow}(\mu_C(e_{kl}), r_{kl}), k = 1, \dots, m-1, l = k+1, \dots, m).$$

DEFINITION 7.3 *The relation of linguistic proximity RP^k of an expert e_k is defined according to this expression*:

$$RP^k = \phi_{Q^1}(LC^{\rightarrow}(\mu_C(e_{kl}), \mu_E(l)), l = 1, \dots, m, l \neq k),$$

knowing that when $\mu_C(e_{kl}) \notin C$ then $\mu_C(e_{kl}) = \mu_C(e_{lk})$.

EXAMPLE 7.2 Working as in Examples 6.2 and 6.3, then

$$RC = M,$$

and

$$\{RP^1 = L, RP^2 = EL, RP^3 = L, RP^4 = M\},$$

respectively.

COMMENT 7.1 In view of the resulting consensus measures, which indicate a medium consensus current state, the consensus reaching process can stop or continue. In the second case, then, the moderator has to make some of the following considerations:

- Advise all the experts to change their opinions on pairs of alternatives, and in particular on the set of pairs of alternatives

$$\{(x_1, x_2), (x_2, x_1), (x_2, x_3), (x_2, x_4), (x_3, x_2), (x_4, x_1), (x_4, x_2)\}.$$

- Advise the experts $\{e_1, e_2, e_3\}$ to diminish their disagreement among them.
- Decide on the usefulness of maintaining the pessimistic effect of the chosen linguistic quantifier to calculate the linguistic consensus measures and of the chosen conjunction function in the following step of the consensus measuring process.

8. CONCLUSIONS

A consensus model is proposed in order to develop a consensus reaching process in a GDM context with heterogeneous groups of experts and a heterogeneous set of alternatives in a heterogeneous linguistic framework. This model contains two types of consensus measures to guide the consensus process from two different perspectives. The first type, called *linguistic consensus degrees*, studies the consensus state from a global perspective, considering all the experts, and the second type, called *linguistic consensus proximity*, studies the consensus state from a particular perspective, i.e., considering particular experts. Furthermore, all the types of measures are applied on three level of actions for representing the current consensus state. Therefore, the consensus model presents three consensus measures of each type. The main features of the consensus model are the following ones:

- its measures are based on a fuzzy characterization of the concept of coincidence defined from an *ad hoc closeness table*;
- it uses different linguistic domains to express the opinions and the consensus measures;
- its measures are calculated by means of the LOWA operator, several conjunction functions, and linguistic quantifiers representing the concept of fuzzy majority;

- it presents a flexible structure, which allows us to use different linguistic quantifiers and different conjunction functions, with a view to inducing different consensus ideas.

In short, a flexible consensus model has been presented.

Finally, we point out two aspects that are outside of our objectives in this paper, but are of interest in a decision process too. They are: (1) the negotiation process between the experts and the moderator for reaching and acceptable consensus level [33], and (2) how to model the possible conflicts between experts or goals and their explicit representation [7, 8], which can help in the negotiation process and can enhance its explanation.

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