Stable models in multi-adjoint normal logic programs

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***Abstract*—Multi-adjoint normal logic programming arises as an extension of multi-adjoint logic programming considering a negation operator in the underlying lattice. In the literature, we can find different semantics for logic programs with negation [3]–[5]. We are interested in considering the stable model semantics in our logic programming framework. This paper summarizes a broad study on the syntax and semantics of multi-adjoint normal logic programming framework which has been recently published in [1]. Specifically, we will analyze the existence and the unicity of stable models for multi-adjoint normal logic programs.**

***Index Terms*—multi-adjoint logic programs, negation operator, stable models**

I. INTRODUCTION

Multi-adjoint logic programming was introduced in [9] as a general logic programming framework in which several implications appear in the rules of a same logic program and any order-preserving operator is allowed in the body of its rules. An interesting consequence of considering order-preserving operators in the body of the rules is associated with the existence of a least model. This fact makes possible to check whether a statement is a consequence of the logic program by simply computing the truth value of the statement under the least model. Therefore, the semantics of a multi-adjoint logic program is based on the least model of the program.

A well known fact in the logic programming literature is that the use of a negation operator increases the flexibility of a logic programming language. We are interested in enriching the multi-adjoint logic programming environment with the inclusion of a negation operator, which will give rise to a new kind of logic programs called multi-adjoint normal logic programs. It is important to emphasize that the existence of minimal models in an arbitrary multi-adjoint normal logic program cannot be ensured, in general. Furthermore, minimal models are not enough in order to prove that a statement is a consequence of a multi-adjoint normal logic program. As a result, the semantics of multi-adjoint normal logic programs will not be based on the notion of minimal model, but on the notion of stable model.

Different semantics such as the well-founded semantics [3], the stable models semantics [4] and the answer sets semantics [5] have been developed for logic programs with negation. In this paper, we will focus on the study of the existence and the unicity of stable models for multi-adjoint normal logic programs. According to the literature, sufficient conditions to ensure the existence of stable models have already been stated in other logical approaches [2], [6], [7], [10]–[13].

This paper will present a brief summary on the syntax and semantics defined for multi-adjoint normal logic programs in [1], including the most important results related to the existence and the unicity of stable models. In particular, we will show sufficient conditions which ensure the existence of stable models for multi-adjoint normal logic programs defined on any convex compact set of an euclidean space. Besides, in what regards the uniqueness of stable models, sufficient conditions for multi-adjoint normal logic programs defined on the set of subintervals $[0,1]\times[0,1]$ will be provided.

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II. MULTI-ADJOINT NORMAL LOGIC PROGRAMS

The syntax of multi-adjoint normal logic programs is based on an algebraic structure composed by a complete bounded lattice together with various adjoint pairs and a negation operator. This algebraic structure is usually known as multi-adjoint normal lattice and it is formally defined as follows.

Definition 1. *The tuple* $(L,\preceq,\leftarrow_1,\&_1,\dots,\leftarrow_n,\&_n,\neg)$ *is a* multi-adjoint normal lattice *if the following properties are verified:*

1) $(L,\preceq)$ *is a bounded lattice, i.e. it has a bottom* $(\bot)$ *and a top* $(\top)$ *element;*
2) $(\&_i,\leftarrow_i)$ *is an adjoint pair in* $(L,\preceq)$*, for* $i\in\{1,\dots,n\}$*;*
3) $\top\,\&_i\,\vartheta=\vartheta\,\&_i\,\top=\vartheta$*, for all* $\vartheta\in L$ *and* $i\in\{1,\dots,n\}$*.*
4) $\neg$ *is a negation operator, that is, a decreasing mapping* $\neg\colon L\to L$ *satisfying the equalities* $\neg(\bot)=\top$ *and* $\neg(\top)=\bot$*.*

A multi-adjoint normal logic program is defined from a multi-adjoint normal lattice as a set of weighted rules.

Definition 2. *Let* $(L,\preceq,\leftarrow_1,\&_1,\dots,\leftarrow_n,\&_n,\neg)$ *be a multi-adjoint normal lattice. A multi-adjoint normal logic program (MANLP)* $\mathbb{P}$ *is a finite set of weighted rules of the form:*

$$\langle p\leftarrow_i @[p_1,\dots,p_m,\neg p_{m+1},\dots,\neg p_n];\vartheta\rangle$$

where $i\in\{1,\dots,n\}$*,* $@$ *is an aggregator operator,* ϑ *is an element of* L *and* $p,p_1,\dots,p_n$ *are propositional symbols such that* $p_j\neq p_k$*, for all* $j,k\in\{1,\dots,n\}$*, with* $j\neq k$*.*

Let $\mathbb{P}$ be a MANLP and $\Pi_{\mathbb{P}}$ the set of propositional symbols in $\mathbb{P}$. Then, an interpretation is any mapping $I\colon \Pi_{\mathbb{P}} \to L$. We will say that an interpretation I satisfies a rule in $\mathbb{P}$ of the form $\langle p \leftarrow_i @[p_1, \dots, p_m, \neg p_{m+1}, \dots, \neg p_n]; \vartheta\rangle$ if and only if its evaluation under I is greater or equal than the confidence factor associated with the rule, that is:

$$\vartheta \preceq \hat{I}\,(p \leftarrow_i @[p_1, \dots, p_m, \neg p_{m+1}, \dots, \neg p_n])$$

A model is an interpretation that satisfies all rules in $\mathbb{P}$. As it was stated previously, the semantics of MANLPs is based on stable models. The notion of stable model is closely related to the notion of reduct given by Gelfond and Lifchitz [4]. Now, we will define the notion of reduct for MANLPs.

Given a MANLP $\mathbb{P}$ and an interpretation I, we build the reduct of $\mathbb{P}$ with respect to I, denoted by $\mathbb{P}_I$, by substituting each rule in $\mathbb{P}$ of the form

$$\langle p \leftarrow_i @[p_1, \dots, p_m, \neg p_{m+1}, \dots, \neg p_n]; \vartheta\rangle$$

by the rule

$$\langle p \leftarrow_i @_I[p_1, \dots, p_m]; \vartheta\rangle$$

where the operator $\dot{@}_I\colon L^m \to L$ is defined as

$$\dot{@}_I[\vartheta_1, \dots, \vartheta_m] = \dot{@}[\vartheta_1, \dots, \vartheta_m, \dot{\neg}\, I(p_{m+1}), \dots, \dot{\neg}\, I(p_n)]$$

for all $\vartheta_1, \dots, \vartheta_m \in L$.

Definition 3. *Given a MANLP $\mathbb{P}$ and an L-interpretation I, we say that I is a* stable model *of $\mathbb{P}$ if and only if I is a minimal model of $\mathbb{P}_I$.*

III. On the existence and unicity of stable models

After introducing the main notions associated with the syntax and semantics of multi-adjoint normal logic programs, sufficient conditions which ensure the existence and the uniqueness of stable models will be provided.

First of all, we will show that any MANLP defined on a non-empty convex compact set in an euclidean space has at least a stable model, whenever the operators appearing in the MANLP are continuous operators. Formally:

Theorem 4. *Let $(K, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ be a multi-adjoint normal lattice where K is a non-empty convex compact set in an euclidean space and $\mathbb{P}$ be a finite MANLP defined on this lattice. If $\&_1, \dots, \&_n$, $\neg$ and the aggregator operators in the body of the rules of $\mathbb{P}$ are continuous operators, then $\mathbb{P}$ has at least a stable model.*

As far as the uniqueness of stable models is concerned, a special algebraic structure is considered and sufficient conditions from which we can ensure the unicity of stable models for multi-adjoint normal logic programs defined on the set of subintervals of $[0,1] \times [0,1]$, denoted by $\mathcal{C}([0,1])$, are given. The considered algebraic structure is mainly composed by conjunctions defined as

$$\&_{\beta\delta}^{\alpha\gamma}([a,b],[c,d]) = [\, a^{\alpha} * c^{\gamma},\, b^{\beta} * d^{\delta}\,]$$

with $a, b, c, d \in \mathbb{R}$, together with their residuated implications [8].

Theorem 5. *Let $\mathbb{P}$ be a finite MANLP defined on $(\mathcal{C}([0,1]), \leq, \leftarrow_{\beta_1\delta_1}^{\alpha_1\gamma_1}, \&_{\beta_1\delta_1}^{\alpha_1\gamma_1}, \dots, \leftarrow_{\beta_m\delta_m}^{\alpha_m\gamma_m}, \&_{\beta_m\delta_m}^{\alpha_m\gamma_m}, \neg)$ such that the only possible operators in the body of the rules are $\&_{\beta\delta}^{\alpha\gamma}$, with $\alpha = \beta = \gamma = \delta = 1$, and $[\vartheta_p^1, \vartheta_p^2] = \max\{[\vartheta^1, \vartheta^2] \mid \langle p \leftarrow_{\beta_w\delta_w}^{\alpha_w\gamma_w} \mathcal{B}; [\vartheta^1, \vartheta^2]\rangle \in \mathbb{P}\}$. If the inequality*

$$\sum_{j=1}^{h} (\vartheta^2)^{\beta_w} \cdot \delta_w \cdot (\vartheta_{q_j}^2)^{\delta_w - 1} \cdot \left(\vartheta_{q_1}^2 \cdots \vartheta_{q_{j-1}}^2 \cdot \vartheta_{q_{j+1}}^2 \cdots \vartheta_{q_h}^2\right)^{\delta_w} + (\vartheta^2)^{\beta_w} \cdot \delta_w \cdot (k-h)(\vartheta_{q_1}^2 \cdots \vartheta_{q_h}^2)^{\delta_w} < 1$$

holds for every rule $\langle p \leftarrow_{\beta_w\delta_w}^{\alpha_w\gamma_w} q_1 * \cdots * q_h * \neg q_{h+1} * \cdots * \neg q_k; [\vartheta^1, \vartheta^2]\rangle \in \mathbb{P}$, with $w \in \{1, \dots, m\}$, then there exists a unique stable model of $\mathbb{P}$.

IV. Conclusions and future work

The philosophy of the multi-adjoint paradigm has been considered in order to define the syntax and semantics of a novel and flexible logic programming framework with negation. Moreover, we have shown under what conditions the existence and the unicity of stable models in multi-adjoint normal logic programs are guaranteed.

As a future work, we are interested in applying the obtained results to other logics with negation operators.

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