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REVIEWS

G-Networks: Development of the Theory of Multiplicative Networks[1]

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Abstract—This is a review on G-networks, which are the generalization of the Jackson and BCMP networks, for which the multi-dimensional stationary distribution of the network state probabilities is also represented in product form. The G-networks primarily differ from the Jackson and BCMP networks in that they additionally contain a flow of the so-called negative customers and/or triggers. Negative customers and triggers are not served. When a negative customer arrives at a network node, one or a batch of positive (ordinary) customers is killed (annihilated, displaced), whereas a trigger displaces a positive customer from the node to some other node. For applied mathematicians, G-networks are of great interest for extending the multiplicative theory of queueing networks and for practical specialists in modeling computing systems and networks and biophysical neural networks for solving pattern recognition and other problems.

1. INTRODUCTION

A central place in queueing network theory belongs to networks admitting product form of the joint stationary distribution of the number of customers at nodes. For the sake of brevity, such networks are often referred to as multiplicative networks. The theory of multiplicative networks takes its origin in [76], in which the multidimensional stationary distribution of a Markov process describing the stochastic behavior of an open homogeneous exponential network with nodes of infinite capacity was first expressed in product form. The network studied in [76] is now referred to as the Jackson network in honor of its author. Subsequent weighty contributions to multiplicative network theory were stimulated by the publication of [26] formulating the so-called BCMP theorem (the abbreviation consists of the first letters of the names of its four authors) formulating the product solution for a large class of open networks that are the generalization of the Jackson network. Necessary conditions for a Jackson network to be expressed in product form are that all its input flows must be Poisson and distributions of their service times must be exponential. For BCMP networks, the second constraint may not always hold, but the service mechanisms (disciplines) at nodes must be of a special type. Subsequent developments in the theory of multiplicative networks resulted from different types of generalizations of the Jackson and BCMP networks concerned with, for example, the dependence of input flows on the number of customers in the network, dependence of probabilities of transitions between network nodes on the state of these nodes, constraints on the number of customers in the network, bypassing of nodes, etc [10, 80, 82, 83]. The theory of Jackson and BCMP networks and their generalizations are described in sufficient detail in many Russian and foreign reviews and monographs [1, 2, 5, 12, 42, 63, 80].

An absolutely new class of open networks generalizing the Jackson and BCMP networks and admitting product solution was introduced by E. Gelenbe [49, 50, 53] (see also [63]). These are

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networks containing, along with ordinary (positive) customers, additional Poisson flows of negative customers and/or triggers. A negative customer differs from an ordinary (positive) customer in that upon arrival at a network node it kills a positive customer if any at this node, thereby reducing the number of positive customers at the node by one. Thereafter the negative customer quits the network, receiving no service. A trigger, unlike a negative customer, does not kill a positive customer, but instantaneously displaces him with a given probability from the present node to some other node. Such a network is called the G-network (here G denotes the first letter in the name, Gelenbe, of the author [49, 50, 53]).

As has been already mentioned, G-networks are multiplicative and the joint stationary distribution of the number of customers at the network nodes is representable in product form. But to find this distribution, we must solve a system of nonlinear algebraic equations for the intensities of customer flows in the network—the main difference between a G-network and Jackson and BCMP networks for which the system of equations for the intensities of flows in the network is linear.

In later works (e.g., [64]), Gelenbe investigated a more general case, namely, he introduced an additional flow of signals that with given probabilities may be either negative customers or triggers.

The study of G-networks has been started only recently; the first papers on this topic appeared in 1989 [49, 50] and are, in general, not related to analytical modeling of computing networks.

G-networks owe their appearance to analytical modeling of biophysical neural networks [49, 50], which are characterized by pulse-like signals. These signals are generated after random time intervals and their motion largely resemble the circulation of customers in a queueing network. Moreover, a signal excitation at the adjacent neuron (network node) increases the neuron potential and is interpreted as a positive customer, whereas signal suppression decreases the neutron potential by one unit and can be regarded as a negative customer that reduces the neuron potential by one unit.

But, as has been found later, G-networks can be used in various applications in modeling computer systems and networks (for example, flow control in computer networks, modeling the effect of viruses in networks, etc.), production systems and networks, in solving problems of pattern recognition, combinatorial optimization, etc. [21, 22, 47, 49–52, 55, 57–60, 61, 63, 65].

By way of illustration, let us examine a simple example [56] demonstrating the possibilities of application of G-networks to flow control in a computer network (note that the BCMP theorem cannot be applied to this example).

Let a computer network consist of an input queue (node I) and S subnetworks $N_1, \ldots, N_S$, which present alternative nonintersecting routes for packets. Packets are served in order of arrival at node I and then routed to a subnetwork N_i.

The packets at a subnetwork N_i, $i = \overline{1, S}$, are routed by two different mechanisms.

(I) Packets at the subnetworks $N_1, \ldots, N_S$ are routed with fixed probabilities $P_1, \ldots, P_S$. The optimal values of the probabilities P_i, $i = \overline{1, S}$, are chosen by the minimal delivery time criterion at the design stage of the computer network [6, 7].

(II) The second routing method is based on the information about free buffer memory at a subnetwork N_i. This information is transformed into a control packet, which is sent to the input queue I and ensures the immediate displacement of a packet from the queue I to a subnetwork N_i (this is an example of the signal-trigger effect used in G-networks).

Diverse applications of G-networks, in turn, stimulated their intensive studies reported in a large number of papers (see, for example, review [14]) and dissertations [29, 66, 86]. Only a few Russian publications [3, 9, 11, 28] are devoted to G-networks. Therefore, this review on G-networks and their applications may serve as an incentive to applied mathematicians to investigate new network models and provide practical specialists both in computer systems and networks and other application fields with new analytical modeling tools.

2. A NETWORK WITH NEGATIVE CUSTOMERS

Let us begin the study of G-networks with a simple case, in which the additional flow arriving at the network is just a flow of negative customers. We refer to such a network as the *base G-network*. Precisely the base G-network was first introduced and investigated by Gelenbe [49, 50, 53].

Let us consider an open queueing network with M single-server nodes of infinite buffer capacity. A flow of positive (ordinary) customers of intensity λ_{0i}^{+} and a Poisson flow of negative customers of intensity λ_{0i}^{-} arrive from outside (node 0) at node i. All input flows are assumed to be independent. The service times of positive customers at node i are distributed exponentially with parameter μ_i, $i=\overline{1,M}$. A negative customer upon arrival at a network node containing at least one positive customer instantaneously kills (annihilates, removes from the network) one positive customer (under the assumption that the service time of positive customers is exponentially distributed; which customer is killed is unimportant if we are interested only in queueing processes at nodes) and then quits the network without receiving any service at the node. Consequently, only positive customers exist at a server or wait in the queue at every network node. Therefore, in the sequel, in discussing the service of positive customers, we sometimes refer to them simply as customers for brevity.

A positive customer, upon completion of service at node i, is jockeyed with probability p_{ij}^{+} to node j as a positive customer, or with probability p_{ij}^{-} as a negative customer, or quits the network with probability $p_{i0}=1-\sum\limits_{j=1}^{M}(p_{ij}^{+}+p_{ij}^{-})$ to outside (node 0).

The stochastic behavior of our queueing network is described by a homogeneous Markov process $\{X(t),\ t\ge 0\}$ over the state set

$$\mathcal{X}=\{(k_1,k_2,\ldots,k_M),\ k_i\ge 0,\ i=\overline{1,M}\}. \tag{2.1}$$

The state $(k_1,k_2,\ldots,k_M)$ denotes that at some instant there are k_1 (positive) customers at node 1, k_2 customers at node 2, …, and k_M customers at node M.

Let us introduce a vector $\mathbf{k}=(k_1,k_2,\ldots,k_M)$. Let us also take $\lambda_0^{+}=\sum\limits_{i=1}^{M}\lambda_{0i}^{+}$ and $\lambda_0^{-}=\sum\limits_{i=1}^{M}\lambda_{0i}^{-}$. Note that λ_0^{+} and λ_0^{-} are the intensities of the total Poisson flows of positive and negative customers arriving at the network from outside, respectively.

Finally, let us introduce two matrices P^{+} and P^{-} with elements p_{ij}^{+} and p_{ij}^{-}, $i,j=\overline{1,M}$, respectively. Furthermore, we take $P=P^{+}+P^{-}$. We assume that the matrix P is indecomposable.

We shall study the stationary operation mode of the network.

Let λ_i^{+} and λ_i^{-}, $i=\overline{1,M}$, denote the intensities of positive and negative customers in the network, respectively. The intensities λ_i^{+} and λ_i^{-} are defined by the system of nonlinear equations

$$\begin{aligned}\lambda_i^{+}&=\lambda_{0i}^{+}+\sum_{j=1}^{M}q_j\mu_j p_{ji}^{+},\\ \lambda_i^{-}&=\lambda_{0i}^{-}+\sum_{j=1}^{M}q_j\mu_j p_{ji}^{-},\quad i=\overline{1,M},\end{aligned} \tag{2.2}$$

where

$$q_i=\lambda_i^{+}/(\lambda_i^{-}+\mu_i). \tag{2.3}$$

Let $p(\mathbf{k})$ denote the stationary probability of the state $\mathbf{k}$.

Theorem 1. *For a G-network with negative customers, if there exists a unique positive solution to the system of Eqs.* (2.2), (2.3) *such that the condition*

$$q_i < 1, \quad i = \overline{1, M}, \tag{2.4}$$

is satisfied, then the stationary distribution $p(\mathbf{k})$ *of the Markov process* $\{X(t),\ t \geq 0\}$ *can be represented in product form as*

$$p(\mathbf{k}) = \prod_{i=1}^{M} p(k_i), \tag{2.5}$$

where

$$p(k_i) = (1 - q_i) q_i^{k_i}, \quad k_i \geq 0, \tag{2.6}$$

for all $i = \overline{1, M}$.

The proof of Theorem 1, like the proofs of all other theorems below, is carried out in several stages. First we find the system of equilibrium equations for the Markov process describing the G-network behavior. (For more complex G-network models, precisely this stage may prove tedious, though it is not complicated in essence). Then substituting formulas (2.5) and (2.6) into the system of equilibrium equations, we demonstrate that the substitution results in a system of identities. Finally, following, for example, the logic of the Foster theorem [44] and since the solution of the system of equations in the form (2.5), (2.6) under condition (2.4) is positive and bounded, we find that the solution thus obtained is the unique stationary distribution of the Markov process $\{X(t),\ t \geq 0\}$.

Obviously, q_i has a probabilistic meaning—the probability that node i is not empty, or, since the network contains only single-server nodes, the utilization coefficient for the server of node i.

What finally remains is to examine the existence of a solution to the system of nonlinear Eqs. (2.2), (2.3). We shall study this question in detail in Section 5 for more general G-networks.

3. A G-NETWORK WITH NEGATIVE CUSTOMERS AND BATCH REMOVAL OF POSITIVE CUSTOMERS

One of the subsequent generalizations of the G-network is the case in which a negative customer may kill a batch of positive customers, where the batch size is random and defined by some probability distribution. Such a model is studied in depth in [56].

First let us consider the base G-network. Using the service mechanism for positive customers and laws as before, according to which a positive customer upon completion of service at node i is jockeyed to node j with probability p_{ij}^{+} as a positive customer, or with probability p_{ij}^{-} as a negative customer or quits the network with probability p_{i0}, let us describe the behavior of negative customers arriving at network nodes.

When a negative customer arrives at a node i containing $k_i \geq B_i$ positive customers, where B_i is an integer random variable, the number of customers at the node decreases by B_i (B_i positive customers are instantaneously killed). If $k_i < B_i$, then the node i is completely emptied (i.e., all positive customers at the node i at this instant are instantaneously killed). The random variable B_i, which actually determines the maximal size of the annihilated batch of positive customers at node i, obeys an arbitrary discrete distribution law $\mathcal{P}\{B_i = m\} = \pi_{im},\ m \geq 1$.

The operation of a G-network with batch removal is also described by the homogeneous Markov process $\{X(t),\ t \geq 0\}$ with a state set of the type (2.1) and with the same physical interpretation for the states of the process.

The balance equations for flow intensities in the network stationary operation mode, from the viewpoint of formal expression, are the same as for the base G-network, i.e., of the form (2.2)

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} q_j \mu_j p_{ji}^+,$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} q_j \mu_j p_{ji}^-, \quad i = \overline{1, M}, \tag{3.1}$$

but here q_i is defined differently, namely,

$$q_i = \frac{\lambda_i^+}{\lambda_i^- f_i(q_i) + \mu_i}, \tag{3.2}$$

where

$$f_i(x) = \frac{1 - \sum\limits_{m=1}^{\infty} \pi_{im} x^m}{1 - x}.$$

Obviously, the system of balance Eqs. (3.1), (3.2) for the flow intensities in the network is also nonlinear.

Let us assume that the substochastic matrix $P^+ + P^-$ is indecomposable.

Theorem 2. *For a G-network with negative customers and batch removal of positive customers, if there exists a unique positive solution* $(\lambda_i^+, \lambda_i^-)$, $i = \overline{1, M}$, *to the system of Eqs.* (3.1), (3.2) *such that* $q_i < 1$, *then the stationary distribution* $p(\mathbf{k})$ *of the Markov process* $\{X(t),\ t \geq 0\}$ *is representable in product form as*

$$p(\mathbf{k}) = \prod_{i=1}^{M} p(k_i), \tag{3.3}$$

where

$$p(k_i) = (1 - q_i) q_i^{k_i}, \quad k_i \geq 0, \tag{3.4}$$

for all $i = \overline{1, M}$, *and* q_i *is defined by formula* (3.2).

Theorem 2 implies that q_i, as for the base G-network, is the stationary probability that the server of the node i is busy, i.e., q_i is the utilization coefficient of the server of the node i.

4. G-NETWORK WITH NEGATIVE CUSTOMERS AND TRIGGERS

As stated in the Introduction, the action of environment on the queueing process of positive customers may affect not only negative customers, which simply kill one or more positive customers at a node, but also triggers arriving from outside, whose action consists in instantaneously moving a positive customer from one node to some other node. G-networks with negative customers and triggers were initially studied in [64], then in [38, 39, 54], and in [73] under additional assumptions.

As in [63, 64], first let us study a network with M single-server nodes of infinite buffer capacity. A Poisson flow of positive (ordinary) customers of intensity λ_{0i}^+ and an additional flow of signals, which is also a Poisson flow of intensity λ_{0i}^-, arrive at a network node i. The service time of a positive customer at node i, as in the models investigated earlier, is exponentially distributed with parameter μ_i. Upon completion of service, a positive customer at node i is jockeyed to node j with probability p_{ij}^+ as a positive customer, or with probability p_{ij}^- as a signal, or quits the network with probability $p_{i0} = 1 - \sum_{j=1}^{M} (p_{ij}^+ + p_{ij}^-)$. Let $P = P^+ + P^-$, where, as before, the matrices P^+ and P^- consist of the probabilities p_{ij}^+ and p_{ij}^-, respectively. Thus, not only positive customers, but also signals circulate in the network, where the matrix P controls the movement of positive customers as well as signals in the network.

A signal arriving at an empty node (i.e., a node containing no positive customers) does not exert any influence on the network and instantaneously quits the network. In the contrary case, i.e., if node j is not empty, the following events may take place when a signal arrives at node j:

(1) The arriving signal instantaneously displaces a positive customer from node j to node s with probability q_{js} and is called the trigger. Let Q denote the matrix with elements q_{js}.

(2) Or the signal with probability $q_{j0} = 1 - \sum_{s=1}^{M} q_{js}$ acts like a negative customer and kills a batch of positive customers at the node j (the batch removal procedure is described in Section 3). The size B_j of the killed batch is a random variable with probability distribution $\pi_{jm},\ m \geq 1$.

In certain papers (e.g., [74]), the concepts of a signal and a trigger are identically used. Clearly, this is a merger of terms. But this fact must be kept in mind in reading papers on G-networks.

We assume that the matrices P and Q are indecomposable; this is essential for formulating the main result. Physically, it implies that an arriving positive customer upon completion of service (i.e., the positive customer that was not killed by a negative customer) necessarily quits the network.

The operation of a G-network with negative customers, triggers, and batch removal of positive customers can also be described by a homogeneous Markov process $\{X(t),\ t \geq 0\}$ with state set (2.1). Here too, the states of the processes are physically interpreted as in Section 2.

As usual, let us examine the stationary operation mode of our queueing network.

Let λ_i^+ and λ_i^-, $i = \overline{1, M}$, be the intensities of flows of positive customers and signals in the network, respectively. Then the balance equations for the intensities λ_i^+ and λ_i^- are

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} q_j \mu_j \left[p_{ji}^+ + \sum_{s=1}^{M} p_{js}^- q_s q_{si} \right] + \sum_{j=1}^{M} \lambda_{0j}^- q_j q_{ji},$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} q_j \mu_j p_{ji}^-, \quad i = \overline{1, M}, \tag{4.1}$$

where q_i is defined by

$$q_i = \frac{\lambda_i^+}{\lambda_i^- (1 - q_{i0}) + \lambda_i^- q_{i0} f_i(q_i) + \mu_i}, \tag{4.2}$$

and

$$f_i(q_i) = \frac{1 - \sum_{m=1}^{\infty} \pi_{im} q_i^m}{1 - q_i}.$$

Theorem 3. *For a G-network with negative customers, triggers, and batch removal of positive customers, if there exists a unique positive solution* $(\lambda_i^+, \lambda_i^-)$, $i = \overline{1, M}$, *to the system of Eqs.* (4.1), (4.2) *such that* $q_i < 1$, *then the stationary distribution* $p(\mathbf{k})$ *of the process* $\{X(t),\ t \geq 0\}$

can be represented in product form as

$$p(\mathbf{k}) = \prod_{i=1}^{M} p(k_i), \tag{4.3}$$

where

$$p(k_i) = (1 - q_i) q_i^{k_i}, \quad k_i \geq 0, \tag{4.4}$$

for all $i = \overline{1, M}$ *and* q_i *is defined by formula* (4.2).

As for the G-network models considered earlier, from Theorem 3 we find that q_i is the stationary probability that node i is not empty, or, which is the same thing, q_i is the utilization coefficient of the server of the node i.

5. SOLUTION OF THE BALANCE EQUATIONS FOR FLOW INTENSITIES AND STABILITY OF G-NETWORKS

The systems of balance equations for the flow intensities for all G-networks studied thus far are nonlinear. Therefore, the key to determining the multidimensional stationary distribution of the number of customers at network nodes (according to Theorems 1–3) is the study of the existence of a unique positive solution for the system of Eqs. (2.2), (2.3); (3.1), (3.2) and (4.1), (4.2). This is not a trivial problem, and we now examine it in detail.

In [53], the existence and uniqueness of the unique positive solution of Eqs. (2.2), (2.3) for flow intensities is demonstrated for the base G-network with a special type of transition probability matrix. In particular, a sufficient condition for the existence and uniqueness of a unique positive solution of Eqs. (2.2), (2.3) is shown to be the condition

$$\mu_i + \lambda_{0i}^- > \lambda_i^+ + \sum_{j=1}^{M} \mu_j p_{ji}^+, \quad i = \overline{1, M}. \tag{5.1}$$

A network for which condition (5.1) holds is said to be hyperstable.

An approach to solving a nonlinear system of balance equations for flow intensities of the type (2.2), (2.3) for the general base G-network is developed in [64]. It is extended in [56] to G-networks with signals and batch removal of positive customers. It is somewhat more general. Here we briefly outline it, using the results and notation of Section 4.

The method developed in [56] for solving a nonlinear system of balance equations for flow intensities is based on the Brouwer's fixed-point theorem.

Theorem 4. *If* $P^+ + Q$ *is a semistochastic indecomposable matrix, then the solution* $(\lambda_i^+, \lambda_i^-)$, $i = \overline{1, M}$, *of the nonlinear system of Eqs.* (4.1) *always exists.*

Let us schematically outline the proof of this theorem. Strictly speaking, this scheme lies at the base of the algorithm for solving the system of Eqs. (4.1).

First let us rewrite Eqs. (4.1) in a slightly different form as

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} \lambda_j^+ g_j p_{ji}^+ + \sum_{j=1}^{M} \lambda_j^+ h_j q_{ji},$$
$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} \lambda_j^+ g_j p_{ji}^-, \quad i = \overline{1, M}, \tag{5.2}$$

where

$$g_i = \frac{\mu_i}{\lambda_i^- q_{i0} f_i(q_i) + \mu_i}, \quad h_i = \frac{\lambda_i^- q_{i0} f_i(q_i)}{\lambda_i^- q_{i0} f_i(q_i) + \mu_i}.$$

Introducing four vectors $\boldsymbol{\lambda}^+$, $\boldsymbol{\lambda}^-$, $\boldsymbol{\lambda}_0^+$, and $\boldsymbol{\lambda}_0^-$ with coordinates λ_i^+, λ_i^-, λ_{0i}^+, and λ_{0i}^-, respectively, and a diagonal matrix G with diagonal elements g_i, let us rewrite (5.1) as

$$\boldsymbol{\lambda}^{\mathrm{T}+}(I - [GP^+ + (I-G)Q]) = \boldsymbol{\lambda}_0^{\mathrm{T}+},$$
$$\boldsymbol{\lambda}^{\mathrm{T}-} = \boldsymbol{\lambda}^{\mathrm{T}+} GP^- + \boldsymbol{\lambda}_0^{\mathrm{T}-}.$$

Hence we obtain

$$\boldsymbol{\lambda}^{\mathrm{T}+} = \boldsymbol{\lambda}_0^{\mathrm{T}+} \sum_{n=0}^{\infty} [GP^+ + (I-G)Q]^n,$$
$$\boldsymbol{\lambda}^{\mathrm{T}-} - \boldsymbol{\lambda}_0^{\mathrm{T}-} = \boldsymbol{\lambda}_0^{\mathrm{T}+} \sum_{n=0}^{\infty} [GP^+ + (I-G)Q]^n GP^-.$$

Let us introduce a vector $\mathbf{y} = \boldsymbol{\lambda}^- - \boldsymbol{\lambda}_0^-$ and a vector function

$$F(\mathbf{y}) = \boldsymbol{\lambda}_0^{\mathrm{T}+} \sum_{n=0}^{\infty} [GP^+ + (I-G)Q]^n GP^-.$$

Note that the function $F(\mathbf{y})$ is expressed through the matrix G, which depends on $\boldsymbol{\lambda}^-$. By the Brouwer's theorem, the equation $\mathbf{y} = F(\mathbf{y})$ has a fixed point $\mathbf{y}^*$. Precisely this fixed point is the solution of the system of Eqs. (5.2)

$$\boldsymbol{\lambda}^-(\mathbf{y}^*) = \boldsymbol{\lambda}_0^- + \mathbf{y}^*,$$
$$\boldsymbol{\lambda}^{\mathrm{T}+}(\mathbf{y}^*) = \boldsymbol{\lambda}_0^{\mathrm{T}+} \sum_{n=0}^{\infty} (F(\mathbf{y}^*)P^+)^n.$$

Computational aspects of solving nonlinear systems of balance equations for flow intensities for certain G-network models are discussed in [45]. For the base G-network, an iterative algorithm for computing the probabilities q_i and concurrent verification of the network stability is developed. Every iteration is shown to be of second-order complexity.

6. A G-NETWORK WITH RANDOM SIGNAL ACTIVATION TIME

In the previous sections, we have studied G-networks with signals (which could be negative customers or triggers) whose action is manifested instantaneously, i.e., the activation time of every signal is zero, and was, therefore, disregarded in the design of the G-network model.

In this section, we assume that a signal arriving at the network is activated not instantaneously, but only after a random time. A similar network with single-server nodes (under general assumptions on service times for positive customers) is investigated in [34] through the quasi-reversibility concept [90]. But the formulas derived in [34] for the product solution are erroneous. Below we state the results of [3, 28] for a G-network with random signal delay for single-server nodes with exponentially distributed service times for positive customers and Markov service for signals. These results show that the product solution is of a different form than that of [34]. In [3, 28], it is also shown that the product solution also holds for the general Markov service for positive customers at nodes, but only in a symmetric network.

Thus, let us once again consider an open queueing network with M nodes of infinite buffer capacity. A Poisson flow of positive customers of intensity λ_{0i}^+ and a Poisson flow of signals of intensity λ_{0i}^- arrive from outside at node i. All customer and signal flows are independent.

The probability that a positive customer is served at node i in time $(t, t+\Delta)$ is $\mu_i^+(k)\Delta + o(\Delta)$ if there are k customers at instant t at the node. A positive customer upon completion of service at node i is jockeyed with probability p_{ij}^+ to node j as a positive customer, or with probability p_{ij}^- as a signal, or quits the network (is jockeyed to node 0) with probability $p_{i0} = 1 - \sum\limits_{j=1}^{M}(p_{ij}^+ + p_{ij}^-)$.

Every arriving signal is activated after a certain random time interval. Furthermore, the probability that a signal arriving at node i is activated in time $(t, t+\Delta)$ is $\mu_i^-(n)\Delta + o(\Delta)$, provided there are n unactivated signals at this node at instant t. Upon expiry of the activation time,

either a signal acts with probability q_{ij}^+ as a trigger and displaces one positive customer from node i to node j such that the positive customer remains positive,

or the signal acts with probability q_{ij}^- once again as a trigger and displaces one positive customer from node i to node j such that the positive customer is transformed into a signal at node j,

or the signal acts with probability $q_{i0} = 1 - \sum\limits_{j=1}^{M}(q_{ij}^+ + q_{ij}^-)$ as a negative customer, which, after killing a positive customer at node i, quits the network.

The customer that is displaced from node i to node j (as a positive customer or signal) is not served at node i.

If, upon activation of a signal, there are no positive customers at the node, then the signal quits the network without exerting any influence on the operation of the network as a whole.

Let us introduce four matrices P^+, P^-, Q^+, and Q^- with elements p_{ij}^+, p_{ij}^-, q_{ij}^+, and q_{ij}^-, $i, j = \overline{1, M}$, respectively. Furthermore, let $P = P^+ + P^-$ and $Q = Q^+ + Q^-$. We assume that the matrices P and Q are indecomposable.

The stochastic behavior of our G-network is described by the homogeneous Markov process $\{X(t),\ t \geq 0\}$ over the state set

$$\mathcal{X} = \left\{((k_1, n_1), (k_2, n_2), \ldots, (k_M, n_M)),\ k_i \geq 0,\ n_i \geq 0,\ i = \overline{1, M}\right\}. \tag{6.1}$$

The state $((k_1, n_1), (k_2, n_2), \ldots, (k_M, n_M))$ denotes that at a certain instant there are k_1 positive customers and n_1 (unactivated) signals at node 1, k_2 customers and n_2 signals at node 2, …, and k_M customers and n_M signals at node M.

Let us introduce two vectors $\mathbf{k} = (k_1, k_2, \ldots, k_M)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_M)$, and take $(\mathbf{k}, \mathbf{n}) = ((k_1, n_1), (k_2, n_2), \ldots, (k_M, n_M))$.

Let $\lambda_0^+ = \sum\limits_{i=1}^{M} \lambda_{0i}^+$ and $\lambda_0^- = \sum\limits_{i=1}^{M} \lambda_{0i}^-$; note that λ_0^+ and λ_0^- are the intensities of the total Poisson flows of positive customers and signals arriving at the network from outside, respectively.

As before, let us study the stationary operation mode of the queueing network. Let $p(\mathbf{k}, \mathbf{n})$ denote the stationary probability of the state $(\mathbf{k}, \mathbf{n})$.

6.1. Single-Server Nodes

We assume that the service times of customers at node i are exponentially distributed with parameter μ_i^+. Then

$$\mu_i^+(k_i) = u(k_i)\mu_i^+, \quad i = \overline{1, M}. \tag{6.2}$$

The intensities λ_i^+ and λ_i^-, $i = \overline{1, M}$, of flows of positive customers and signals in the network are determined from the following system of nonlinear equations:

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} q_j \left(\mu_j^+ p_{ji}^+ + \lambda_j^- q_{ji}^+\right),$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} q_j \left(\mu_j^+ p_{ji}^- + \lambda_j^- q_{ji}^-\right), \quad i = \overline{1, M}, \tag{6.3}$$

where $q_i = \lambda_i^+ / \left(\lambda_i^- + \mu_i^-\right)$.

Let us also take

$$\rho_i^-(j) = \lambda_i^- / \mu_i^-(j),$$

$$\Lambda_0 = \sum_{i=1}^{M} q_i \mu_i^+ p_{i0} + \sum_{i=1}^{M} q_i \lambda_i^- q_{i0}.$$

Note that the equality

$$\Lambda_0 + \sum_{i=1}^{M} \lambda_i^- = \lambda_0^+ + \lambda_0^- \tag{6.4}$$

holds and can be used to check computations in solving the system of Eqs. (6.3).

Theorem 5. *For a G-network with random signal delay, if equality* (6.2) *holds and the conditions*

$$\lambda_i^+ < \lambda_i^- + \mu_i^+, \quad G_i = \sum_{n_i=0}^{\infty} \prod_{j=1}^{n_i} \rho_i^-(j) < \infty, \quad i = \overline{1, M}, \tag{6.5}$$

are satisfied, then the stationary distribution $p(\mathbf{k}, \mathbf{n})$ *of the Markov process* $\{X(t),\ t \geq 0\}$ *is representable in product form as*

$$p(\mathbf{k}, \mathbf{n}) = \prod_{i=1}^{M} p(k_i, n_i), \quad i = \overline{1, M}, \tag{6.6}$$

where

$$p(k_i, n_i) = (1 - q_i) q_i^{k_i} G_i^{-1} \prod_{j=1}^{n_i} \rho_i^-(j), \quad k_i, n_i \geq 0, \quad i = \overline{1, M}, \quad \text{and} \quad \prod_{j=1}^{0} \equiv 1. \tag{6.7}$$

Note that here q_i is the stationary probability that node i contains at least one positive customer.

6.2. Symmetric G-Networks

Let us consider the G-network described above for which

$$p_{ij}^+ = q_{ij}^+, \quad p_{ij}^- = q_{ij}^-, \quad p_{i0} = q_{i0}, \quad i, j = \overline{1, M}. \tag{6.8}$$

Such a G-network is said to be *symmetric*.

The intensities of flows of positive customers and signals λ_i^+ and λ_i^-, $i, j = \overline{1, M}$, are defined by the system of equations

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} \lambda_j^+ p_{ji}^+,$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} \lambda_j^+ p_{ji}^-, \quad i, j = \overline{1, M}, \tag{6.9}$$

which, unlike for the G-networks considered earlier, is linear. Moreover, we take

$$\Lambda_0 = \sum_{i=1}^{M} \lambda_i^+ p_{i0}.$$

The equality

$$\Lambda_0 + \sum_{i=1}^{M} \lambda_i^- = \lambda_0^+ + \lambda_0^-, \tag{6.10}$$

holds and is identical to relation (6.3) derived for single-server nodes, but the intensities of flows of customers and signals are defined by different systems of equations.

In what follows, we take $q_i(j) = \lambda_i^+/(\lambda_i^- + \mu^+(j))$.

Theorem 6. *For a G-network with signal delay, if relation* (6.8) *holds and the conditions*

$$F_i = \sum_{k_i=0}^{\infty} \prod_{j=1}^{k_i} q_i(j) < \infty, \quad G_i = \sum_{n_i=0}^{\infty} \prod_{j=1}^{n_i} \rho_i^-(j) < \infty, \quad i = \overline{1, M}, \tag{6.11}$$

are satisfied, then the stationary distribution $p(\mathbf{k}, \mathbf{n})$ of the Markov process $\{X(t),\ t \geq 0\}$ is representable in product form as

$$p(\mathbf{k}, \mathbf{n}) = \prod_{i=1}^{M} p(k_i, n_i), \tag{6.12}$$

where

$$p(k_i, n_i) = F_i^{-1} G_i^{-1} \prod_{j=1}^{k_i} q_i(j) \prod_{l=1}^{n_i} \rho_i^-(l), \quad k_i, n_i \geq 0, \quad i = \overline{1, M}. \tag{6.13}$$

6.3. Discussion of the Results

In G-networks described in Sections 6.1 and 6.2, the service is partially (only for signals) or completely (both for positive customers and signals) is assumed to be Markovian under which the service intensities $\mu_i^+(k_i)$ at node i for positive customers and $\mu_i^-(n_i)$ for signals depend on the number k_i of customers and n_i signals at node i, respectively. This assumption is rather general and is sufficient in the sense that, defining the intensities $\mu_i^+(k_i)$ and $\mu_i^-(n_i)$ by different expressions, we can determine different service mechanisms as particular cases.

Service of signals. Let us study a few concrete signal service (activation) mechanisms.

Service of signals without waiting. Let a signal arriving at node i, $i = \overline{1, M}$, be activated after a random time having an exponential distribution with parameter μ_i^-, irrespective of other (unactivated) signals at the node.

In reality, such an activation mechanism for signals at node i is equivalent to service of signals on an infinite number of identical servers, where every server functions independently of others, and the service time at any server is exponentially distributed with parameter μ_i^-, $i = \overline{1, M}$.

Service of signals with waiting. An extension of this signal activation mechanism is the service of signals at node i with waiting at m_i^- identical servers with a common buffer of infinite capacity. Assuming that the service time at any server is exponentially distributed with parameter μ_i^-, we obtain $\mu_i^-(n_i) = \mu_i^- \min(n_i, m_i^-)$, $i = \overline{1, M}$. Such a service mechanism presupposes that signals may wait for service at the buffer.

Service of impatient signals. Let us assume that the maximal sojourn time at node i for a signal (only for servers while serving without waiting in the queue or for servers and queue while serving with waiting) is bounded by a random variable having an exponential distribution with parameter γ_i^-, $i = \overline{1, M}$. A signal, upon expiry of this duration, instantaneously quits the node i and its further behavior obeys the same laws that govern signals that receive service. Such a service mechanism for signals is called the service of "impatient" customers. Then, for service without waiting we obtain $\mu_i^-(n_i) = (\mu_i^- + \gamma_i^-)n_i$, and for service with waiting, $\mu_i^-(n_i) = \mu_i^- \min(n_i, m_i^-) + \gamma_i^- n_i$, $i = \overline{1, M}$.

Service of positive customers. This problem under Markov service for positive customers has been solved only for a symmetric network. In analogy with what has been said about signal service, we examine three cases.

Service of positive customers without waiting. Let us assume that node i has an infinite number of identical servers for serving positive customers, and the service at every server is exponentially distributed with parameter μ_i^+, $i = \overline{1, M}$. In this case, $\mu_i^+(k_i) = \mu_i^+ k_i$, $i = \overline{1, M}$.

Service with waiting for positive customers. If the number of servers at node i for serving positive customers is finite and equal to m_i^+ and the service time at every server is exponentially distributed with parameter μ_i^+, then $\mu_i^+(k_i) = \mu_i^+ \min(k_i, m_i^+)$, $i = \overline{1, M}$.

Service of impatient positive customers. Let the maximal sojourn time at node i for a positive customer be bounded by a random variable, which is exponentially distributed with parameter γ_i^+. Then a customer (at a server or in the queue) for which this duration has expired quits the node i, and the further route of this impatient customer is determined by the same scheme as for a positive customer that has been successfully served at node i. In this case, $\mu_i^+(k_i) = \mu_i^+ \min(k_i, m_i^+) + \gamma_i^+ k_i$, $m_i^+ \leq \infty$, $i = \overline{1, M}$.

7. G-NETWORKS WITH SEVERAL CLASSES OF POSITIVE CUSTOMERS AND SIGNALS

Generalizations of G-networks to the case of several classes of positive and negative customers and signals are described in many papers [34, 35, 38, 39, 46, 48, 57, 62, 64, 74, 84].

In [48, 57, 64], the base G-network is extended to the case of several classes of positive customers under the assumption that the number of classes of both types of customers is identical. Each of these papers describes different variants for the effect of negative customers with their types. In [64], negative customers of a fixed class are assumed to affect only positive customers of the same class. In [57], the positive customer is chosen at random, i.e., if a negative customer arrives at a node i containing $k_i > 0$ positive customers (with no regard for their type), then a positive customer of the class c is killed with probability k_{ci}/k_i. In [48], a G-network with different disciplines (FIFO (first-in-first-out), PS (processor-sharing), and LIFO/PR (last-in-first-out with preemptive resumption)) is investigated. A positive customer is chosen "for slaughter" according to the service mechanism defined for the node. Moreover, a negative customer of the class m at node i may kill a positive customer of the class k with probability K_{imk}. In [62], the results of [48] are extended to the case in which there are also several classes of triggers.

We now briefly state the main results of [62], in which a somewhat simplified analog of the BCMP theorem is demonstrated for G-networks with FIFO, PS, and LIFO/PR service mechanisms at single-server nodes (types of nodes are expressed in the terminology of the BCMP theorem).

For the sake of brevity of presentation, we shall not state the complete description of the G-network and use the material of previous sections wherever possible.

Let us consider a queueing network consisting of M single-server nodes of infinite buffer capacity. R Poisson flows of positive customers of intensity λ^+_{0ik}, $i = \overline{1,M}$, $k = \overline{1,R}$, and S Poisson flows of signals of intensity λ^-_{0im}, $i = \overline{1,M}$, $m = \overline{1,S}$ arrive at the network from outside. All flows arriving at the network are independent.

A positive customer upon completion of service at a network node may change his class and node type, or may be converted into a signal, or quit the network.

A signal arriving at a nonempty node chooses a positive customer at the node as a "target" (according to the service discipline at the node). A signal arriving at an empty node quits the network without exerting any action on the network.

A signal of the class m, after choosing a positive customer of the class k, is activated with probability K_{imk} as a trigger and displaces the positive customer to node i, and such a displacement does not take place with additional probability $1 - K_{imk}$. After the attempts to displace the target customer, the signal vanishes.

A positive customer of the class k, upon completion of service at node i, is jockeyed with probability $p^+_{ij,kl}$ to node k as a positive customer of the class l, or with probability $p^-_{ij,km}$ as a signal of the class m, or quits the network with probability

$$p_{ik0} = 1 - \sum_{j=1}^{M}\sum_{l=1}^{R} p^+_{ij,kl} - \sum_{j=1}^{M}\sum_{m=1}^{S} p^-_{ij,kl}, \quad i = \overline{1,M}, \quad k = \overline{1,R}. \tag{7.1}$$

The service times of a positive customer of the class k at node i of any type are exponentially distributed with parameter μ_{ik}.

We assume that the G-network satisfies the following properties.

Property 1. Nodes of type 1 (under the FIFO discipline) satisfy the condition

$$\mu_{ik} + \sum_{m=1}^{S} K_{imk}\lambda^-_{0im} = c_i, \quad k = \overline{1,R}. \tag{7.2}$$

Property 2. Node i of type 1 and a signal of the class m are such that

$$\sum_{j=1}^{M}\sum_{l=1}^{R} p^-_{ji,lm} > 0,$$

and the condition

$$K_{ima} = K_{imb}, \quad i = \overline{1,M}, \quad m = \overline{1,R}, \tag{7.3}$$

is satisfied for any $a, b = \overline{1,R}$. This condition implies that a trigger signal of the class m in its attempt to displace a positive customer from some node does not "know" the type of the customer, i.e., the signal does not distinguish positive customers by their type.

Property 3. For a node of type 2, the probability that a positive customer is chosen as a target for slaughter by the negative customer arriving at the node is $1/c$, provided there are c customers (without regard for their type) at the node.

Note that conditions (7.2) and (7.3) for a node of type 1 can be replaced by a more restrictive condition of the type

$$\mu_{ia} = \mu_{ib}, \quad K_{ima} = K_{imb}, \quad i = \overline{1, M},$$
$$m = \overline{1, S}, \quad a, b = \overline{1, R}.$$

The stochastic behavior of the G-network with several classes of positive customers and signals is described by the homogeneous Markov process $\{\mathbf{X}(t) = (\mathbf{X}_1(t), \ldots, \mathbf{X}_M(t)),\ t \geq 0\}$, where the component $\mathbf{X}_i(t)$ describes the state of the node i at instant t. The state set of the process $\{\mathbf{X}(t),\ t \geq 0\}$ is of the form

$$\mathcal{X} = \{(\mathbf{x}_1, \ldots, \mathbf{x}_M):$$
$$\mathbf{x}_i = (x_{i,j_1}, \ldots, x_{i,j_{|x_i|}}), \quad j_i = \overline{1, R}, \quad i = \overline{1, M}, \text{ for nodes of type 1, 4;}$$
$$\mathbf{x}_i = (x_{i1}, \ldots, x_{iR}), \quad x_{ik} \geq 0, \quad i = \overline{1, M}, \quad k = \overline{1, R}\}, \text{ for nodes of type 2.}$$

Here $\mathbf{x}_i$ is the state of the node i at some instant t: for nodes of types 2 and 4, the component $x_{i,j}$ of the vector $\mathbf{x}_i = (x_{i,1}, \ldots, k_{i,|x_i|})$ shows the class of the customer waiting at the jth place in queue at node i (the order in the queue is determined by the FIFO and LIFO disciplines, respectively), and $|x_i|$ is the number of customers at node i without regard for their class. For a node of type 2, we have $\mathbf{x}_i = (x_{i1}, \ldots, x_{iR})$ and its component x_{ik} determines the number of customers at node i without regard for their classes.

Let $p(\mathbf{k})$ denote the stationary probability of the state $\mathbf{k}$.

Theorem 7. *Let properties* 1–3 *hold for a G-network with several classes of positive customers and signals. Then, if the system of nonlinear equations*

$$q_{ik} = \frac{\lambda^+_{0ik} + \lambda^+_{ik}}{\mu_{ik} + \sum\limits_{m=1}^{S} K_{imk}(\lambda^-_{0ik} + \lambda^-_{ik})},$$

$$\lambda^+_{ik} = \sum_{j=1}^{M}\sum_{l=1}^{R} p^+_{ji,lk}\mu_{jl}q_{jl} + \sum_{j=1}^{M}\sum_{l=1}^{R}\sum_{h=1}^{M}\sum_{m=1}^{R}\sum_{s=1}^{M} \mu_{jl}q_{jl}p^-_{jh,lm}K_{hms}q_{hs}p^+_{hi,sk},$$

$$\lambda^-_{ik} = \sum_{j=1}^{M}\sum_{l=1}^{R} p^-_{ji,lm}\mu_{jl}q_{jl}, \quad i = \overline{1, M}, \quad k = \overline{1, R}, \tag{7.4}$$

has a solution such that $q_{ik} > 0$ *for every pair* i, k *and* $\sum\limits_{k=1}^{R} q_{ik} < 1$ *for every node* i, *then the stationary distribution* $p(\mathbf{k})$ *of the Markov process* $\{\mathbf{X}(t),\ t \geq 0\}$ *is expressed in product form as*

$$p(\mathbf{k}) = G\prod_{i=1}^{M} g(\mathbf{x}_i). \tag{7.5}$$

Moreover, $g(\mathbf{x}_i)$ *depends on the node type and takes the form*

$$g(\mathbf{x}_i) = \prod_{h=1}^{|x_i|} q_{i,x_{ih}} \quad \text{for nodes of type 1,} \tag{7.6}$$

$$g(\mathbf{x}_i) = |x_i|!\prod_{k=1}^{R} \frac{(q_{i,k})^{x_{ik}}}{x_{ik}!} \quad \text{for nodes of type 2,} \tag{7.7}$$

$$g(\mathbf{x}_i) = \prod_{h=1}^{|x_i|} q_{i,x_{ih}} \quad \text{for nodes of type 4,} \tag{7.8}$$

and G *is the normalization constant.*

Note that the conditions $q_{ik} > 0$ and $\sum_{k=1}^{R} q_{ik} < 1$ guarantee the existence of a stationary operation mode for the network.

The proof of Theorem 8 is based on the logic described in Section 2, but the proof stages for Theorem 8 are rather cumbersome.

8. OTHER MODELS AND METHODS OF ANALYSIS OF G-NETWORKS

The main G-network model and its generalization due to the introduction of batch removal of positive customers and triggers for displacing positive customers to other nodes, including several classes of positive customers and signals, are investigated largely by E. Gelenbe or in his joint papers with coauthors. These models and works are the main models and works concerned with G-networks. But in the literature there are works of other authors devoted to different modifications and refinements of the main G-network models and development of new methods of analysis of G-networks. Below we outline these works, using the notation introduced in the previous sections.

A base G-network in which a negative customer upon arrival at an empty node remains at the node is investigated in [32]. Therefore, the queue length at the node may become negative. A sufficient condition for the existence of an invariant measure p of the Markov process describing the network, as shown in [32], is the existence of a positive solution to the balance equations

$$q_i\mu_i + \sum_{j=1}^{M} q_i q_j \mu_j p_{ji}^- + q_i\lambda_{0i}^- = \sum_{j=1}^{M} q_j\mu_j p_{ji}^+ + \lambda_{0i}^+, \quad i = \overline{1, M}, \tag{8.1}$$

for the unknowns q_i, $i = \overline{1, M}$. If this condition is satisfied, the invariant measure p can be represented in product form as

$$p(\mathbf{k}) = \prod_{i=1}^{M} q_i^{k_i}, \quad \mathbf{k} \in \mathbf{Z}^M. \tag{8.2}$$

By the assumption that the queue length may be negative, the states $\mathbf{k}$ of the set $\mathbf{Z}^M$ may also take any negative integral values. Therefore, the invariant measure (8.2) cannot be normalized. This problem is solved in [32] by introducing lower and upper bounds for the queue length at every node and modifying the transition probabilities such that these bounds are not violated.

In [73, 74], admitting that the queue length may take negative values for the base G-network, the service intensities and transition probabilities are assumed to depend on the network states, whereas batch arrival of negative customers is assumed in [75].

In many papers, as has been already mentioned, the concept of a signal is identified with the concept of a trigger in the sense as introduced by Gelenbe in [54, 56]. For example, in the G-network studied in [74], the so-called effective signals of type 0 play the role of a trigger. The concepts of a signal and a trigger are fused together to some extent in [34, 35, 38–41]. In these papers, signals are assumed to circulate in a network.

In [34], signal delay (service, activation in the sense of Section 6) at a node for a random time is introduced for G-networks with several classes of positive customers and trigger signals. Positive customers and signals are routed along the network according to the description given in Section 6 (of course, with due regard for several classes of positive customers). The service time for every type of positive customers is distributed either exponentially or according to an arbitrary law, but only for special symmetric service disciplines (e.g., processor sharing and LCFS/PR disciplines). The network is investigated using the two-stage approach of [38] for the base G-network. In the first stage, an isolated node is shown to satisfy the quasi-reversibility condition (for an isolated node in equilibrium mode, future arrivals of positive customers and signals, the current network state and the past processes of departures of positive customers and signals from a node must be independent). In the second stage, the approach of [90] is applied to extend the results obtained for isolated nodes to the network as a whole. For the stationary distribution of probabilities of the network states that coincide with the states of the set $\mathcal{X}$ (6.1), a product form is derived, but, as noted in Section 6, formula (3) in [34] for marginal distributions of the states of network nodes is not correct. (Probably, the error is unimportant and can be corrected).

In [40], the results obtained in [34] for a G-network with instantaneous signal activation are extended to networks in which the transition probabilities for positive customers and signals depend on the network prehistory. The network is studied by the approach of [34, 38]. Two cases are examined. In case I, for a customer of a given type that quits the node upon completion of service, his transition probability is assumed to depend on the time spent on his service due to service interruption as consequence of arrival of a signal at the node. In case II, the transition probability of a positive customer that has been served at a node or is displaced to another node by a newly arrived signal depends on the number of service interruptions by signals. In [40], the stationary joint distribution of the number of customers at network nodes is expressed in product form.

In [11], an approach based on the quasi-reversibility concept [80] is applied to study the base G-network with bypasses for nodes. In [11], a positive customer jockeyed to node i with probability $f_i(k_i)$, where k_i is the number of positive customers at node i, joins the queue at this node, and with additional probability $1 - f_i(k_i)$ is assumed to have been served at this node and leave the node. Below we state the main result of [11].

The operation of a G-network with bypasses is described by a homogeneous Markov process $\{\mathbf{X}(t) = (X_1(t), \dots, X_M(t)),\ t \geq 0\}$ over the state set $\mathcal{X} = \mathcal{X}_1, \times \mathcal{X}_2 \times \dots \times \mathcal{X}_M$, where $\mathcal{X}_i = \{0, 1, 2, \dots\}$ if $f_i(k_i) > 0$, $k_i = 0, 1, \dots$, and $\mathcal{X}_i = \{0, 1, 2, \dots n_i\}$ if $f_i(k_i) > 0$, $k_i = \overline{0, n_i - 1}$, and $f_i(n_i) = 0$ for some $n_i \geq 1$, $i = \overline{1, M}$.

The equations for the intensities of flows of positive and negative customers in the network are the same as for the base G-network without bypasses, namely,

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} q_j \mu_j p_{ji}^+,$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} q_j \mu_j p_{ji}^-, \quad i = \overline{1, M}, \tag{8.3}$$

where

$$q_i = \lambda_i^+ / (\lambda_i^- + \mu_i). \tag{8.4}$$

Theorem 8. *If there exists a positive solution* $(\lambda_i^+, \lambda_i^-)$, $i = \overline{1, M}$, *to the system of Eqs.* (8.3), (8.4) *such that the condition*

$$\sum_{k_i \in \mathcal{X}_i} \prod_{n=1}^{k_i} q_i f_i(n - 1) < \infty, \quad i = \overline{1, M},$$

is satisfied, then the Markov process $\{\mathbf{X}(t) = (X_1(t), \dots, X_M(t)),\ t \geq 0\}$ *is ergodic and its stationary distribution* $p(\mathbf{k})$ *is representable in product form as*

$$p(\mathbf{k}) = \prod_{i=1}^{M} p_i(k_i), \quad \mathbf{k} \in \mathcal{X}, \tag{8.5}$$

where

$$p_i(k_i) = p_i(0) \prod_{n=1}^{k_i} q_i f_i(n-1), \quad k_i \geq 0, \quad i = \overline{1,M}, \tag{8.6}$$

q_i *is defined by formula* (8.4), *and*

$$p_i(0) = \left[\sum_{k_i \in \mathcal{X}_i} \prod_{n=1}^{k_i} q_i f_i(n-1) \right]^{-1}, \quad i = \overline{1,M}. \tag{8.7}$$

The G-network investigated in [9] is a combination of two nonintersecting subnetworks: a base G-network and a network with bypasses. Two cases are distinguished for the network with bypasses: exponential time of service at nodes under the FCFS discipline or general service time distribution under the FCFS/PR discipline. The stationary probability distribution for the network states is expressed in product form for both cases. This distribution under the FCFS/PR discipline depends only on the first moments of service time at the network nodes.

The so-called "network" Markov process describing the stochastic behavior of a large class of queueing networks, including particular networks with negative customers (triggers, signals), is investigated in [37]. The main result of [37] is the necessary and sufficient conditions for studying isolated nodes and expressing the stationary distribution of the network as a whole in product form through the stationary state distributions of nodes. This result in effect determines the procedure of derivation of the product-form stationary distribution (if it exists) for the network.

A rather general Markov model, the "string" network, in which the effect of negative customers (triggers, signals) underlies the design of the respective Markov process, is studied in [88]. The state of the Markov process describing such a network is defined by a vector of the number of customers at network nodes (several types of customers is also possible), and the transition intensities of the process include the possibility for choosing a set of vectors defining states in which changes take place and vectors defining positive or negative increments of components of the vector set. Such a model includes many of the network models described above as particular cases. The stationary distribution of the "string" network is determined in [88]. A key point for finding the stationary distribution is the traffic equation containing transition intensities of the "string" network. Sufficient conditions for the existence of a solution to traffic equations are formulated and the relationship of traffic equations with partial balance equations is derived in [88].

9. G-NETWORKS WITH CATASTROPHES

Let us examine one more type of G-networks—G-networks with catastrophes. They differ from the base G-network described in Section 2 in the following. A Poisson flow of catastrophes of intensity λ_{0i}, $i = \overline{1,M}$, arrive at node i from outside. Upon arrival of a catastrophe at node i from outside or another node, all positive customers at the node are killed (unlike in the case of a negative customer, which kills only one positive customer). All other assumptions stipulated for the base G-network also hold for a G-network with catastrophes. Such a G-network with catastrophes is studied in [35]; here we only state its main results.

Let us study the stationary operation mode of the queueing network. The balance equations for the intensities of flows of positive customers and catastrophes in the network are of the form

$$\lambda_i^+ = \lambda_{0i}^+ + \sum_{j=1}^{M} q_j \mu_j p_{ji}^+,$$

$$\lambda_i^- = \lambda_{0i}^- + \sum_{j=1}^{M} q_j \mu_j p_{ji}^-, \quad i = \overline{1,M}, \tag{9.1}$$

where

$$q_i = \frac{\lambda_i^+ + \lambda_i^- + \mu_i - \sqrt{(\lambda_i^+ + \lambda_i^- + \mu_i)^2 - 4\lambda_i^+\mu_i}}{2\mu_i}. \tag{9.2}$$

Note that Eqs. (9.1) coincide in form with Eqs. (2.2) for the base G-network, but the stationary utilization factor q_i for the server i is defined by another formula.

Theorem 9. *For a G-network with catastrophes, if there exists a solution to the nonlinear system of Eqs.* (9.1), (9.2) *such that* $\lambda_i^- > 0$ *or* $\lambda_i^- = 0$ *and* $\lambda_i^+ < \mu_i$, $i = \overline{1, M}$, *then the stationary probability distribution for the network states is representable in product form as*

$$p(\mathbf{k}) = \prod_{i=1}^{M} (1 - q_i) q_i^{k_i}. \tag{9.3}$$

The proof of Theorem 9 is based on the approach developed in [38].

10. TWO-PHASE G-SYSTEMS

10.1. Two-Phase G-Systems with Negative Customers

All results pertaining to G-networks primarily consist in determining the product form for the stationary probability distribution of the network states. Naturally, other performance indexes of a network or its nodes are also of interest. In particular, an important network performance index is the response or sojourn time distribution for a customer in the network. It is not easy to find this characteristic. This problem in general formulation has thus far not been solved either for the Jackson networks or G-networks. Therefore, the results found even for narrow particular cases of G-networks are of great interest. Below we outline the results obtained for the response time distribution in [71] for a tandem G-network, which we refer to as the two-phase G-system.

The case of $M = 2$, $p_{1,2}^+ = 1$ and $p_{2,0}^+ = 1$ is investigated in [71] in terms of the base G-network under the assumption that customers are served in every phase (at every node) according to the FCFS discipline. To analyze the response time distribution, it is also important to know precisely which customer is killed by an arriving negative customer. A negative customer, upon arrival in any service phase, is assumed to kill (displaces to outside) the customer at the queue end. In foreign literature, this procedure is known as the RCE (removal of the customer at the end) procedure.

As a consequence of Theorem 1 in [71], we obtain

$$q_1 = \frac{\lambda_{0,1}^+}{\lambda_{0,1}^- + \mu_1}, \quad q_2 = \frac{\lambda_{0,2}^+ + \mu_1 q_1}{\lambda_{0,2}^- + \mu_2},$$

and if the condition $q_i < 1$, $i = 1, 2$, is satisfied, then the stationary probability distribution for the states of the two-phase G-system takes of the form

$$p(k_1, k_2) = (1 - q_1)(1 - q_2) q_1^{k_1} q_2^{q_2}, \quad k_1, k_2 \geq 0. \tag{10.1}$$

Since the service time at every server is exponentially distributed, the stationary distribution of the number of customers in phases does not depend on which customer in the queue is killed. But this fact is quite important in analyzing the sojourn (response) time distribution. Indeed, the sojourn time of a labelled customer depends on the arrival of future positive customers, since they are allocated in the queue behind the labelled customer and precisely they are killed according to the RCE procedure by the negative customers arriving at this phase.

Let $W^*(s)$ denote the Laplace–Stieltjes transform of the total sojourn time of a positive customer in the two-phase G-system and the probability that this customer is not killed. Then

$$W^*(s) = (1 - q_1)(1 - q_2)\frac{\mu_1}{\lambda_1^+}y_1(s)G(q_2, 0, y_1(s), s), \tag{10.2}$$

where G satisfies the functional equation (for any x, y, z, and s for which $|x| < 1$, $|y| < 1$, $|z| < 1$, and $R(s) \geq 0$)

$$RG = \left(\lambda_{0,1}^- + \mu_1 - \frac{\lambda_{0,1}^+}{z}\right) - \frac{\mu_1 z + \lambda_{0,2}^+}{y}G(x, 0, z, s) + \frac{\mu_2}{(1-y)(1-z)},$$

$$R = s + \lambda_{0,1}^-\lambda_{0,2}^-\mu_1 + \mu_2(1 - x) - \frac{\lambda_{0,1}^+}{z} - \lambda_{0,1}^- z - \frac{\lambda_{0,2}^+}{y} - \lambda_{0,2}^- y - \mu_1\frac{z}{y},$$

and $y_1(s)$ is the least root of the quadratic equation

$$\lambda_{0,1}^- y^2 - (y + \lambda_{0,1}^- + \mu_1(1 - q_1))y + \lambda_{0,1}^+ = 0.$$

The probability that a customer is not killed is

$$W^*(0) = \frac{\mu_1}{\lambda_{0,1}^- + \mu_1}\,\frac{\mu_2}{\lambda_{0,2}^- + \mu_2}, \tag{10.3}$$

i.e., is representable in product form.

For the particular case of $\lambda_{0,2}^+ = \lambda_{0,2}^- = 0$, the sojourn times of a positive customer in different ph‘ases are shown to be independent in [71].

10.2. Two-Phase G-Systems with Catastrophes

In the two-phase system with catastrophes investigated in [68], the first phase consists of one server with infinite buffer and the second consists of one server with no waiting room. Its input is a Markov flow of customers. Service times have a phase-type distribution for the first server and arbitrary for the second server. If the second server is busy at the instant of completion of service of a customer in the first phase, then the customer that has been served at the first server remains at the server, thereby blocking it till the second server becomes free. A Markov flow of catastrophes also arrives at this system. When a disaster arrives, the system is completely emptied. In [68], the Markov renewal process describing the stochastic behavior of the two-phase system with catastrophes have been introduced and expressions have been derived in terms of generating functions for the stationary distribution of the Markov chain imbedded at customer departure instants. The stationary distribution of the states of the system for arbitrary instants has also been determined and the departure flow has been studied.

11. SINGLE-PHASE G-SYSTEMS

The study of queueing systems with negative customers or G-systems as we shall refer to them (G-queues in English terminology [58]) was begun almost concurrently with G-networks. We now briefly outline the main trends of research on G-systems. Apparently, G-systems do not come under the scope of our review, but this is not true. The results of certain individual queueing systems may, for example, be required as models for the operation of isolated nodes in an approximate analysis of G-networks in analogy with the usual queueing systems, in particular, by the decomposition method with regard for one or, possibly, two distribution moments describing customer flows and service times (see, for example, [1]).

11.1. Single-Phase G-Systems with Negative Customers

A single-server queueing system with infinite buffer and negative customers was first studied by Gelenbe *et al.* [60], introducing an additional flow of negative arrivals under broad assumptions on the input flow of positive customers and their service times. Positive customers are served according to the FCFS discipline and two strategies are introduced for negative arrivals for choosing a positive customer for removal from the system:

(1) the RCE strategy, i,e., removal of the customer at the end (see Section 10) and

(2) the RCH strategy (removal of the customer at the head).

According to [60], a condition for the existence of a stationary operation mode for this queueing system depends, not on the intensities of input flow and service, as is the case for an ordinary queueing system without negative customers, but on the distributions of inter-arrival intervals and service times of customers.

In [70], the Laplace–Stieltjes transform for the joint sojourn time distribution for a positive customer and the probability that a positive customer is not killed are derived for $M/M/1/\infty$ systems under FCFS or LCFS service discipline and systems with negative customers under RCE and RCH strategies. The queueing process for the $M/G/1/\infty$ system was investigated by these authors in [72] for the combinations FCFS-RCE, FCES-RCH, and LCFS-RCH of service disciplines and removal strategies. (Note that the combinations FCFS-RCE and LCFS-RCH give the same results in analyzing the queueing process). For all these three cases, the generating functions for the stationary queue distribution are derived. It must be mentioned that these generating functions differ if the service times of positive customers are not exponentially distributed. For example, while a first-order Fredholm integral equation is to be solved for determining the generating function for the FCFS-RCE combination, the generating function $Q(z)$ for the queue length in the $M/G/1/\infty/$ FCFS-RCH system is expressed in explicit form closely resembling the Pollaczek–Khinchin formula:

$$Q(z) = (1-\rho)\frac{\lambda^- + \lambda^+(1-z)\beta(\lambda^+(1-z)+\lambda^-)}{\lambda^- + \lambda^+(\beta(\lambda^+(1-z)+\lambda^- - z)},$$

where λ^+ and λ^- are the intensities of arrivals of positive and negative customers, respectively, $\rho = \lambda^+(1-\beta(\lambda^-))/\lambda^-$, and $\beta(s)$ is the Laplace–Stieltjes transform for the service times of positive customers.

In [33], the stationary queue length distribution for a multi-server MM $CPP/GE/c/\infty$ system is determined, in which positive and negative customers arrive in batch Poisson flows with a geometric distribution for the batch size and controlled Markov processes with a finite state set. The sojourn time for a positive customer in a similar system under RCE and RCH strategies is determined [69]. It is the generalization of the results of [70].

An $M/G/1/\infty$ system with negative customers of a slightly different type than that of [60] and other cited papers is investigated in [27] under the assumption that a negative customer acts not instantaneously upon arrival, but only at the instants of completion of service of positive customers. The queueing process at service completion and arbitrary instants is studied in [27].

11.2. Single-Phase G-Systems with Catastrophes

In this section, we study a queueing system with catastrophes or clearing. These concepts are defined in Section 9 and require no explanation. Generally speaking, (complete or partial) clearing systems were investigated long ago (see, for example, [87, 89] or the comparatively recent paper [81]) in connection with optimization of systems in which clearing could be controlled without any reference to negative customers.

The concept of a negative customer gave a new impulse to the study of G-systems with catastrophes. For example, a random process—a job that an $M/G/1/\infty$ system with catastrophes must implement at a certain instant—is investigated in [78], in which the Pollaczek–Khinchin formula is generalized to the stationary case. In [30], the model of [78] is extended to a system with removal of a certain number of works; here removal need not necessarily be associated with an integral number of customers. The models of [30, 78] are refined in [79, 31]. A queueing process for the BMAP/SM/1/N system with a Markov flow of catastrophes and finite buffer is investigated in [41].

11.3. G-Systems with Retrial Customers

A large number of papers on G-systems deal with retrial models [13–20, 23–25, 66, 67].

There are models in which positive customers are served at a bufferless server. A customer arriving at the system when the server is busy joins a batch of retrial customers, called the orbit. Every customer in the orbit makes an attempt after certain exponentially distributed time intervals to receive service and the process continues till the retrial customer finds an idle server. In principle, such a model can be interpreted as a two-phase system (two-node network), in which phase 1 consists of a bufferless server and phase 2 consists of an infinite number of servers. Furthermore, in phase 2 servers with repeated service can be blocked (such type of blocking for a two-phase system with single-server nodes is investigated, for example, in [4]). Negative customers in such models may only kill a customer in the orbit or at a server.

In the M/M/1/0 system with repeated attempts and a Poisson flow of negative arrivals studied in [15, 16], a negative customer only kills the customers in the orbit under the assumption that the interval between repeated attempts from the orbit is exponentially distributed with parameter $\alpha(1-\delta_{0j})+j\nu$ if there are j retrial customers in the orbit, where δ_{0j} is the Kronecker delta. This is the so-called linear service discipline for retrial customers. In [15, 16], this G-system is studied in-depth and several important characteristics, namely, stationary distribution and factorial moments of the number of customers in the orbit, distribution of sojourn time on the orbit under the FCFS-RCE discipline, busy period, and the stationary distribution of the Markov chain imbedded at departure instants, are determined.

In [17, 20], an M/G/1/0 system with repeated attempts and Poisson flow of catastrophes is investigated, the method of the supplementary variable is applied to find the stationary probability distribution for the states of the system, and the time of sojourn of a customer in the system is determined. A numerical method of computing the stationary state probabilities for a G-system with retrials is developed in [18]. Many stationary characteristics of the M/M/1/0 system with recurrent flow of catastrophes are determined in [18].

For the M/G/1/0 system with repeated and negative customers under the LCFS/PR discipline, the stationary state probability distribution is determined in terms of the generating function in [23, 24]. These results are extended to several Poisson flows of positive customers in [25].

12. NEGATIVE CUSTOMERS AND SIGNALS: AN ALTERNATIVE INTERPRETATION

In conclusion, let us once again examine the concept of a negative customer and state the alternative interpretation for this concept in queueing theory [3].

As has been already mentioned, the concept of a negative customer was introduced comparatively recently by Gelenbe. But queueing systems with bounded sojourn time or, which is the same thing, systems with impatient customers (see, for example, [18]), which, in essence, are not different from negative customers, have been known since long in queueing theory.

Indeed, let us consider negative customers that exert instantaneous action on positive customers. We can consider an alternative service mechanism not based on the concept of a negative customer, but leads to the same effect for the service of ordinary customers. Let us assume that every positive customer in the queue or at a server (or servers) quit the queueing system in time $(t, t+\Delta)$ with probability $\gamma_k^- \Delta + o(\Delta)$ without completing his service, provided there were k customers in the system at instant t, i.e., customers may be impatient. Then, assuming that $\gamma_k^- = \gamma^-/k$, we find that if there were k customers at instant t in the system, then one customer quits the node in time $(t, t+\Delta)$ with probability $\gamma^- \Delta + o(\Delta)$. This, by the assumption that negative customers do not exert any action on the system in the absence of positive customers, is equivalent to the effect induced on the service of positive customers by negative customers of a Poisson flow with parameter γ^-.

Within the framework of the network model with signals (under the assumption that signals are instantaneously activated), an impatient customer who quits node i without completing his service, may go to node j (which is equivalent to the action of a trigger) or quit the network (which is equivalent to the action of a negative customer) with given probabilities, which may differ from the respective probabilities for the behavior of a served positive customer.

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