Operations between fuzzy multisets

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***Abstract*—For fuzzy multisets the membership values are multisets in $[0,1]$. These sets are a mathematically generalization of the hesitant fuzzy sets, but in this general environment, the information about repetition is not lost, so that, the opinions given by the experts are better managed. Moreover, the order of the different opinions is also considered and this information is not lost either. In particular, we have studied in detail the basic operations for these sets: complement, union and intersection.**

***Index Terms*—fuzzy multiset, complement, aggregated union, aggregated intersection.**

I. Introduction

Fuzzy sets where introduced by Lotfi A. Zadeh (see [7]) as a way to deal with real-life situations where there is either limited knowledge or some sort of implicit ambiguity about whether an element should be considered a member of a set. Thus, the membership degree for any element is a value in the real interval $[0, 1]$. However, it could be paradoxical that the membership value itself should be one precise real number. Then, different generalizations appeared as a way to solve this paradox. In that cases, the membership degree could be, for example, an interval (interval-valued fuzzy sets [2]), a function (type-2 fuzzy sets [2]) or an arbitrary subsets of $[0, 1]$ (hesitant fuzzy sets [5]). When the subsets are finite, the hesitant fuzzy sets are called typical hesitant fuzzy sets and they are the ones that have attracted the most attention from researchers ([1], [3], [6]). However, for hesitant fuzzy sets the order of the elements in the set is not important and moreover, the repetition are not allowed. Clearly, this could be an important drawback. In fact, the need to account for repeated membership values has been recognised in the literature about hesitant fuzzy sets and, in fact, multiset-based hesitant fuzzy sets were already mentioned in the original paper that introduced the hesitant fuzzy sets [5]. Thus, fuzzy multisets can be considered as an appropriate tool to deal with repetitions. In that case, the membership degree is a multiset in the $[0, 1]$ interval. But despite the similarities, we cannot regard the typical hesitant fuzzy sets as a particular case of the fuzzy multisets and neither can we identify the fuzzy multisets with the multiset-based hesitant fuzzy sets because the definitions for the intersection and union are different in each theory. In [4] we have established the appropriate mathematical definitions for the main operations for fuzzy multisets and show how the hesitant theory definitions can be worked out from an extension of the fuzzy multiset definitions. The main concepts and results obtained in [4] are summarized in the next two sections.

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II. Fuzzy multisets

As we mentioned in the introduction, the values that make up a hesitant element in a hesitant fuzzy set are typically the result of applying several criteria on membership. In a common use case, it is assumed that there are a number of "experts" or "decision-makers" for a hesitant fuzzy set who produce a membership value for each element in the universe. A problem with the hesitant fuzzy sets in the experts' model is that the information about repetition is lost. For example, if there are five experts and four of them assign a membership value of 0.1 to an element whereas the fifth expert assigns the value 0.2, the hesitant element will be $\{0.1, 0.2\}$, regardless of the fact that 0.1 was four times more popular among the experts. This information loss can be avoided by using fuzzy multisets [3] (also called fuzzy bags [6]), which we are going to discuss now.

Definition 2.1: [3] Let X be the universe. A fuzzy multiset $\hat{A}$ over X is characterized by a function $\hat{A}\colon X \to \mathbb{N}^{[0,1]}$. The family of all the fuzzy multisets over X is called the fuzzy power multiset over X and is denoted by $\mathscr{FM}(X)$.

Example 2.2: Say we have a single-element universe $X = \{x\}$. We can define a fuzzy multiset $\hat{A}$ as $\hat{A}(x) = \langle 0.1, 0.2, 0.2\rangle$ in angular-bracket notation. Or in other words, using Definition 2.1, the element x is being mapped into a function $Count_{\hat{A}(x)}\colon [0,1] \to \mathbb{N}$ defined as $Count_{\hat{A}(x)}(0.1) = 1$, $Count_{\hat{A}(x)}(0.2) = 2$ and $Count_{\hat{A}(x)}(t) = 0$ for any $t \neq 0.1$ and $t \neq 0.2$. This function $Count_{\hat{A}(x)}$ characterizes a crisp multiset for any x in X.

III. Operations between fuzzy multisets

The complement for the fuzzy multisets is quite intuitive.

Definition 3.1: [3] Let X be a universe and let $\hat{A} \in \mathscr{FM}(X)$ be a fuzzy multiset. The complement of $\hat{A}$ is the fuzzy multiset $\hat{A}^c$ defined by the following count function:

$$Count_{\hat{A}^c(x)}(t) = Count_{\hat{A}(x)}(1-t), \quad \forall x \in X, \quad \forall t \in [0,1]$$

where $Count : M :\to \mathbb{N}$ mapping each element of the universe to a natural number (including 0).

Example 3.2: If we have a two-element universe $X = \{x, y\}$, then a fuzzy multiset $\hat{A}$ with $\hat{A}(x) = \langle 0.3 \rangle$ and $\hat{A}(y) = \langle 0.5, 0.8, 0.8 \rangle$ has the complement

$$Count_{\hat{A}^c(x)}(t) = \begin{cases} 1, & \text{if } t = 0.7, \\ 0, & \text{otherwise,} \end{cases}$$

$$Count_{\hat{A}^c(y)}(t) = \begin{cases} 1, & \text{if } t = 0.5, \\ 2, & \text{if } t = 0.2, \\ 0, & \text{otherwise,} \end{cases}$$

that is, $\hat{A}^c(x) = \langle 0.7 \rangle$ and $\hat{A}(y) = \langle 0.5, 0.2, 0.2 \rangle$.

By taking the multiset union of all the combinations, we can define what we will call the aggregated intersection and union of two fuzzy multisets, which do not privilege any particular ordering.

Definition 3.3: Let X be a universe and let $\hat{A}, \hat{B} \in \mathscr{F}\mathscr{M}(X)$ be two fuzzy multisets. The aggregated intersection of $\hat{A}$ and $\hat{B}$ is a fuzzy multiset $\hat{A} \cap^a \hat{B}$ such that for any element $x \in X$, $\hat{A} \cap^a \hat{B}(x)$ is the union, in the crisp multiset sense, of the regularised (s_A, s_B)-ordered intersections for all the possible pairs of ordering strategies (s_A, s_B), that is,

$$\hat{A} \cap^a \hat{B}(x) = \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}^r) \\ s_{\hat{B}} \in \mathcal{OS}(\hat{B}^r)}} \hat{A} \cap^r_{(s_A, s_B)} \hat{B}(x), \quad \forall x \in X.$$

Definition 3.4: Let X be a universe and let $\hat{A}, \hat{B} \in \mathscr{F}\mathscr{M}(X)$ be two fuzzy multisets. The aggregated union of $\hat{A}$ and $\hat{B}$ is a fuzzy multiset $\hat{A} \cup^a \hat{B}$ such that for any element $x \in X$, $\hat{A} \cup^a \hat{B}(x)$ is the union, in the crisp multiset sense, of the regularised (s_A, s_B)-ordered unions for all the possible pairs of ordering strategies (s_A, s_B), that is,

$$\hat{A} \cup^a \hat{B}(x) = \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}^r) \\ s_{\hat{B}} \in \mathcal{OS}(\hat{B}^r)}} \hat{A} \cup^r_{(s_A, s_B)} \hat{B}(x), \quad \forall x \in X.$$

Example 3.5: For two fuzzy multisets $\hat{E}(x) = \langle 0.1, 0.4 \rangle$ and $\hat{F}(x) = \langle 0.2, 0.3 \rangle$, the Miyamoto intersection and union are $\hat{E} \cap \hat{F}(x) = \langle 0.1, 0.3 \rangle$ and $\hat{E} \cup \hat{F}(x) = \langle 0.2, 0.4 \rangle$. In order to calculate their aggregated intersection and union, we need to first calculate the intersections and unions for all the possible ordering strategies. There are two possible ordering strategies for $\hat{E}$, resulting in the sequences $(0.1, 0.4)$ and $(0.4, 0.1)$, and two possible ordering strategies for $\hat{F}$, resulting in the sequences $(0.2, 0.3)$ and $(0.3, 0.2)$. This leads to the four sequences of pairwise minima, $(0.1, 0.3)$, $(0.1, 0.2)$, $(0.2, 0.1)$, $(0.3, 0.1)$, which result in two ordered intersections, $\langle 0.1, 0.3 \rangle$, $\langle 0.1, 0.2 \rangle$; and to the four sequences of pairwise maxima, $(0.2, 0.4)$, $(0.3, 0.4)$, $(0.4, 0.3)$, $(0.4, 0.2)$, which result in two ordered unions, $\langle 0.2, 0.4 \rangle$, $\langle 0.3, 0.4 \rangle$. By taking the union, in the crisp multiset sense, we get the aggregated intersection and union: $\hat{E} \cap^a \hat{F}(x) = \langle 0.1, 0.2, 0.3 \rangle$ and $\hat{E} \cup^a \hat{F}(x) = \langle 0.2, 0.3, 0.4 \rangle$. We have found the striking result that the numeric values match those of the hesitant fuzzy set intersection and union in the previous examples, a hint that the hesitant theory is equivalent to the fuzzy multiset theory when the aggregated operations are used, as we will prove in the next section.

These definitions are coherent with the existing ones for hesitant fuzzy sets and fuzzy sets, which can be seen as particular cases of fuzzy multisets. These relations can be summed up in the following diagram:

Fuzzy sets	$\mathscr{F}(X)$	$\xrightarrow{\phi_{\mathrm{FH}}}$	$\mathscr{F}\mathscr{H}^1(X)$		
			$\cap$		
Hesitant fuzzy sets			$\mathscr{F}\mathscr{H}(X)$	$\xrightarrow{\phi_{\mathrm{MH}}^{-1}}$	$\mathscr{F}\mathscr{M}(X)/\sim_r$
					$\cap$
Fuzzy multisets					$\mathscr{F}\mathscr{M}(X)$

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