Galois connections between a fuzzy preordered structure and a general fuzzy structure

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Extended abstract

Galois connections (both in isotone and in antitone forms) can be found in different areas, and it is common to find papers dealing with them either from a practical or a theoretical point of view. In the literature, one can find numerous papers on theoretical developments on (fuzzy) Galois connections [1], [19], [21] and also on applications thereof [13], [14], [22], [25], [28], [30], [33]. One important specific field of application is that of (Fuzzy) Mathematical Morphology, in which the (fuzzy) erosion and dilation operations are known to form a Galois connection, consider [6], [11], [20], [31], [32]; another important source of applications of Galois connections is within the field of (Fuzzy) Formal Concept Analysis, in which the concept-forming operators form either an antitone or isotone Galois connection (depending on the specific definition); in this research direction, one still can find recent papers on the theoretical background of the discipline [2]–[4], [8], [24], [29] and a number of applications [10], [26], [27].

Concerning the generalization of Galois connections to the fuzzy case, to the best of our knowledge, after the initial approach by Bělohlávek [1], a number of authors have introduced different approaches to so-called fuzzy (isotone or antitone) Galois connections; see [5], [14], [15], [19], [21], [23], [34]. It is remarkable that the mappings forming the Galois connection in all the above-mentioned approaches are crisp rather than fuzzy. In our opinion the term 'fuzzy Galois connection' should be reserved for the case in which the involved mappings are actually fuzzy mappings, and that is why we prefer to stick to the term 'Galois connection' rather than 'fuzzy Galois connection', notwithstanding the fact that we are working in the context of fuzzy structures.

In previous works, some of the authors have studied the problem of constructing a right adjoint (or residual mapping) associated to a given mapping $f\colon \mathbb{A} \to B$ where $\mathbb{A}$ is endowed with some order-like structure and B is unstructured: in [18], we consider $\mathbb{A}$ to be a crisp partially (pre)ordered set $\langle A, \leq_A\rangle$; later, in [7], we considered $\mathbb{A}$ to be a fuzzy preposet $\langle A, \rho_A\rangle$.

In this paper, we consider the case in which there are two underlying fuzzy equivalence relations in both the domain and the codomain of the mapping f, more specifically, f is a *morphism* between the *fuzzy structures* $\langle A, \approx_A\rangle$ and $\langle B, \approx_B\rangle$ where, in addition, $\langle A, \approx_A\rangle$ has a fuzzy preordering relation ρ_A. Firstly, we have to characterize when it is possible to endow B with the adequate structure (namely, enrich it to a fuzzy pre-ordered structure) and, then, construct a mapping g from B to A compatible with the fuzzy equivalence relations such that the pair (f, g) forms a Galois connection.

Although all the obtained results are stated in terms of the existence and construction of right adjoints (or residual mappings), they can be straightforwardly modified for the existence and construction of left adjoints (or residuated mappings). On the other hand, it is worth remarking that the construction developed in this paper can be extended to the different types of Galois connections (see [16]).

The core of the paper starts after introducing the preliminary notions on Galois connections between fuzzy preordered structures. Specifically, given a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preordered structure $\mathbb{A}$ into a fuzzy structure $\langle B, \approx_B\rangle$, we characterize when it is possible to construct a fuzzy relation ρ_B that induces a suitable fuzzy preorder structure on B and such that there exists a mapping $g\colon B \to \mathbb{A}$ such that the pair (f, g) constitutes a Galois connection. In the case of existence of right adjoint, it is worth remarking that the right adjoint need not be unique since, actually, its construction is given with several of degrees of freedom, in particular for extending the fuzzy ordering from the image of f to the entire codomain. Although a convenient extension has been given, our results do not imply that every right adjoint can be constructed in this way, and there may exist other constructions that are adequate as well. This is a first topic for future work.

Then, we follow the structure of [17] where we consider a mapping $f\colon \langle A, \rho_A\rangle \to B$ (and ρ_A is a fuzzy relation satisfying reflexivity, $\otimes$-transitivity and the weakest form of antisymmetry, namely, $\rho_A(a, b) = \rho_A(b, a) = \top$ implies $a = b$, for all $a, b \in A$); a further step was given in [7] for the same case $f\colon \langle A, \rho_A\rangle \to B$, in which antisymmetry was dropped. Both cases above can be seen as fuzzy preordered structures, in the sense of this paper, just by considering the so-called symmetric kernel relation (the conjunction of $\rho_A(a, b)$ and $\rho_A(a, b)$); the relationship between these and other kinds of structures can be found in [35]. Summarizing, the problem in [7] can be seen as constructing a right adjoint of a mapping $f\colon \langle A, \rho_A\rangle \rightarrow B$ which involves the construction of ρ_B, whereas in this paper our problem is to find a right adjoint to a mapping $f\colon \langle A, \approx_A, \rho_A\rangle \rightarrow \langle B, \approx_B\rangle$ in which the fuzzy equivalence $\approx_B$ has to be preserved; therefore, the main result in [7] is not exactly a particular case. We have considered a fuzzy mapping as a morphism $\langle A, \approx_A\rangle \rightarrow \langle B, \approx_B\rangle$ between fuzzy structures, adopting the approach of [12], while our long-term goal is to study fuzzy Galois connections constituted of truly fuzzy mappings.

In a few words, our approach is based on the canonical decomposition of Galois connections in our framework, followed by an analysis of conditions for the existence of the right adjoint. As a consequence of the canonical decomposition, we propose a two-step procedure for verifying the existence of the right adjoint in a constructive manner.

Conclusions and future work

Galois connections have found applications in areas such as formal concept analysis, and in mathematical morphology where, respectively, the intent and extent operators, and the erosion and the dilation operations are required to form a Galois connection. The results presented in this work pave the way to build specific settings of mathematical morphology parameterized by a fixed candidate to be an erosion (or dilation) operator; and the same approach would also apply to the development of new settings of formal concept analysis. In general, the construction of new Galois connections is of interest in fields in which there are two approaches to certain reality and one has more information about one of them, since the existence of a Galois connection allows to retrieve the unknown information in the other approach. In this respect, we will explore the application of the obtained results in the area of compression of data (images, etc.) in which the existence of the right adjoint of a given compressing mapping might allow to recover as much information as possible.

Last but not least, it is worth to study the two following extensions: on the one hand, we could consider an even more general notion of fuzzy mapping, for instance that proposed in [9]; on the other hand, we could consider $\mathbb{L}$-valued sets as a suitable generalization of our fuzzy structures.

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