Aggregation on relaxed indistinguishability operators based on different triangular norms

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***Abstract*—We consider the aggregation problem of relaxed T_i–indistinguishability operators where each member of the collection to be merged is a relaxed indistinguishability operator with respect to a different t-norm T_i. Thus we provide sufficient conditions to merge a collection such an indistinguishability operators. Moreover, we give a characterization of those functions that allow to merge a collection of relaxed indistinguishability operators into a new one whenever all involved t–norms are continuos, Archimedean and strict. Namely, we characterize these functions in terms of the additive generator of the involved t-norms and those functions that transforms n-dimensional triangular triplets of the non-negative real line into a 1-dimensional triplet of the non-negative real line.**

I. Introduction

Aggregation functions constitute an important tool in the field of information fusion (see [4], [5], [7]). The information can be given by means of fuzzy relations what depends on different applications (see, for instance, [1], [13], [19], [26], [27], [34]). Sometimes the aggregation procedure used to take a working decision requires that the nature of the aggregated data be preserved. In the last decades, the problem of aggregating fuzzy relations has grown considerably among researchers in fuzzy mathematics (see [14], [21], [22], [28], [29], [31]). Motivated by the preceding fact, in [9], [10], we have carried out the study of the class preservation of relaxed T–indistinguishability operators by means of aggregation.

Let us recall that, on account of [32], a T-indistinguishability operator on a (non-empty) set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x, y, z \in X$ the following: (i) $E(x,x) = 1$, (ii) $E(x,y) = E(y,x)$, and (iii) $T(E(x,y), E(y,z)) \leq E(x,z)$. In general, a T–indistinguishability operator allows to classify objects when a measure presents some kind of uncertainty. They are also known as measures of similarity, in fact, the greater $E(x,y)$ the most similar are x and y. We assume that the reader is familiar with the basics of t–norms (see, for instance, [22]).

In the literature, we can found several authors that have analyzed the functions that allows to merge a collection of indistinguishability operators into a single one (see [28], [29], [31]). In particular, in [28] a characterization of those functions that aggregate indistinguishability operators have been given in terms of triangle triplets (compare [30]). Recently, the aforementioned description of the functions that aggregates indistinguishability operators have been extended by Mayor and Recasens in [24], now, in terms of T-triangular triplets.

Inspired by [24], recent results on the aggregation of relaxed T-indistinguishability operators have been developed in [9]. Now, let us recall that a relaxed T–indistinguishability operator on a (non–empty) set X is a fuzzy relation $E : X \times X \to [0,1]$ verifying for all $x, y, z \in X$ the following properties: (i) $E(x,y) = E(y,x)$ and (ii) $T(E(x,y), E(y,z)) \leq E(x,z)$. A relaxed T-indistinguishability operator E on a nonempty set X will be called a relaxed T-equality provided that it satisfies the following additional property for any $x, y, z \in X$: (i') $E(x,y) = E(x,x) = E(y,y) \Leftrightarrow x = y$.

There exist another measures very useful to classify objects that are related to (relaxed) T–indistinguishability operators, the so-called (extended) pseudo–metrics (see [12]). However, the extended pseudo-metrics are measures of dissimilarity. In the recent literature, different techniques to generate extended pseudo-metrics from indistinguishability operators, and vice-versa, have been developed (see, for example, [2], [3], [17], [20], [22], [25], [31], [33]).

It is clear that T–indistinguishability operators are relaxed T–indistinguishability operators. Nevertheless, the following is an instance of relaxed and partial indistinguishability operator that is not an indistinguishability one.

Example 1. *Let Σ be a non–empty alphabet. Denote by Σ^∞ the set of all finite and infinite sequences over Σ. Given $v \in \Sigma^\infty$ denote by $\mathrm{l}(v)$ the length of v. Thus $\mathrm{l}(v) \in \mathbb{N} \cup \{\infty\}$ for all $v \in \Sigma^\infty$. Moreover, if $\Sigma_F = \{v \in \Sigma^\infty : \mathrm{l}(v) \in \mathbb{N}\}$ and $\Sigma_\infty = \{v \in \Sigma^\infty : \mathrm{l}(v) = \infty\}$, then $\Sigma^\infty = \Sigma_F \cup \Sigma_\infty$ and we will write $v = v_1 v_2 \ldots v_{l(v)}$ and $w = w_1 w_2 \ldots$ whenever $v \in \Sigma_F$ and $w \in \Sigma_\infty$, respectively.*

Next, given $x, y \in \Sigma_\infty$, denote by $l(x,y)$ the longest common prefix of x and y (of course if x and y have not a common prefix then $l(x,y) = 0$).

Define the fuzzy relation E_Σ on $\Sigma^\infty \times \Sigma^\infty$ by $E_\Sigma(u,v) = 1 - 2^{-\mathrm{l}(v,w)}$ for all $u, v \in \Sigma^\infty$. Then it is a simple matter to check that E_Σ is a relaxed $T_{\min}$–indistinguishability operator on Σ^∞. In fact, it is a relaxed $T_{\min}$-equality on Σ^∞. Note that $T_{\min}$ denotes the minimum t–norm.

Note that E_Σ is not a indistinguishability operator, since $E_\Sigma(u,u) = 1 \Leftrightarrow u \in \Sigma_\infty$.

As we have mentioned before, the study of the class preservation of relaxed T–indistinguishability operators by means of aggregation was carried out in [9], [10]. Now we focus our attention on the similar preservation problem when a collection of T_i–indistinguishability operators belonging to the aforesaid class under consideration and in such a way that each member of the collection to be merged is a relaxed indistinguishability operator with respect to different t-norm T_i and all of them are defined on the same set.

Thus, in this work, we provide sufficient conditions to merge a collection of relaxed indistinguishability operators in Section II. Concretely, we see that a dominance condition is sufficient to guarantee that a function aggregates relaxed T_i–indistinguishability operators. Furthermore, we give a characterization of those functions that allow to merge a collection of relaxed indistinguishability operators into a new one whenever all involved t–norms are continuos, Archimedean and strict. Namely, we characterize these functions in terms of the additive generator of the involved t-norms and those functions that transforms n-dimensional triangular triplets of the positive real line into a 1-dimensional triplet of the positive real line. Finally, in Section III, some conclusions are given and future work is proposed.

II. The Aggregation of Relaxed T_i–Indistinguishability Operators

First of all we introduce the notion of relaxed T_i–indistinguishability operator aggregation function as follows.

Definition 1. *A function $F : [0,1]^n \to [0,1]$ aggregates relaxed T_i–indistinguishability operators into a relaxed T-indistinguishability operator if $F(E_1,\ldots,E_n)$ is a relaxed T–indistinguishability operators on X for any set X and any collection $(E_1,\ldots,E_n)$ of relaxed T_i–indistinguishability operators on X, where $F(E_1,\ldots,E_n)$ is the fuzzy binary relation given by $F(E_1,\ldots,E_n)(x,y) = F(E_1(x,y),\ldots,E_n(x,y))$.*

Notice that Definition 1 extends the notion given in [9]. In the light of the preceding concept we have the next result.

Proposition 1. *Let T be a t–norm and let $E_i, i = 1,\ldots,n$, be a collection of relaxed T_i–indistinguishability operators. If a function $F : [0,1]^n \to [0,1]$ is increasing and satisfies $T(F(a),F(b)) \leq F(T_1(a_1,b_1)),\ldots,T_n(a_n,b_n))$ for all $\mathbf{a},\mathbf{b} \in [0,1]^n$, then $F(E_1,\ldots,E_n)$ is a relaxed T–indistinguishability operator.*

Proof. To prove that $F(E_1,\ldots,E_n)$ is a relaxed T-indistinguishability operator, we only need to show the T-transitivity condition of $F(E_1,\ldots,E_n)$ because the symmetry follows directly from the symmetry of each E_i relaxed T_i–indistinguishability operator.

Now, as each E_i is a relaxed T_i–indistinguishability operator we have that $T_i(E_i(x,y),E_i(y,z)) \leq E_i(x,z)$ for all $i = 1,\ldots,n$. Since F is an increasing function we get $F(T_1(E_1(x,y),E_1(y,z)),\ldots,T_n(E_n(x,y),E_n(y,z))) \leq F(E_1(x,z),\ldots,E_n(x,z))$.

Calling $\mathbf{a} = (E_1(x,y),\ldots,E_n(x,y))$, $\mathbf{b} = (E_1(y,z),\ldots,E_n(y,z))$ and $\mathbf{c} = (E_1(x,z),\ldots,E_n(x,z))$ we obtain that $T(F(E_1,\ldots,E_n)(x,y),F(E_1,\ldots,E_n)(y,z)) = T(F(E_1(x,y),\ldots,E_n(x,y)),F(E_1(y,z),\ldots,E_n(y,z))) = T(F(a),F(b)) \leq F(T_1(a_1,b_1)),\ldots,T_n(a_n,b_n)) = F(T_1(E_1(x,y),E_1(y,z)),\ldots,T_n(E_n(x,y),E_n(y,z))) \leq F(E_1(x,z),\ldots,E_n(x,z))$. Therefore, $F(E_1,\ldots,E_n)$ is a relaxed T-indistinguishability operator. □

According to the previous result, it seems natural to wonder whether the converse of Proposition 1 is true in general. However, the next example, which was given in [9], shows that there are relaxed T_i-indistinguishability aggregation functions that are not increasing.

Example 2. *Denote by T_D the Drastic t-norm. Consider the function $F : [0,1]^2 \to [0,1]$ defined by*

$$F(\mathbf{a}) = \begin{cases} 1, & \mathbf{a} = (1,1) \\ 0, & \mathbf{a} = (1/2,1/2) \\ 1/2, & \textit{otherwise} \end{cases}$$

for all $\mathbf{a} \in [0,1]^2$. It is not hard to check that F aggregates T_D–indistinguishability operators. However F is not an increasing function due to $F(1/2,0) \leq F(0,0)$.

When the collection of $n+1$–t-norms T, T_i, are such that $T \leq T_i$, for all $i = 1,\ldots,n$, the following result can be stated.

Proposition 2. *Let T be a t–norm and let $E_i, i = 1,\ldots,n$, be a collection of relaxed T_i–indistinguishability operators. If $T \leq T_i$ for all $i = 1,\ldots,n$, then $T(E_1,\ldots,E_n)$ is a relaxed T–indistinguishability operator.*

Proof. First of all we show the transitivity condition. On the one hand, by the commutativity and associativity of T, we have that

$$T(T(E_1(x,y),\ldots,E_n(x,y)),T(E_1(y,z),\ldots,E_n(y,z))) = T(T(E_1(x,y),E_1(y,z)),\ldots,T(E_n(x,y),E_n(y,z))).$$

On the other hand, the monotony of T and the fact that $T \leq T_i$ provide that

$$T(T(E_1(x,y),E_1(y,z)),\ldots,T(E_n(x,y),E_n(y,z))) \leq T(T_1(E_1(x,y),E_1(y,z)),\ldots,T_n(E_n(x,y),E_n(y,z))).$$

Since $T_i(E_i(x,y),E_i(y,z)) \leq E_i(x,z)$ for all $i = 1,\ldots,n$ we have that

$$T(T_1(E_1(x,y),E_1(y,z)),\ldots,T_n(E_n(x,y),E_n(y,z))) \leq T(E_1(x,z),\ldots,E_n(x,z)).$$

It follows that

$$T(T(E_1,\ldots,E_n)(x,y),T(E_1,\ldots,E_n)(y,z)) \leq T(E_1,\ldots,E_n)(x,z).$$

This means that $T(E_1,\ldots,E_n)$ is a T-transitive relation. As the symmetry of this relation is always guaranteed by the symmetry of T, we conclude that $T(E_1,\ldots,E_n)$ is a relaxed T-indistinguishability operator. □

As an example, in the spirit of the above proposition, we can choose $T = T_D$ and $T_i = T_P$ for all $i = 1,\ldots,n$. Then, by Proposition 2, $T_D(E_1,\ldots,E_n)$ is a relaxed T_D–indistinguishability operator when E_i is a collection of T_P–indistinguishability operators, since $T_D \leq T_P$.

In order to introduce the announced characterization of those functions that merge relaxed T_i–indistinguishability operators, we need to recall the following notion which was provided in [6].

Definition 2. *A triplet $(a,b,c) \in [0,\infty]^3$ is said to be (1-dimensional) triangular if and only if $a \le b+c$, $b \le a+c$ and $c \le a+b$. Being $\mathbf{a},\mathbf{b},\mathbf{c} \in [0,\infty]^n$, $n \ge 1$, we say that $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a (n-dimensional) triangular triplet if (a_i,b_i,c_i) is a triangular triplet for all $i = 1,\dots,n$, where $\mathbf{a} = (a_1,\dots,a_n)$, $\mathbf{b} = (b_1,\dots,b_n)$ and $\mathbf{c} = (c_1,\dots,c_n)$.*

Taking into account the preceding notion we provide the following characterization.

Theorem 1. *Let $T, T_1, \dots T_n$ be strict and continuos Archimedean t–norms. The following assertions are equivalent:*

1) *A function $F : [0,1]^n \to [0,1]$ aggregates a collection of n relaxed T_i –indistinguishability operators E_i into a relaxed T–indistinguishability operator.*
2) *There exists a function $G : [0,+\infty]^n \to [0,+\infty]$ which trasnforms n–dimensional triangular triplets into 1–dimensional triangular triplets and satisfies*
$$G = t \circ F \circ (t_1^{-1} \times \dots \times t_n^{-1}),$$
where t and t_i are additive generators of t–norms T and T_i, respectively.

Proof. (1) $\to$ (2) Let us suppose that F aggregates a collection of relaxed T_i-indistinguishability operators into a relaxed T–indistinguishability operator. We have to prove that G transforms n-dimensional triangular triplets into 1-dimensional triangular triplets. To this end, assume that $(\mathbf{a},\mathbf{b},\mathbf{c}) \in [0,\infty]^n$ is a n-dimensional triplet. Define the fuzzy relations E_i, $i = 1,\dots,n$, on a non-empty set $X = \{x,y,z\}$ (of different elements) by $E_i(x,y) = E_i(y,x) = t_i^{-1}(b_i)$, $E_i(x,z) = E_i(z,x) = t_i^{-1}(a_i)$, $E_i(y,z) = E_i(z,y) = t_i^{-1}(c_i)$, $E_i(x,x) = E_i(y,y) = E_i(z,z) = 1$. It is not hard to check that each E_i is a relaxed T_i–indistinguishability operator, since $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a n-dimensional triplet. Next, let us prove that $(G(\mathbf{a}),G(\mathbf{b}),G(\mathbf{c}))$ is a 1–dimensional triangular triplet.

Since F aggregates the collection of relaxed T_i–indistinguishability operators E_i into a relaxed T–indistinguishability operator we have that

$$T(F(E_1,\dots,E_n)(x,y), F(E_1,\dots,E_n)(y,z)) \le F(E_1,\dots,E_n)(x,z).$$

It follows that

$$t^{-1}(t \circ F(E_1,\dots,E_n)(x,y)) + t \circ (F(E_i,\dots,E_n)(y,z)) \le F(E_1,\dots,E_n)(x,z).$$

Whence we deduce that

$$t \circ (F(E_1,\dots,E_n)(x,y)) + t \circ (F(E_i,\dots,E_n)(y,z)) \ge t(F(E_1,\dots,E_n)(x,z)).$$

The fact that $E_i(x,y) = t^{-1}(b_i)$, $E_i(x,z) = t^{-1}(a_i)$ and $E_i(y,z) = t^{-1}(c_i)$ provides that

$$t \circ (F(t_1^{-1}(b_1),\dots,t_n^{-1}(b_n))) + t \circ (F(t_1^{-1}(c_1),\dots,t_n^{-1}(c_n))) \ge t \circ (F(t_1^{-1}(a_1),\dots,t_n^{-1}(a_n))).$$

Hence we conclude that $G(\mathbf{a}) \le G(\mathbf{b}) + G(\mathbf{c})$ and, thus, that G transforms a n-dimensional triangular triplet into a 1-dimensional triangular triplet.

(2) $\to$ (1) Assuming that G transforms n-dimensional triangular triplets into 1-dimensional triangle triplets, we must prove that $F(E_1,\dots,E_n)$ is a relaxed T-indistinguishability operator for all collection E_i, $i = 1,\dots,n$, of relaxed T_i-indistinguishability operators. To this end. consider a collection E_i, $i = 1,\dots,n$, of relaxed T-indistinguishability operators on a non-empty set X. Then, for each $x,y,z \in X$, we set $E_i(x,y) = E_i(y,x) = a_i$, $E_i(y,z) = E_i(z,y) = b_i$, and $E_i(x,z) = E_i(z,x) = c_i$. Since $T_i(E_i(x,y),E_i(y,z)) \le E_i(x,z)$, $T_i(E_i(x,z),E_i(z,y)) \le E_i(x,y)$ and $T_i(E_i(y,x),E_i(x,z)) \le E_i(y,z)$ we have that $t_i(a_i) + t_i(b_i) \ge t_i(c_i)$, $t_i(b_i) + t_i(c_i) \ge t_i(a_i)$ and $t_i(a_i) + t_i(c_i) \ge t_i(b_i)$ and, thus, that $(t_i(a_i),t_i(b_i),t_i(c_i))$ is a 1-dimensional triangle triplet. So $(\mathbf{t}(\mathbf{a}),\mathbf{t}(\mathbf{b}),\mathbf{t}(\mathbf{c}))$ is a n-dimensional triangle triplet. The fact that G transforms n-dimensional triangular triplets into 1-dimensional triangle triplets provides that $(G(\mathbf{t}(\mathbf{a})),G(\mathbf{t}(\mathbf{b})),G(\mathbf{t}(\mathbf{c})))$ is a 1-dimensional triangle triplet, where $\mathbf{t}(\mathbf{a}) = (t_1(a_1),\dots,t_n(a_n))$, $\mathbf{t}(\mathbf{b}) = (t_1(b_1),\dots,t_n(b_n))$ and $\mathbf{t}(\mathbf{c}) = (t_1(c_1),\dots,t_n(c_n))$. Thus $G(\mathbf{t}(\mathbf{a})) \le G(\mathbf{t}(\mathbf{b})) + G(\mathbf{t}(\mathbf{c}))$, $G(\mathbf{t}(\mathbf{b})) \le G(\mathbf{t}(\mathbf{a})) + G(\mathbf{t}(\mathbf{c}))$ and $G(\mathbf{t}(\mathbf{c})) \le G(\mathbf{t}(\mathbf{a})) + G(\mathbf{t}(\mathbf{b}))$.

Since $G = t \circ F \circ (t_1^{-1} \times \dots \times t_n^{-1})$ we have

$$\begin{aligned} t(F(\mathbf{a})) &= G(\mathbf{t}(\mathbf{a})) \le G(\mathbf{t}(\mathbf{b})) + G(\mathbf{t}(\mathbf{c})) \\ &= t(F(\mathbf{b})) + t(F(\mathbf{c})). \end{aligned}$$

From the preceding inequality we deduce that

$$F(\mathbf{a}) \ge t^{-1}(t(F(\mathbf{b})) + t(F(\mathbf{c}))) = T(F(\mathbf{b}),F(\mathbf{c})).$$

Therefore we obtain that

$$T(F(E_1,\dots,E_n)(z,y), F(E_1,\dots,E_n)(x,z)) \le F(E_1,\dots,E_n)(x,y).$$

Similarly we can show that

$$T(F(E_1,\dots,E_n)(x,y), F(E_1,\dots,E_n)(y,z)) \le F(E_1,\dots,E_n)(x,z)$$

and

$$T(F(E_1,\dots,E_n)(y,x), F(E_1,\dots,E_n)(x,z)) \le F(E_1,\dots,E_n)(y,z).$$

Moreover, $F(E_1,\dots,E_n)(x,y) = F(E_1,\dots,E_n)(y,x)$ due to every E_i is a relaxed T_i–indistinguishability operator. Consequently, we conclude that F aggregates relaxed T_i–indistinguishability operators into a relaxed T–indistinguishability operator. $\square$

In order to finish the paper we recall two pertinent properties of those functions that transforms n-dimensional triangular triplets into 1-dimensional triangular triplets (see [6]).

Proposition 3. *Consider a function $G : [0, \infty]^n \longrightarrow [0, \infty]$, $n \geq 1$. Then:*

i) *If G transforms n–dimensional triangular triplets into 1-dimensional triangular triplets, then it is subadditive.*
ii) *If G is increasing and subadditive, then it transforms n–dimensional triangular triplets into 1-dimensional triangular triplets.*

Taking into account Propositions 1 and 3 we can state the next result.

Corollary 1. *Let $T, T_1, \dots T_n$ be strict and continuos Archimedean t–norms ($n \geq 1$) and let $F : [0, 1]^n \longrightarrow [0, 1]$ be an increasing function. Then the following assertions are equivalent:*

1) *$F : [0, 1]^n \longrightarrow [0, 1]$ aggregates relaxed T_i-indistinguishability operators into a relaxed T-indistinguishability operator.*
2) *The function $G : [0, +\infty]^n \to [0, +\infty]$ given by $G = t \circ F \circ (t_1^{-1} \times \dots \times t_n^{-1})$ is subadditive.*

Proof. $(1) \to (2)$ Taking into account Theorem 1 we know that G transforms n-dimensional triangular triplets into a 1-dimensional triangular triplet and, in addition, from Proposition 3 we can ensure that G is a subadditive function.

$(2) \to (1)$. The monotony of F yields that G is increasing. Proposition 3 guarantees that the function G transforms n-dimensional triangular triplets into 1-dimensional triangular triplet. Now, from Theorem 1, it follows that F aggregates relaxed T_i-indistinguishability operators into a relaxed T-indistinguishability operator. □

III. Conclusions

We have addressed the aggregation problem of a collection of relaxed indistinguishability operators where each member of the collection to be merged is a relaxed indistinguishability operator with respect to a different t-norm. A sufficient condition to merge a collection of the indistinguishability operators under consideration has been given. Moreover, we have characterized those functions that allow to merge a collection of relaxed T_i–indistinguishability operators into a new with respecto to a new t–norm T. Namely, the aforesaid characterization is provided in terms of the additive generator of the all involved t-norms and another functions that transforms a n-dimensional triangular triplet of the positive real line into a 1-dimensional triplet of the positive real line. As future work we will discuss whether some concrete families of aggregation functions (OWA operators, overlap functions, etc) could be used as relaxed T-indistinguishability operators.

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