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Equivalence relations on fuzzy subgroups*

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***Abstract*—We compare four equivalence relations defined in fuzzy subgroups: Isomorphism, fuzzy isomorphism and two equivalence relations defined using level subset notion. We study if the image of two equivalent fuzzy subgroups through aggregation functions is a fuzzy subgroup, when it belongs to the same class of equivalence and if the supreme property is preserved in the class of equivalence and through aggregation functions.**

***Index Terms*—Aggregation function, fuzzy subgroup, level subgroup, isomorphism, fuzzy isomorphism, sup property**

An alternative axiomatization for a fuzzy modal logic of preferences

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***Abstract*—In a recent paper, the authors have proposed an axiomatic system for a modal logic of gradual preference on fuzzy propositions that was claimed to be complete with respect to the intended semantics. Unfortunately, the completeness proof has a flaw, that leaves still open the question of whether the proposed system is actually complete. In this paper, we propose an alternative axiomatic system with a multi-modal language, where the original modal operators are definable and their semantics are preserved, and for which completeness results are proved.**

***Index Terms*—fuzzy preferences, fuzzy modal logic, completeness**

I. Introduction

Reasoning about preferences is a topic that has received a lot of attention in Artificial Intelligence since many years, see for instance [HGY12], [DHKP11], [Kac11]. Two main approaches to representing and handling preferences have been developed: the relational and the logic-based approaches.

In the classical setting, every preorder (i.e. reflexive and transitive) relation $R \subseteq W \times W$ on a set of alternatives W can be regarded as a (weak) preference relation by understanding $(a, b) \in R$ as denoting b *is not less preferred than* a. From R one can define three disjoint relations:

- the *strict preference* $P = R \cap R^d$,
- the *indifference relation* $I = R \cap R^t$, and
- the *incomparability relation* $J = R^c \cap R^d$.

where $R^d = \{(a, b) \in R : (b, a) \notin R\}$, $R^t = \{(a, b) : (b, a) \in R\}$ and $R^c = \{(a, b) \in R : (a, b) \notin R\}$. It is clear that P is a strict order (irreflexive, antisymmetric and transitive), I is an equivalence relation (reflexive, symmetric and transitive) and J is irreflexive and symmetric. The triple (P, I, J) is called a *preference structure*, where the initial weak preference relation can be recovered as $R = P \cup I$.

In the fuzzy setting, preference relations can be attached degrees (usually belonging to the unit interval $[0, 1]$) of fulfilment or strength, so they become *fuzzy relations*. A weak fuzzy preference relation on a set X will be now a fuzzy preorder $R : X \times X \to [0, 1]$, where $R(a, b)$ is interpreted as the degree in which b is at least as preferred as a. Given a t-norm $\odot$, a fuzzy $\odot$-preorder satisfies reflexivity ($R(a, a) = 1$ for each $a \in X$) and $\odot$-transitivity ($R(a, b) \odot R(b, c) \leq R(a, c)$ for each $a, b, c \in X$). The most influential reference is the book by Fodor and Roubens [FR94], that was followed by many other works like, for example [DBM07], [DBM10], [DMB04], [DBM08], [DGLM08]. In this setting, many questions have been discusssed, like e.g. the definition of the strict fuzzy order associated to a fuzzy preorder (see for example [Bod08a], [Bod08b], [BD08], [EGV18]).

The basic assumption in logical-based approaches is that preferences have structural properties that can be suitably described in a fomalized language. This is the main goal of the so-called *preference logics*, see e.g. [HGY12]. The first logical systems to reason about preferences go back to S. Halldén [Hal57] and to von Wright [vW63], [vW72], [Liu10]. Others related works are [EP06], [vBvOR05]. More recently van Benthem et al. in [vBGR09] have presented a modal logic-based formalization of representing and reasoning with preferences. In that paper the authors first define a basic modal logic with two unary modal operators $\Diamond^{\leq}$ and $\Diamond^{<}$, together with the universal and existential modalities, A and E respectively, and axiomatize them. Using these primitive modalities, they consider several (definable) binary modalities to capture different notions of preference relations on classical propositions, and show completeness with respect to the intended preference semantics. Finally they discuss their systems in relation to von Wright axioms for *ceteris paribus* preferences [vW63]. On the other hand, with the motivation of formalising a comparative notion of likelihood, Halpern studies in [Hal97] different ways to extend preorders on a set X to preorders on subsets of X and their associated strict orders. He studies their properties and relations among them, and he also provides an axiomatic system for a *logic of relative likelihood*, that is proved to be complete with respect to what he calls *preferential structures*, i.e. Kripke models with preorders as accessibility relations. All these works relate to the classical (modal) logic and crisp preference (accessibility) relations.

In the fuzzy setting, as far as the authors are aware, there are not many formal logic-based approaches to reasoning with fuzzy preference relations, see e.g. [BEFG01]. More recently, in the first part of [EGV18] we studied and characterized

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different forms to define fuzzy relations on the set $\mathcal{P}(\mathcal{W})$ of subsets of W, from a fuzzy preorder on W, in a similar way to the one followed in [Hal97], [vBGR09] for classical preorders, while in the second part we have semantically defined and axiomatized several two-tiered graded modal logics to reason about different notions of preferences on crisp propositions, see also [EGV17]. On the other hand, in [VEG17a] we considered a modal framework over a many-valued logic with the aim of generalizing Van Benthem et al.'s modal approach to the case of both fuzzy preference accessibility relations and fuzzy propositions. To do that, we first extended the many-valued modal framework for only a necessity operator $\Box$ of [BEGR11], by defining an axiomatic system with both necessity and possibility operators $\Box$ and $\Diamond$ over the same class of models. Unfortunately, in the last part of that paper, there is a mistake in the proof of Theorem 3 (particularly, equation (4)). This leaves open the question of properly axiomatizing the logic of graded preferences defined there.

In this paper we address this problem, and propose an alternative approach to provide a complete axiomatic system for a logic of fuzzy preferences. Namely, given a finite MTL-chain $\mathbf{A}$ (i.e. a finite totally ordered residuated lattice) as set of truth values, and given an A-valued preference Kripke model (W, R, e), with R a fuzzy preorder valued on A, we consider the a-cuts R_a of the relation R for every $a \in A$, and for each a-cut R_a, we consider the corresponding modal operators $\Box_a, \Diamond_a$. These operators are easier to be axiomatized, since the relations R_a are not fuzzy any longer, they are a nested set of classical (crisp) relations. The good news is that, in the our rich (multi-modal) logical framework, we can show that the original modal operators $\Box$ and $\Diamond$ are definable, and vice-versa if we expand the logic with Monteiro-Baaz's Δ operator. So we obtain a different, but equivalent, system where the original operators can be properly axiomatized in an indirect way through the graded operators.

The paper is structured as follows. After this introduction, in Section II we present the multi-modal language and the intended semantics given by *graded preference Kripke models*, which allows the formalization of different notions dealing with preferences taking values in some arbitrary MTL-chain $\mathbf{A}$. In Section III, we discuss different possibilities to formalize notions of preferences on fuzzy propositions in preference Kripke models. In Section IV we will exhibit a complete axiomatization of an alternative preference logic that is not, however, equivalent to the one from [VEG17a], since the language is intrinsically different. Nevertheless, we will see in Section V how, by the addition to the logic of the so-called Monteiro-Baaz Δ operation, we can also provide an axiomatization of the original logic of graded preference models pursued in [VEG17a]. We finish with some conclusions and open problems.

II. A multi-modal preference logic: language and semantics

Let us begin by defining the formal language of our underlying many-valued propositional setting. Let $\mathbf{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a *finite* and linearly ordered (bounded, integral, commutative) residuated lattice (equivalently, a finite MTL-chain) [GJKO07], and consider its canonical expansion $\mathbf{A^c}$ by adding a new constant $\overline{a}$ for every element $a \in A$ (canonical in the sense that the interpretation of $\overline{a}$ in $\mathbf{A^c}$ is a itself). A negation operation $\neg$ can always be defined as $\neg x = x \rightarrow 0$.

The logic associated with $\mathbf{A^c}$ will be denoted by $\mathbf{\Lambda(A^c)}$, and its logical consequence relation $\models_{\mathbf{A^c}}$ is defined as follows: for any set $\Gamma \cup \{\varphi\} \subseteq \mathbf{Fm}$ of formulas built in the usual way from a set of propositional variables $\mathcal{V}$ in the language of residuated lattices (we will use the same symbol to denote connectives and operations), including constants $\{\overline{a} : a \in A\}$,

- $\Gamma \models_{\mathbf{A^c}} \phi$ if, and only if, $\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}^c)$, if $h[\Gamma] \subseteq \{1\}$ then $h(\phi) = 1$,

where $\mathrm{Hom}(\mathbf{Fm}, \mathbf{A}^c)$ denotes the set of evaluations of formulas on $\mathbf{A^c}$.

Lifting to the modal level, we extend the propositional language by graded modal operators $\Box_a, \Diamond_a$, one pair for each element a of the algebra $\mathbf{A}$. We let the set $\mathbf{MFm}$ of *multi-modal formulas* defined as usual from a set $\mathcal{V}$ of propositional variables, residuated lattice operations $\{\wedge, \vee, \odot, \rightarrow\}$, truth constants $\{\overline{a} : a \in A\}$, and modal operators $\{\Box_a, \Diamond_a : a \in A\}$.

We are now ready to introduce $\mathbf{A}$-valued preference Kripke models.

Definition II.1. *An* $\mathbf{A}$*-preference model* is a triple $\mathfrak{M} = \langle W, R, e\rangle$ such that

- W is a set of worlds,
- $R\colon W \times W \rightarrow A$ is an A-valued fuzzy pre-order, i.e. a reflexive and $\odot$-transitive A-valued binary relation between worlds, and
- $e\colon W \times \mathcal{V} \rightarrow A$ is a world-wise $\mathbf{A}$-evaluation of variables. This evaluation is uniquely extended to formulas of $\mathbf{MFm}$ by using the operations in $\mathbf{A}$ for what concerns propositional connectives, and letting for each $a \in A$,

$$\begin{aligned} e(v, \Box_a\varphi) &= \bigwedge_{w: v \preceq_a w} \{e(w, \varphi)\} \\ e(v, \Diamond_a\varphi) &= \bigvee_{w: v \preceq_a w} \{e(w, \varphi)\} \end{aligned}$$

  where $v \preceq_a w$ stands for $R(v, w) \geq a$.

We will denote by $\mathbb{P}_{\mathbf{A}}$ the class of $\mathbf{A}$-preference models. Given an $\mathbf{A}$-preference model $\mathfrak{M} \in \mathbb{P}_{\mathbf{A}}$ and $\Gamma \cup \{\varphi\} \subseteq \mathbf{MFm}$, we write $\Gamma \Vdash_{\mathfrak{M}} \varphi$ whenever for any $v \in W$, if $e(v, \gamma) = 1$ for all $\gamma \in \Gamma$, then $e(v, \varphi) = 1$ too. Analogously, we write $\Gamma \Vdash_{\mathbb{P}_{\mathbf{A}}} \varphi$ whenever $\Gamma \Vdash_{\mathfrak{M}} \varphi$ for any $\mathfrak{M} \in \mathbb{P}_{\mathbf{A}}$.

We will denote by differentiated names some particular definable modal operators that enjoy a special meaning in our models. Namely:

- $\Box\varphi := \bigwedge_{a \in A} \overline{a} \rightarrow \Box_a\varphi$ and $\Diamond\varphi := \bigvee_{a \in A} \overline{a} \odot \Box_a\varphi$.

  It is easy to check that the evaluation of these operators in a preference model as defined here, coincides with the usual one for fuzzy Kripke models, i.e.,

$$e(v, \Box\varphi) = \bigwedge_{w \in W} \{R(v, w) \to e(w, \varphi)\}$$

$$e(v, \Diamond\varphi) = \bigvee_{w \in W} \{R(v, w) \odot e(w, \varphi)\}$$

- $A\varphi := \Box_0\varphi$ and $E\varphi := \Diamond_0\varphi$.

  Again, it is easy to see that these operators coincide with the global necessity and possibility modal operators respectively, i.e.,

$$e(v, A\varphi) = \bigwedge_{w \in W} \{e(w, \varphi)\}, \quad e(v, E\varphi) = \bigvee_{w \in W} \{e(w, \varphi)\}.$$

III. Modeling fuzzy preferences on propositions

The preference models introduced above are a very natural setting to formally address and reason over graded or fuzzy preferences over non-classical contexts. They are similar to the (classical) preference models studied by van Benthem et. al in [vBGR09], but offering a lattice of values (and so, a many-valued framework) where to evaluate both the truth degrees of formulas and the accessibility (preference) relation. The latter can be naturally interpreted as a graded preference relation between possible worlds or states (assignments of truth-values to variables). The question is then how to lift a (fuzzy) preference relation $\leq$ on worlds to (fuzzy) preference relations among formulas.

In the classical case, for instance in [vBGR09], [EGV18] the following six extensions are considered, where $[\varphi]$ and $[\psi]$ denote the set of models of propositions φ and ψ respectively:

- $\varphi \leq_{\exists\exists} \psi$ iff $\exists u \in [\varphi], v \in [\psi]$ such that $u \leq v$
- $\varphi \leq_{\exists\forall} \psi$ iff $\exists u \in [\varphi]$, such that $\forall v \in [\psi]$, $u \leq v$
- $\varphi \leq_{\forall\exists} \psi$ iff $\forall u \in [\varphi]$, $\exists v \in [\psi]$ such that $u \leq v$
- $\varphi \leq_{\forall\forall} \psi$ iff $\forall u \in [\varphi]$ and $v \in [\psi]$, $u \leq v$
- $\varphi \leq_{\exists\forall 2} \psi$ iff $\exists v \in [\psi]$, such that $\forall u \in [\varphi]$, $u \leq v$
- $\varphi \leq_{\forall\exists 2} \psi$ iff $\forall v \in [\psi]$, $\exists u \in [\varphi]$ such that $u \leq v$

However, not all these extensions can be expressed in our framework. For instance, we can express the orderings $\leq_{\exists\exists}$ and $\leq_{\forall\forall}$ as follows:

- $\varphi \leq_{\exists\exists} \psi := E(\varphi \wedge \Diamond\psi)$
- $\varphi \leq_{\forall\exists} \psi := A(\varphi \to \Diamond\psi)$

Some others would need to consider the inverse order $\geq$ of $\leq$ in the models or to assume the order $\leq$ be total, and some other are not just expressible (see [vBGR09]). On the other hand, not all the extensions above are also equally reasonable, for instance some of them are not even preorders. This is not the case of $\leq_{\forall\exists}$ and $\leq_{\forall\exists 2}$, that are indeed preorders.

In the fuzzy case, the formulas

$$E(\varphi \wedge \Diamond\psi),$$
$$A(\varphi \to \Diamond\psi)$$

make full sense as a fuzzy generalizations of the $\leq_{\exists\exists}$ and $\leq_{\forall\exists}$ preference orderings respectively, and moreover, as shown in [VEG17a], the expression $A(\varphi \to \Diamond\psi)$ models a fuzzy preorder in formulas (i.e. it satisfies reflexivity and $\odot$-transitivity).

Using the graded modalities $\Diamond_a$, one could also consider other intermediate extensions like

$$E(\varphi \wedge \Diamond_a\psi),$$
$$A(\varphi \to \Diamond_a\psi)$$

which would correspond to the fuzzy extensions of the following preference orderings $\leq^a_{\exists\exists}$ $\leq^a_{\forall\exists}$ on crisp propositions defined from the a-cut of the fuzzy preorder R:

- $\varphi \leq^a_{\exists\exists} \psi$ iff $\exists u \in [\varphi]$, $\exists v \in [\psi]$ such that $R(u, v) \geq a$.
- $\varphi \leq^a_{\forall\exists} \psi$ iff $\forall u \in [\varphi]$, $\exists v \in [\psi]$ such that $R(u, v) \geq a$.

Indeed, given an $\mathbf{A}$-valued preference model $\mathfrak{M} = \langle W, R, e\rangle$, one can define the following fuzzy preference relations on formulas:

- $\varphi \preceq^a_{\exists\exists} \psi$ iff there are worlds $v, w \in W$ such that $R(v, w) \geq a$ and $e(v, \varphi) \leq e(w, \psi)$
- $\varphi \preceq^a_{\forall\exists} \psi$ iff for each world $v \in W$, there is a world $w \in W$ such that $R(v, w) \geq a$ and $e(v, \varphi) \leq e(w, \psi)$.

Then, it is not difficult to check that

$$\Vdash_{\mathfrak{M}} E(\varphi \wedge \Diamond_a\psi) \text{ iff } \varphi \preceq^a_{\exists\exists} \psi$$
$$\Vdash_{\mathfrak{M}} A(\varphi \to \Diamond_a\psi) \text{ iff } \varphi \preceq^a_{\forall\exists} \psi \ .$$

So, we think our many-valued logical framework is expressive enough to capture many notions of (fuzzy) preferences among formulas. In the next section we provide an axiomatisation for this fuzzy multi-modal preference logic.

IV. Axiomatizing fuzzy preference models

In [VEG17a], we proposed the following axiomatic system $\mathsf{P}_{\mathbf{A}}$, in the language only with $\Box$ and $\Diamond$ modal operators (i.e. without the $\Box_a$'s and $\Diamond_a$'s):

- The axioms and rules of the minimal modal logic $\mathsf{BM}_{\mathbf{A}}$ for the pairs $(\Box, \Diamond)$ and (A, E) of modal operators (see [VEG17a, Def. 2])
- T: $\Box\varphi \to \varphi, \varphi \to \Diamond\varphi, \quad A\varphi \to \varphi, \varphi \to E\varphi$
- 4: $\Box\varphi \to \Box\Box\varphi, \Diamond\Diamond\varphi \to \Diamond\varphi,\ A\varphi \to AA\varphi, EE\varphi \to E\varphi$
- B: $\varphi \to AE\varphi$
- The *inclusion* axioms: $\mathsf{A}\varphi \to \Box\varphi$, $\Diamond\varphi \to \mathsf{E}\varphi$

In [VEG17a, Th. 3], this system was claimed to be complete with respect to the class $\mathbb{P}_{\mathbf{A}}$ of preference models. Unfortunately, we have discovered there is a flaw at the end of the proof, so the claim of the theorem remains unproved. In this section we remedy this problem by considering an alternative axiomatic system, based on the use of the graded modalities $\Box_a$ and $\Diamond_a$, for $a \in A$, introduced in Section II.

To this end, we introduce next the axiomatic system $\mathsf{mM}_{\mathbf{A}}$ defined by the following axioms and rules:

1) For each $a \in A$,
   - Axioms of minimum modal logic $\mathsf{BM}_{\mathbf{A}}$ for each pair of operators $(\Box_a, \Diamond_a)$ (see [VEG17a, Def. 2])
2) For each $a \in A$, the axiom
   - C_a: $\Box_a(\overline{k} \vee \varphi) \to \overline{k} \vee \Box_a\varphi$
3) For each $a, b \in A$, axioms K, T and 4:

- $K_a\colon \Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$
- $T_a\colon \Box_a\varphi \to \varphi, \qquad \varphi \to \Diamond_a\varphi$
- $4_{a,b}\colon \Box_{a\odot b}\varphi \to \Box_a\Box_b\varphi, \qquad \Diamond_a\Diamond_b\varphi \to \Diamond_{a\odot b}\varphi$

4) For each $a \le b$, nestedness axioms:
   - $\Box_a\varphi \to \Box_b\varphi, \quad \Diamond_b\varphi \to \Diamond_a\varphi$
5) For $a = 0$, axiom
   - $B_0\colon \varphi \to \Box_0\Diamond_0 A$
6) Rules: Modus Ponens and the necessitation for $\Box_0$:[1]
   - from φ derive $\Box_0\varphi$

Letting $\vdash_{\mathsf{mM_A}}$ be the consequence relation of the previous axiomatic system defined as usual, we can show that it is indeed complete with respect to our intented semantics given by the class of preference structures $\Vdash_{\mathbb{P}_\mathbf{A}}$. Formally,

Theorem IV.1. *For any* $\Gamma, \varphi \subseteq \mathbf{MFm}$,

$\Gamma \vdash_{\mathsf{mM_A}} \varphi$ *if and only if* $\Gamma \Vdash_{\mathbb{P}_\mathbf{A}} \varphi$.

Proof. Soundness (left to right direction) is easy to check. For what concerns completeness (right to left direction), we can define a canonical model

$$\mathfrak{M}^c = (W^c, \{R^c_a\}_{a\in A}, e^c)$$

with a set of crisp accessibility relations as follows, where $Th(\mathsf{mM_A}) = \{\varphi : \vdash_{\mathsf{mM_A}} \varphi\}$ denotes the set of theorems of $\mathsf{mM_A}$:

- $W^c = \{v \in Hom(\mathbf{MFm}, \mathbf{A})\colon v(Th(\mathsf{mM_A})) = \{1\}\}$,
- $R^c_a(v, w)$ if and only if $v(\Box_a\varphi) = 1 \Rightarrow w(\varphi) = 1$ for all $\varphi \in \mathbf{MFm}$,
- $e^c(v, p) = v(p)$, for any propositional variable p.

It is clear (since the only modal inference rules affects only theorems of the logic) that if $\Gamma \not\vdash_{\mathsf{mM_A}} \varphi$, then there is $v \in W^c$ such that $v(\Gamma) \subset \{1\}$ and $v(\varphi) < 1$. It is then only necessary to prove that the evaluation in the model can be defined in that way, namely, to prove the corresponding Truth Lemma, which follows from [BEGR11] and [VEG17a], i.e., for each formula $\varphi \in \mathbf{MFm}$ and each $v \in W^c$, it holds that

$$e^c(v, \Box_a\varphi) = \bigwedge_{R^c_a(v,w)} w(\varphi) \text{ and } e^c(v, \Diamond_a\varphi) = \bigvee_{R^c_a(v,w)} w(\varphi).$$

The nestedness axioms allow us to easily prove that for any $a \le b \in A$, it holds that $R^c_b \subseteq R^c_a$. Consider then the fuzzy relation R^c defined by

$$R^c(v, w) = \max\{a \in A\colon R^c_a(v, w)\}.$$

It is clear that $R^c(w, v) \ge a$ if and only if $R^c_a(v, w)$. Then, the truth lemma for the original model directly implies both

$$e^c(v, \Box_a\varphi) = \bigwedge_{w\in W^c, R^c(v,w)\ge a} w(\varphi),$$

$$e^c(v, \Diamond_a\varphi) = \bigvee_{w\in W^c, R^c(v,w)\ge a} w(\varphi).$$

[1]Observe that, together with the nestedness axioms, this rule implies the necessitation rule for each $\Box_a$.

It follows from axioms T_a that each R^c_a is reflexive, and so, R^c is a reflexive relation as well. Moreover, from axioms $4_{a,b}$, we get that R^c is $\odot$-transitive. The only remaining step is to prove is that we can obtain an equivalent model (in the sense of preserving the truth-values of formulas) in which R^c_0 is the total relation (in order to really get that $\Box_0$ and $\Diamond_0$ are global modalities). Observe that in the model defined above, thanks to axioms $T_0, 4_{0,0}$ and B_0, R^c_0 can be proven to be an equivalence relation, even though it is not necessarily the case that $R^c_0 = W^c \times W^c$. Nevertheless, since $R^c_b \subseteq R^c_0$ for all $b \in A$, for any arbitrary $v \in W^c$, we can define the model $\mathfrak{M}^c_v$ from $\mathfrak{M}^c$ by restricting the universe to $W^c_v = \{u \in W^c\colon R^c_0(v, u)\}$ and get that, for any $u \in W^c_v$ and any formula $\varphi \in \mathbf{MFm}$,

$$e^c(u, \varphi) = e^c_v(u, \varphi).$$

All the previous considerations allow us to prove that if $\Gamma \not\vdash_{\mathsf{mM_A}} \varphi$ there is $v \in W^c_v$ such that $e^c_v(v, \Gamma) \subseteq 1$ and $e^c_v(v, \varphi) < 1$. Given that the model $\mathfrak{M}^c_v$ defined above is indeed an $\mathbf{A}$-preference model, this concludes the completeness proof. $\Box$

V. Closing the loop: from graded to fuzzy modalities

In the previous section, we have seen that we have been able to provide a complete axiomatic system $\mathsf{mM_A}$ for the graded preference modalities $\Box_a$'s and $\Diamond_a$'s, and in Section II we have seen that the original fuzzy modalities $\Box$ and $\Diamond$ can be expressed from them. Thus, the system $\mathsf{mM_A}$ can be considered in fact as a sort of indirect axiomatization of the modalities $\Box$ and $\Diamond$ as well. In this section, generalising an approach introduced in [BEGR09], we will see that, by enriching our language with the well-known Monteiro-Baaz Δ connective (see e.g. [Háj98]), the graded modalities $\Box_a, \Diamond_a$ can also be expressed in terms of the original modal operators $\Box$ and $\Diamond$. Surprisingly enough we can do it using only the $\Diamond$ operator, while it is not clear using only $\Box$ would suffice.

Recall that the Monteiro-Baaz Δ operation over a linearly ordered MTL-chain $\mathbf{A}$ is the operation defined as

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in A$.

In the following, we will write $\varphi \equiv \psi$ to denote that φ and ψ are logically equivalent in the class of preferece models $\mathbb{P}_\mathbf{A}$. We will also denote by $\varphi \approx \overline{b}$ the formula $\Delta(\varphi \leftrightarrow \overline{b})$.

Lemma V.1.

$$\Box_a\varphi \equiv \bigwedge_{b\in A} (\Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to \overline{b})$$

$$\Diamond_a\varphi \equiv \bigvee_{b\in A} (\Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b}))\&\overline{b})$$

Proof. As in [BEGR09] we can check that

$$e(v, \Diamond(\varphi \approx \overline{b})) = \bigvee_{e(w,\varphi)=b} R(v, w).$$

Then $e(v, \Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b}))) = \Delta(a \to \bigvee_{e(w,\varphi)\approx b} R(v,w))$, and thus

$$e(v, \Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b}))) = \begin{cases} 1, & \text{if } a \leq \bigvee_{e(w,\varphi)=b} R(v,w) \\ 0, & \text{otherwise} \end{cases}.$$

Letting $S = \{b \in A\colon a \leq \bigvee_{e(w,\varphi)=b} R(v,w)\}$, the previous trivially implies both that

$$e(v, \Delta(\overline{a} \to \Diamond(\varphi = \overline{b})) \to \overline{b}) = \begin{cases} b, & \text{if } b \in S \\ 1, & \text{otherwise} \end{cases}$$

$$e(v, \Delta(\overline{a} \to \Diamond(\varphi = \overline{b}))\&\overline{b}) = \begin{cases} b, & \text{if } b \in S \\ 0, & \text{otherwise} \end{cases}.$$

Moreover, it is easy to see that

$$\{b \in A\colon a \leq \bigvee_{e(w,\varphi)=b} R(v,w)\} = \{e(w,\varphi)\colon a \leq Rvw\}.$$

Then, we have
$e(v, \bigwedge_{b\in A}(\Delta(\overline{a} \to \Diamond(\varphi = \overline{b})) \to \overline{b})) = \bigwedge S = \bigwedge_{a\leq R(v,w)} e(w,\varphi) = e(v, \Box_a\varphi)$
and $e(v, \bigvee_{b\in A}(\Delta(\overline{a} \to \Diamond(\varphi = \overline{b}))\&\overline{b})) = \bigvee S = \bigvee_{a\leq R(v,w)} e(w,\varphi) = e(v, \Diamond_a\varphi)$, concluding the proof. □

It is then the case that it is possible to provide an axiomatization for the fragment with only $\Box, \Diamond, A$ and E of the logic $\Vdash_{\mathbb{P}_\mathbf{A}}$ plus Δ. First, it is easy to provide an axiomatic system for the whole logic $\Vdash_{\mathbb{P}_\mathbf{A}}$ plus Δ by adding to $\mathsf{mM}_\mathbf{A}$ an axiomatization for Δ (see eg. [Háj98], [VEG17b]) and the interaction axioms

$$\Delta\Box_a\varphi \to \Box_a\Delta\varphi.$$

From here, it is clear that we can use the interdefinability of $\Box_a, \Diamond_a$ from $\Diamond$ proven above, and obtain in that way an axiomatic system complete with respect to the intended semantics.

This system is, however, quite more involved than the one presented in [VEG17a] (that did not achieve completeness with respect to its intended semantics). An open problem for future works is to study possible simplifications of this axiomatization, since the $(\Box, \Diamond, A, E)$-fragment is possibly the best suited to formalise graded preference relations while maintaining a lower level of elements in the language (and so, probably a lower complexity level).

As a side result, the previous characterization allows us to get a definition of the $\Box$ operation in terms of the $\Diamond$ very different from the usual one arising in classical modal logic. In particular, we get the following result.

Lemma V.2.

$$\Box\varphi \equiv \bigwedge_{a\in A}\bigwedge_{b<a\in A} \Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to (\overline{a \to b})$$

Proof. We know by definition that

$$\Box\varphi \equiv \bigwedge_{a\in A} \overline{a} \to \Box_a\varphi.$$

Then, using the previously proven equivalences, we prove the lemma by the following chain of equalities

$$\begin{aligned} \Box\varphi &\equiv \bigwedge_{a\in A}(\overline{a} \to \bigwedge_{b\in A}(\Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to \overline{b})) \\ &\equiv \bigwedge_{a\in A}\bigwedge_{b\in A} \overline{a} \to (\Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to \overline{b}) \\ &\equiv \bigwedge_{a\in A}\bigwedge_{b\in A} \Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to (\overline{a} \to \overline{b}) \\ &\equiv \bigwedge_{a\in A}\bigwedge_{b<a\in A} \Delta(\overline{a} \to \Diamond(\varphi \approx \overline{b})) \to (\overline{a \to b}) \end{aligned}$$

□

VI. Conclusions and Ongoing work

The aim of this work is to provide a formal framework generalising the treatment of preferences in the style of eg. [vBGR09] to a fuzzy context. We have presented an axiomatic system encompassing reflexive and transitive modalities plus global operators, that is shown to be the syntactical counterpart of many-valued Kripke models with (reflexive and transitive) graded (weak) preference relations between possible worlds or states. It is based on considering the cuts of the relations over the elements of the algebra of evaluation, solving in this way some problems arising from [VEG17a], for what concerns systems extended with the projection connective Δ. This logical framework stands towards the use of modal many-valued logics in the representation and management of graded preferences, in the same fashion that (classical) modal logic has served in the analogous Boolean preference setting.

The generalization of the previous logical system to cases when strict preferences are taken into account is part of ongoing work. The addition of those operators would allow a richer axiomatic definition of preference relations between formulas, in the sense of Section III. Moreover, further study of the introduced preference models should be pursued towards the formalisation of particular notions like indifference or incomparability, and aiming towards the incorporation of these systems in graded reasoners or recommender systems.

On the other hand, the study of the previous systems over other classes of algebras of truth-values (e.g. including infinite algebras like those defined on the real unit interval $[0,1]$ underlying Łukasiewicz, Product or Gödel fuzzy logics) is also of great interest, both from a theoretical point of view and towards the modelization of situations needing of continuous sets of values.

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Stable models in multi-adjoint normal logic programs

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***Abstract*—Multi-adjoint normal logic programming arises as an extension of multi-adjoint logic programming considering a negation operator in the underlying lattice. In the literature, we can find different semantics for logic programs with negation [3]–[5]. We are interested in considering the stable model semantics in our logic programming framework. This paper summarizes a broad study on the syntax and semantics of multi-adjoint normal logic programming framework which has been recently published in [1]. Specifically, we will analyze the existence and the unicity of stable models for multi-adjoint normal logic programs.**

***Index Terms*—multi-adjoint logic programs, negation operator, stable models**

I. Introduction

Multi-adjoint logic programming was introduced in [9] as a general logic programming framework in which several implications appear in the rules of a same logic program and any order-preserving operator is allowed in the body of its rules. An interesting consequence of considering order-preserving operators in the body of the rules is associated with the existence of a least model. This fact makes possible to check whether a statement is a consequence of the logic program by simply computing the truth value of the statement under the least model. Therefore, the semantics of a multi-adjoint logic program is based on the least model of the program.

A well known fact in the logic programming literature is that the use of a negation operator increases the flexibility of a logic programming language. We are interested in enriching the multi-adjoint logic programming environment with the inclusion of a negation operator, which will give rise to a new kind of logic programs called multi-adjoint normal logic programs. It is important to emphasize that the existence of minimal models in an arbitrary multi-adjoint normal logic program cannot be ensured, in general. Furthermore, minimal models are not enough in order to prove that a statement is a consequence of a multi-adjoint normal logic program. As a result, the semantics of multi-adjoint normal logic programs will not be based on the notion of minimal model, but on the notion of stable model.

Different semantics such as the well-founded semantics [3], the stable models semantics [4] and the answer sets semantics [5] have been developed for logic programs with negation. In this paper, we will focus on the study of the existence and the unicity of stable models for multi-adjoint normal logic programs. According to the literature, sufficient conditions to ensure the existence of stable models have already been stated in other logical approaches [2], [6], [7], [10]–[13].

This paper will present a brief summary on the syntax and semantics defined for multi-adjoint normal logic programs in [1], including the most important results related to the existence and the unicity of stable models. In particular, we will show sufficient conditions which ensure the existence of stable models for multi-adjoint normal logic programs defined on any convex compact set of an euclidean space. Besides, in what regards the uniqueness of stable models, sufficient conditions for multi-adjoint normal logic programs defined on the set of subintervals $[0,1]\times[0,1]$ will be provided.

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II. Multi-adjoint normal logic programs

The syntax of multi-adjoint normal logic programs is based on an algebraic structure composed by a complete bounded lattice together with various adjoint pairs and a negation operator. This algebraic structure is usually known as multi-adjoint normal lattice and it is formally defined as follows.

Definition 1. *The tuple* $(L,\preceq,\leftarrow_1,\&_1,\dots,\leftarrow_n,\&_n,\neg)$ *is a* multi-adjoint normal lattice *if the following properties are verified:*

1) $(L,\preceq)$ *is a bounded lattice, i.e. it has a bottom* $(\bot)$ *and a top* $(\top)$ *element;*
2) $(\&_i,\leftarrow_i)$ *is an adjoint pair in* $(L,\preceq)$*, for* $i\in\{1,\dots,n\}$*;*
3) $\top\&_i\vartheta=\vartheta\&_i\top=\vartheta$*, for all* $\vartheta\in L$ *and* $i\in\{1,\dots,n\}$.
4) $\neg$ *is a negation operator, that is, a decreasing mapping* $\neg\colon L\to L$ *satisfying the equalities* $\neg(\bot)=\top$ *and* $\neg(\top)=\bot$.

A multi-adjoint normal logic program is defined from a multi-adjoint normal lattice as a set of weighted rules.

Definition 2. *Let* $(L,\preceq,\leftarrow_1,\&_1,\dots,\leftarrow_n,\&_n,\neg)$ *be a multi-adjoint normal lattice. A multi-adjoint normal logic program (MANLP)* $\mathbb{P}$ *is a finite set of weighted rules of the form:*

$$\langle p\leftarrow_i @[p_1,\dots,p_m,\neg p_{m+1},\dots,\neg p_n];\vartheta\rangle$$

where $i\in\{1,\dots,n\}$*,* $@$ *is an aggregator operator,* ϑ *is an element of* L *and* $p,p_1,\dots,p_n$ *are propositional symbols such that* $p_j\neq p_k$*, for all* $j,k\in\{1,\dots,n\}$*, with* $j\neq k$.

Let $\mathbb{P}$ be a MANLP and $\Pi_{\mathbb{P}}$ the set of propositional symbols in $\mathbb{P}$. Then, an interpretation is any mapping $I\colon \Pi_{\mathbb{P}} \to L$. We will say that an interpretation I satisfies a rule in $\mathbb{P}$ of the form $\langle p \leftarrow_i @[p_1, \dots, p_m, \neg p_{m+1}, \dots, \neg p_n]; \vartheta\rangle$ if and only if its evaluation under I is greater or equal than the confidence factor associated with the rule, that is:

$$\vartheta \preceq \hat{I}\,(p \leftarrow_i @[p_1, \dots, p_m, \neg p_{m+1}, \dots, \neg p_n])$$

A model is an interpretation that satisfies all rules in $\mathbb{P}$. As it was stated previously, the semantics of MANLPs is based on stable models. The notion of stable model is closely related to the notion of reduct given by Gelfond and Lifchitz [4]. Now, we will define the notion of reduct for MANLPs.

Given a MANLP $\mathbb{P}$ and an interpretation I, we build the reduct of $\mathbb{P}$ with respect to I, denoted by $\mathbb{P}_I$, by substituting each rule in $\mathbb{P}$ of the form

$$\langle p \leftarrow_i @[p_1, \dots, p_m, \neg p_{m+1}, \dots, \neg p_n]; \vartheta\rangle$$

by the rule

$$\langle p \leftarrow_i @_I[p_1, \dots, p_m]; \vartheta\rangle$$

where the operator $\dot{@}_I\colon L^m \to L$ is defined as

$$\dot{@}_I[\vartheta_1, \dots, \vartheta_m] = \dot{@}[\vartheta_1, \dots, \vartheta_m, \dot{\neg}\, I(p_{m+1}), \dots, \dot{\neg}\, I(p_n)]$$

for all $\vartheta_1, \dots, \vartheta_m \in L$.

Definition 3. *Given a MANLP $\mathbb{P}$ and an L-interpretation I, we say that I is a* stable model *of $\mathbb{P}$ if and only if I is a minimal model of $\mathbb{P}_I$.*

III. On the existence and unicity of stable models

After introducing the main notions associated with the syntax and semantics of multi-adjoint normal logic programs, sufficient conditions which ensure the existence and the uniqueness of stable models will be provided.

First of all, we will show that any MANLP defined on a non-empty convex compact set in an euclidean space has at least a stable model, whenever the operators appearing in the MANLP are continuous operators. Formally:

Theorem 4. *Let $(K, \preceq, \leftarrow_1, \&_1, \dots, \leftarrow_n, \&_n, \neg)$ be a multi-adjoint normal lattice where K is a non-empty convex compact set in an euclidean space and $\mathbb{P}$ be a finite MANLP defined on this lattice. If $\&_1, \dots, \&_n$, $\neg$ and the aggregator operators in the body of the rules of $\mathbb{P}$ are continuous operators, then $\mathbb{P}$ has at least a stable model.*

As far as the uniqueness of stable models is concerned, a special algebraic structure is considered and sufficient conditions from which we can ensure the unicity of stable models for multi-adjoint normal logic programs defined on the set of subintervals of $[0,1] \times [0,1]$, denoted by $\mathcal{C}([0,1])$, are given. The considered algebraic structure is mainly composed by conjunctions defined as

$$\&_{\beta\delta}^{\alpha\gamma}([a,b],[c,d]) = [\, a^\alpha * c^\gamma,\, b^\beta * d^\delta \,]$$

with $a, b, c, d \in \mathbb{R}$, together with their residuated implications [8].

Theorem 5. *Let $\mathbb{P}$ be a finite MANLP defined on $(\mathcal{C}([0,1]), \leq, \leftarrow_{\beta_1\delta_1}^{\alpha_1\gamma_1}, \&_{\beta_1\delta_1}^{\alpha_1\gamma_1}, \dots, \leftarrow_{\beta_m\delta_m}^{\alpha_m\gamma_m}, \&_{\beta_m\delta_m}^{\alpha_m\gamma_m}, \neg)$ such that the only possible operators in the body of the rules are $\&_{\beta\delta}^{\alpha\gamma}$, with $\alpha = \beta = \gamma = \delta = 1$, and $[\vartheta_p^1, \vartheta_p^2] = \max\{[\vartheta^1, \vartheta^2] \mid \langle p \leftarrow_{\beta_w\delta_w}^{\alpha_w\gamma_w} \mathcal{B}; [\vartheta^1, \vartheta^2]\rangle \in \mathbb{P}\}$. If the inequality*

$$\sum_{j=1}^{h} (\vartheta^2)^{\beta_w} \cdot \delta_w \cdot (\vartheta^2_{q_j})^{\delta_w - 1} \cdot \left(\vartheta^2_{q_1} \cdots \vartheta^2_{q_{j-1}} \cdot \vartheta^2_{q_{j+1}} \cdots \vartheta^2_{q_h}\right)^{\delta_w} + (\vartheta^2)^{\beta_w} \cdot \delta_w \cdot (k-h)(\vartheta^2_{q_1} \cdots \vartheta^2_{q_h})^{\delta_w} < 1$$

holds for every rule $\langle p \leftarrow_{\beta_w\delta_w}^{\alpha_w\gamma_w} q_1 * \cdots * q_h * \neg q_{h+1} * \cdots * \neg q_k; [\vartheta^1, \vartheta^2]\rangle \in \mathbb{P}$, with $w \in \{1, \dots, m\}$, then there exists a unique stable model of $\mathbb{P}$.

IV. Conclusions and future work

The philosophy of the multi-adjoint paradigm has been considered in order to define the syntax and semantics of a novel and flexible logic programming framework with negation. Moreover, we have shown under what conditions the existence and the unicity of stable models in multi-adjoint normal logic programs are guaranteed.

As a future work, we are interested in applying the obtained results to other logics with negation operators.

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Galois connections between a fuzzy preordered structure and a general fuzzy structure

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Extended abstract

Galois connections (both in isotone and in antitone forms) can be found in different areas, and it is common to find papers dealing with them either from a practical or a theoretical point of view. In the literature, one can find numerous papers on theoretical developments on (fuzzy) Galois connections [1], [19], [21] and also on applications thereof [13], [14], [22], [25], [28], [30], [33]. One important specific field of application is that of (Fuzzy) Mathematical Morphology, in which the (fuzzy) erosion and dilation operations are known to form a Galois connection, consider [6], [11], [20], [31], [32]; another important source of applications of Galois connections is within the field of (Fuzzy) Formal Concept Analysis, in which the concept-forming operators form either an antitone or isotone Galois connection (depending on the specific definition); in this research direction, one still can find recent papers on the theoretical background of the discipline [2]–[4], [8], [24], [29] and a number of applications [10], [26], [27].

Concerning the generalization of Galois connections to the fuzzy case, to the best of our knowledge, after the initial approach by Bělohlávek [1], a number of authors have introduced different approaches to so-called fuzzy (isotone or antitone) Galois connections; see [5], [14], [15], [19], [21], [23], [34]. It is remarkable that the mappings forming the Galois connection in all the above-mentioned approaches are crisp rather than fuzzy. In our opinion the term 'fuzzy Galois connection' should be reserved for the case in which the involved mappings are actually fuzzy mappings, and that is why we prefer to stick to the term 'Galois connection' rather than 'fuzzy Galois connection', notwithstanding the fact that we are working in the context of fuzzy structures.

In previous works, some of the authors have studied the problem of constructing a right adjoint (or residual mapping) associated to a given mapping $f\colon \mathbb{A} \to B$ where $\mathbb{A}$ is endowed with some order-like structure and B is unstructured: in [18], we consider $\mathbb{A}$ to be a crisp partially (pre)ordered set $\langle A, \leq_A\rangle$; later, in [7], we considered $\mathbb{A}$ to be a fuzzy preposet $\langle A, \rho_A\rangle$.

In this paper, we consider the case in which there are two underlying fuzzy equivalence relations in both the domain and the codomain of the mapping f, more specifically, f is a *morphism* between the *fuzzy structures* $\langle A, \approx_A\rangle$ and $\langle B, \approx_B\rangle$ where, in addition, $\langle A, \approx_A\rangle$ has a fuzzy preordering relation ρ_A. Firstly, we have to characterize when it is possible to endow B with the adequate structure (namely, enrich it to a fuzzy pre-ordered structure) and, then, construct a mapping g from B to A compatible with the fuzzy equivalence relations such that the pair (f, g) forms a Galois connection.

Although all the obtained results are stated in terms of the existence and construction of right adjoints (or residual mappings), they can be straightforwardly modified for the existence and construction of left adjoints (or residuated mappings). On the other hand, it is worth remarking that the construction developed in this paper can be extended to the different types of Galois connections (see [16]).

The core of the paper starts after introducing the preliminary notions on Galois connections between fuzzy preordered structures. Specifically, given a mapping $f\colon \mathbb{A} \to B$ from a fuzzy preordered structure $\mathbb{A}$ into a fuzzy structure $\langle B, \approx_B\rangle$, we characterize when it is possible to construct a fuzzy relation ρ_B that induces a suitable fuzzy preorder structure on B and such that there exists a mapping $g\colon B \to \mathbb{A}$ such that the pair (f, g) constitutes a Galois connection. In the case of existence of right adjoint, it is worth remarking that the right adjoint need not be unique since, actually, its construction is given with several of degrees of freedom, in particular for extending the fuzzy ordering from the image of f to the entire codomain. Although a convenient extension has been given, our results do not imply that every right adjoint can be constructed in this way, and there may exist other constructions that are adequate as well. This is a first topic for future work.

Then, we follow the structure of [17] where we consider a mapping $f\colon \langle A, \rho_A\rangle \to B$ (and ρ_A is a fuzzy relation satisfying reflexivity, $\otimes$-transitivity and the weakest form of antisymmetry, namely, $\rho_A(a, b) = \rho_A(b, a) = \top$ implies $a = b$, for all $a, b \in A$); a further step was given in [7] for the same case $f\colon \langle A, \rho_A\rangle \to B$, in which antisymmetry was dropped. Both cases above can be seen as fuzzy preordered structures, in the sense of this paper, just by considering the so-called symmetric kernel relation (the conjunction of $\rho_A(a, b)$ and $\rho_A(a, b)$); the relationship between these and other kinds of structures can be found in [35]. Summarizing, the problem in [7] can be seen as constructing a right adjoint of a mapping $f\colon \langle A, \rho_A\rangle \rightarrow B$ which involves the construction of ρ_B, whereas in this paper our problem is to find a right adjoint to a mapping $f\colon \langle A, \approx_A, \rho_A\rangle \rightarrow \langle B, \approx_B\rangle$ in which the fuzzy equivalence $\approx_B$ has to be preserved; therefore, the main result in [7] is not exactly a particular case. We have considered a fuzzy mapping as a morphism $\langle A, \approx_A\rangle \rightarrow \langle B, \approx_B\rangle$ between fuzzy structures, adopting the approach of [12], while our long-term goal is to study fuzzy Galois connections constituted of truly fuzzy mappings.

In a few words, our approach is based on the canonical decomposition of Galois connections in our framework, followed by an analysis of conditions for the existence of the right adjoint. As a consequence of the canonical decomposition, we propose a two-step procedure for verifying the existence of the right adjoint in a constructive manner.

Conclusions and future work

Galois connections have found applications in areas such as formal concept analysis, and in mathematical morphology where, respectively, the intent and extent operators, and the erosion and the dilation operations are required to form a Galois connection. The results presented in this work pave the way to build specific settings of mathematical morphology parameterized by a fixed candidate to be an erosion (or dilation) operator; and the same approach would also apply to the development of new settings of formal concept analysis. In general, the construction of new Galois connections is of interest in fields in which there are two approaches to certain reality and one has more information about one of them, since the existence of a Galois connection allows to retrieve the unknown information in the other approach. In this respect, we will explore the application of the obtained results in the area of compression of data (images, etc.) in which the existence of the right adjoint of a given compressing mapping might allow to recover as much information as possible.

Last but not least, it is worth to study the two following extensions: on the one hand, we could consider an even more general notion of fuzzy mapping, for instance that proposed in [9]; on the other hand, we could consider $\mathbb{L}$-valued sets as a suitable generalization of our fuzzy structures.

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Operations between fuzzy multisets

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***Abstract*—For fuzzy multisets the membership values are multisets in $[0, 1]$. These sets are a mathematically generalization of the hesitant fuzzy sets, but in this general environment, the information about repetition is not lost, so that, the opinions given by the experts are better managed. Moreover, the order of the different opinions is also considered and this information is not lost either. In particular, we have studied in detail the basic operations for these sets: complement, union and intersection.**

***Index Terms*—fuzzy multiset, complement, aggregated union, aggregated intersection.**

I. Introduction

Fuzzy sets where introduced by Lotfi A. Zadeh (see [7]) as a way to deal with real-life situations where there is either limited knowledge or some sort of implicit ambiguity about whether an element should be considered a member of a set. Thus, the membership degree for any element is a value in the real interval $[0, 1]$. However, it could be paradoxical that the membership value itself should be one precise real number. Then, different generalizations appeared as a way to solve this paradox. In that cases, the membership degree could be, for example, an interval (interval-valued fuzzy sets [2]), a function (type-2 fuzzy sets [2]) or an arbitrary subsets of $[0, 1]$ (hesitant fuzzy sets [5]). When the subsets are finite, the hesitant fuzzy sets are called typical hesitant fuzzy sets and they are the ones that have attracted the most attention from researchers ([1], [3], [6]). However, for hesitant fuzzy sets the order of the elements in the set is not important and moreover, the repetition are not allowed. Clearly, this could be an important drawback. In fact, the need to account for repeated membership values has been recognised in the literature about hesitant fuzzy sets and, in fact, multiset-based hesitant fuzzy sets were already mentioned in the original paper that introduced the hesitant fuzzy sets [5]. Thus, fuzzy multisets can be considered as an appropriate tool to deal with repetitions. In that case, the membership degree is a multiset in the $[0, 1]$ interval. But despite the similarities, we cannot regard the typical hesitant fuzzy sets as a particular case of the fuzzy multisets and neither can we identify the fuzzy multisets with the multiset-based hesitant fuzzy sets because the definitions for the intersection and union are different in each theory. In [4] we have established the appropriate mathematical definitions for the main operations for fuzzy multisets and show how the hesitant theory definitions can be worked out from an extension of the fuzzy multiset definitions. The main concepts and results obtained in [4] are summarized in the next two sections.

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II. Fuzzy multisets

As we mentioned in the introduction, the values that make up a hesitant element in a hesitant fuzzy set are typically the result of applying several criteria on membership. In a common use case, it is assumed that there are a number of "experts" or "decision-makers" for a hesitant fuzzy set who produce a membership value for each element in the universe. A problem with the hesitant fuzzy sets in the experts' model is that the information about repetition is lost. For example, if there are five experts and four of them assign a membership value of 0.1 to an element whereas the fifth expert assigns the value 0.2, the hesitant element will be $\{0.1, 0.2\}$, regardless of the fact that 0.1 was four times more popular among the experts. This information loss can be avoided by using fuzzy multisets [3] (also called fuzzy bags [6]), which we are going to discuss now.

Definition 2.1: [3] Let X be the universe. A fuzzy multiset $\hat{A}$ over X is characterized by a function $\hat{A}\colon X \to \mathbb{N}^{[0,1]}$. The family of all the fuzzy multisets over X is called the fuzzy power multiset over X and is denoted by $\mathscr{FM}(X)$.

Example 2.2: Say we have a single-element universe $X = \{x\}$. We can define a fuzzy multiset $\hat{A}$ as $\hat{A}(x) = \langle 0.1, 0.2, 0.2\rangle$ in angular-bracket notation. Or in other words, using Definition 2.1, the element x is being mapped into a function $Count_{\hat{A}(x)}\colon [0, 1] \to \mathbb{N}$ defined as $Count_{\hat{A}(x)}(0.1) = 1$, $Count_{\hat{A}(x)}(0.2) = 2$ and $Count_{\hat{A}(x)}(t) = 0$ for any $t \neq 0.1$ and $t \neq 0.2$. This function $Count_{\hat{A}(x)}$ characterizes a crisp multiset for any x in X.

III. Operations between fuzzy multisets

The complement for the fuzzy multisets is quite intuitive.

Definition 3.1: [3] Let X be a universe and let $\hat{A} \in \mathscr{FM}(X)$ be a fuzzy multiset. The complement of $\hat{A}$ is the fuzzy multiset $\hat{A}^c$ defined by the following count function:

$$Count_{\hat{A}^c(x)}(t) = Count_{\hat{A}(x)}(1 - t), \quad \forall x \in X, \quad \forall t \in [0, 1]$$

where $Count : M :\to \mathbb{N}$ mapping each element of the universe to a natural number (including 0).

Example 3.2: If we have a two-element universe $X = \{x, y\}$, then a fuzzy multiset $\hat{A}$ with $\hat{A}(x) = \langle 0.3 \rangle$ and $\hat{A}(y) = \langle 0.5, 0.8, 0.8 \rangle$ has the complement

$$Count_{\hat{A}^c(x)}(t) = \begin{cases} 1, & \text{if } t = 0.7, \\ 0, & \text{otherwise,} \end{cases}$$

$$Count_{\hat{A}^c(y)}(t) = \begin{cases} 1, & \text{if } t = 0.5, \\ 2, & \text{if } t = 0.2, \\ 0, & \text{otherwise,} \end{cases}$$

that is, $\hat{A}^c(x) = \langle 0.7 \rangle$ and $\hat{A}(y) = \langle 0.5, 0.2, 0.2 \rangle$.

By taking the multiset union of all the combinations, we can define what we will call the aggregated intersection and union of two fuzzy multisets, which do not privilege any particular ordering.

Definition 3.3: Let X be a universe and let $\hat{A}, \hat{B} \in \mathscr{F}\mathscr{M}(X)$ be two fuzzy multisets. The aggregated intersection of $\hat{A}$ and $\hat{B}$ is a fuzzy multiset $\hat{A} \cap^a \hat{B}$ such that for any element $x \in X$, $\hat{A} \cap^a \hat{B}(x)$ is the union, in the crisp multiset sense, of the regularised (s_A, s_B)-ordered intersections for all the possible pairs of ordering strategies (s_A, s_B), that is,

$$\hat{A} \cap^a \hat{B}(x) = \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}^r) \\ s_{\hat{B}} \in \mathcal{OS}(\hat{B}^r)}} \hat{A} \cap^r_{(s_A, s_B)} \hat{B}(x), \quad \forall x \in X.$$

Definition 3.4: Let X be a universe and let $\hat{A}, \hat{B} \in \mathscr{F}\mathscr{M}(X)$ be two fuzzy multisets. The aggregated union of $\hat{A}$ and $\hat{B}$ is a fuzzy multiset $\hat{A} \cup^a \hat{B}$ such that for any element $x \in X$, $\hat{A} \cup^a \hat{B}(x)$ is the union, in the crisp multiset sense, of the regularised (s_A, s_B)-ordered unions for all the possible pairs of ordering strategies (s_A, s_B), that is,

$$\hat{A} \cup^a \hat{B}(x) = \bigcup_{\substack{s_{\hat{A}} \in \mathcal{OS}(\hat{A}^r) \\ s_{\hat{B}} \in \mathcal{OS}(\hat{B}^r)}} \hat{A} \cup^r_{(s_A, s_B)} \hat{B}(x), \quad \forall x \in X.$$

Example 3.5: For two fuzzy multisets $\hat{E}(x) = \langle 0.1, 0.4 \rangle$ and $\hat{F}(x) = \langle 0.2, 0.3 \rangle$, the Miyamoto intersection and union are $\hat{E} \cap \hat{F}(x) = \langle 0.1, 0.3 \rangle$ and $\hat{E} \cup \hat{F}(x) = \langle 0.2, 0.4 \rangle$. In order to calculate their aggregated intersection and union, we need to first calculate the intersections and unions for all the possible ordering strategies. There are two possible ordering strategies for $\hat{E}$, resulting in the sequences $(0.1, 0.4)$ and $(0.4, 0.1)$, and two possible ordering strategies for $\hat{F}$, resulting in the sequences $(0.2, 0.3)$ and $(0.3, 0.2)$. This leads to the four sequences of pairwise minima, $(0.1, 0.3)$, $(0.1, 0.2)$, $(0.2, 0.1)$, $(0.3, 0.1)$, which result in two ordered intersections, $\langle 0.1, 0.3 \rangle$, $\langle 0.1, 0.2 \rangle$; and to the four sequences of pairwise maxima, $(0.2, 0.4)$, $(0.3, 0.4)$, $(0.4, 0.3)$, $(0.4, 0.2)$, which result in two ordered unions, $\langle 0.2, 0.4 \rangle$, $\langle 0.3, 0.4 \rangle$. By taking the union, in the crisp multiset sense, we get the aggregated intersection and union: $\hat{E} \cap^a \hat{F}(x) = \langle 0.1, 0.2, 0.3 \rangle$ and $\hat{E} \cup^a \hat{F}(x) = \langle 0.2, 0.3, 0.4 \rangle$. We have found the striking result that the numeric values match those of the hesitant fuzzy set intersection and union in the previous examples, a hint that the hesitant theory is equivalent to the fuzzy multiset theory when the aggregated operations are used, as we will prove in the next section.

These definitions are coherent with the existing ones for hesitant fuzzy sets and fuzzy sets, which can be seen as particular cases of fuzzy multisets. These relations can be summed up in the following diagram:

Fuzzy sets	$\mathscr{F}(X)$	$\xrightarrow{\phi_{\mathrm{FH}}}$	$\mathscr{F}\mathscr{H}^1(X)$		
			$\cap$		
Hesitant fuzzy sets			$\mathscr{F}\mathscr{H}(X)$	$\xrightarrow{\phi_{\mathrm{MH}}^{-1}}$	$\mathscr{F}\mathscr{M}(X)/\sim_r$
					$\cap$
Fuzzy multisets					$\mathscr{F}\mathscr{M}(X)$

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Un marco semántico general para la Lógica de Simplificación

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***Resumen*—Presentamos una generalización de la Lógica de Simplificación para el razonamiento con reglas "si-entonces" sobre atributos difusos. Las implicaciones y la lógica propuesta están parametrizadas por sistemas de conexiones de Galois isótonas que permiten manejar diferentes interpretaciones de dependencias entre datos. Describimos la semántica de las reglas y el sistema axiomático de la lógica.**

***Index Terms*—Teoría de retículos, lógica difusa, implicaciones**

I. PARAMETRIZACIONES POR CONEXIONES DE GALOIS ISÓTONAS

En este trabajo resumimos el presentado en [9] que se enmarca dentro del Análisis Formal de Conceptos (AFC) [1] en su versión difusa. Ésta considera un retículo completo residuado $\mathbb{L}$ y define un $\mathbb{L}$-contexto como una terna $\mathbf{I} = \langle X, Y, I\rangle$ donde X e Y son conjuntos no vacíos de objetos y atributos respectivamente e I es una $\mathbb{L}$-relación difusa de X en Y. Para cada objeto $x \in X$, se considera el conjunto difuso $I_x \in L^Y$ tal que $I_x(y) = I(x, y)$ para todo $y \in Y$. Una implicación de atributos es una expresión $A \Rightarrow B$ donde $A, B \in L^Y$ y se dice que el contexto $\mathbf{I}$ la satisface si $A \subseteq I_x$ implica $B \subseteq I_x$ para todo $x \in X$.

Nuestra propuesta explora sistemas de inferencia generales para razonar con implicaciones entre atributos difusos. Tomamos como punto de partida la generalización presentada en [3], donde el autor considera, como parámetros, un conjunto S de conexiones de Galois isotonas que es cerrado bajo composición y contiene a la identidad. Propone una axiomatización completa basada en los Axiomas de Armstrong. En este marco general, una implicación $A \Rightarrow B$ es cierta en I_x si, para todo $\langle \boldsymbol{f}, \boldsymbol{g}\rangle \in S$, se cumple que $\boldsymbol{f}(A) \subseteq I_x$ implica $\boldsymbol{f}(B) \subseteq I_x$.

Como alternativa a los bien conocidos Axiomas de Armstong [7], en [8] los autores propusieron una Lógica de Simplificación y nuevos métodos para la manipulación automática de implicaciones [10], [11]. Posteriormente, en [4], se propuso la lógica *FASL (Fuzzy Attribute Simplification Logic)* para implicaciones de atributos con grados y parametrizados por *"hedges"*.

En este resumen mostramos una generalización de la Lógica de Simplificación, equivalente a la citada [3], para implicaciones con grados cuya semántica está parametrizada por conexiones de Galois isótonas.

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II. MARCO TEÓRICO

En este marco general, consideramos, como estructura para los grados, un retículo co-residuado completo, es decir, un álgebra $\mathbb{L} = \langle L, \leq, \oplus, \ominus, 0, 1\rangle$ satisfaciendo las siguientes condiciones:

- $\langle L, \leq, 0, 1\rangle$ es un retículo completo donde 0 es el mínimo y 1 es el máximo. Como es usual, usamos los símbolos $\vee$ y $\wedge$ para denotar respectivamente supremo e ínfimo.
- $\langle L, \oplus, 0\rangle$ es un monoide conmutativo.
- El par $\langle \oplus, \ominus\rangle$ satisface la siguiente propiedad de adjunción: para todo $a, b, c \in L$,

$$a \leq b \oplus c \quad \text{si y solo si} \quad a \ominus b \leq c. \tag{1}$$

L^Y denota el conjunto de todos los $\mathbb{L}$-conjuntos difusos en el universo Y. Las operaciones en $\mathbb{L}$ se extienden elemento a elemento a los $\mathbb{L}$-conjuntos difusos en la forma habitual: Para $A, B \in L^Y$ los $\mathbb{L}$-conjuntos difusos $A \oplus B$ and $A \ominus B$ se definen como $(A \oplus B)(y) = A(y) \oplus B(y)$ y $(A \ominus B)(y) = A(y) \ominus B(y)$ para todo $y \in Y$.

Las parametrizaciones [3] que se usan en nuestra propuesta se definen en términos de conexiones de Galois isótonas en $\langle L^Y, \subseteq\rangle$. En particular, consideramos pares $\langle \boldsymbol{f}, \boldsymbol{g}\rangle$ donde $\boldsymbol{f}, \boldsymbol{g} \colon L^Y \to L^Y$ son tales que, para todo $A, B \in L^Y$,

$$\boldsymbol{f}(A) \subseteq B \quad \text{si y solo si} \quad A \subseteq \boldsymbol{g}(B). \tag{2}$$

Es bien conocido que esta definición es equivalente a pedir que ambas funciones sean isótonas, que $\boldsymbol{g} \circ \boldsymbol{f}$ se inflacionaria y que $\boldsymbol{f} \circ \boldsymbol{g}$ sea deflacionaria. Como consecuencia, $\boldsymbol{g} \circ \boldsymbol{f}$ es un *operador de cierre* y $\boldsymbol{f} \circ \boldsymbol{g}$ es un *operador de núcleo* (*operador interior*).

Además, para cualquier isomorfismo $\boldsymbol{f}$ in $\langle L^Y, \subseteq\rangle$, el par $\langle \boldsymbol{f}, \boldsymbol{f}^{-1}\rangle$ es una conexión de Galois isótona y, en particular, la función identidad $\boldsymbol{I}_Y \colon L^Y \to L^Y$ lo es. Otro ejemplo interesante es $\langle \mathbf{0}_Y, \mathbf{1}_Y\rangle$ donde $\mathbf{0}_Y(A)(y) = 0$ y $\mathbf{1}_Y(A)(y) = 1$, para cualquier $A \in L^Y$ e $y \in Y$.

Por último, dadas dos conexiones de Galois isótonas $\langle \boldsymbol{f}_1, \boldsymbol{g}_1\rangle$ y $\langle \boldsymbol{f}_2, \boldsymbol{g}_2\rangle$, su *composición* $\langle \boldsymbol{f}_1 \circ \boldsymbol{f}_2, \boldsymbol{g}_2 \circ \boldsymbol{g}_1\rangle$ es también una conexión de Galois.

Definición 1 ([3]): Una familia de conexiones de Galois isótonas S in $\langle L^Y, \subseteq \rangle$ es una $\mathbb{L}$-parametrización si $\mathbb{S} = \langle S, \circ, \langle \boldsymbol{I}_Y, \boldsymbol{I}_Y \rangle \rangle$ es un monoide. En otras palabras, si S es es cerrada para la composición y contiene a la identidad.

III. LÓGICA DE SIMPLIFICACIÓN PARAMETRIZADA

Dado un *alfabeto* Y no vacío, cuyos elementos se denominan *atributos*, el conjunto de fórmulas bien formadas del lenguaje es:

$$\mathcal{L}_Y = \{A \Rightarrow B \mid A, B \in L^Y\}.$$

Las fórmulas del lenguaje se denominan *implicaciones* y para cada implicación, la primera y segunda componente se denomina *premisa* y *conclusion* respectivamente. Finalmente, los conjuntos de implicaciones $\Sigma \subseteq \mathcal{L}$ se denominan *teorías*.

Sobre este lenguaje, definimos la Lógica de Simplificación presentando la semántica y un sistema axiomático. Finalmente, en la publicación de referencia del presente resumen [9], se prueba que la visión semántica y sintáctica coinciden, probando la corrección y completitud de la lógica propuesta.

Antes de definir la interpretación de las fórmulas, introducimos el concepto de $\mathbb{L}$-conjuntos difusos S-aditivos que juegan un papel fundamental en los modelos.

Definición 2: Sea Y un conjunto no vacío y S una $\mathbb{L}$-parametrización. Un $\mathbb{L}$-conjunto difuso $A \in L^Y$ se dice *S-aditivo* si, para todo $B, C \in L^Y$ y $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$,

$$\boldsymbol{f}(B) \subseteq A \text{ y } \boldsymbol{f}(C) \subseteq A \text{ implica } \boldsymbol{f}(B \oplus C) \subseteq A.$$

La proposición siguiente es directa a partir de la Defintión 2 y (2).

Proposición 1: Sea Y un conjunto no vacío y S una $\mathbb{L}$-parametrización. Un $\mathbb{L}$-conjunto difuso $A \in L^Y$ es S-aditivo si y solo si $\boldsymbol{g}(A) \oplus \boldsymbol{g}(A) = \boldsymbol{g}(A)$.

Dada una $\mathbb{L}$-parametrización S, los modelos de la lógica se definen en términos de $\mathbb{L}$-conjuntos S-aditivos de la siguiente forma:

Definición 3: Sea $A \Rightarrow B \in \mathcal{L}_Y$. Un conjunto S-aditivo $M \in L^Y$ es un *modelo* para $A \Rightarrow B$ si, para todo $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$, $\boldsymbol{f}(A) \subseteq M$ implica que $\boldsymbol{f}(B) \subseteq M$.

Denotamos el conjuntos de los modelos de $A \Rightarrow B$ por $\mathcal{M}od(A \Rightarrow B)$. De forma usual, el conjunto de modelos para una teoría $\Sigma \subseteq \mathcal{L}_Y$ se define como

$$\mathcal{M}od(\Sigma) = \bigcap_{A \Rightarrow B \in \Sigma} \mathcal{M}od(A \Rightarrow B).$$

Por extensión, un $\mathbb{L}$-contexto $\mathbf{I} = \langle X, Y, I \rangle$ es un modelo de $A \Rightarrow B$ cuando $\{I_x \mid x \in X\} \subseteq \mathcal{M}od(A \Rightarrow B)$.

Definición 4: Sea $A \Rightarrow B \in \mathcal{L}_Y$ y $\Sigma \subseteq \mathcal{L}_Y$. La implicación $A \Rightarrow B$ se dice *semánticamente derivada* de la teoría Σ, denotado por $\Sigma \models A \Rightarrow B$, si $\mathcal{M}od(\Sigma) \subseteq \mathcal{M}od(A \Rightarrow B)$.

Introducimos por último en el presente resumen el sistema axiomático de la lógica.

Definición 5: El sistema axiomático está formado por un esquema de axioma y tres reglas de inferencia:

Reflexividad: infiere $A \Rightarrow A$,
Composición: de $A \Rightarrow B, A \Rightarrow C$ infiere $A \Rightarrow B \oplus C$,
Simplificación: de $A \Rightarrow B, C \Rightarrow D$ infiere $A \oplus (C \ominus B) \Rightarrow D$,
Extensión: de $A \Rightarrow B$ infiere $\boldsymbol{f}(A) \Rightarrow \boldsymbol{f}(B)$.

para todo $A, B, C, D \in L^Y$ y $\langle \boldsymbol{f}, \boldsymbol{g} \rangle \in S$.

Del modo habitual, se dice que una implicación $A \Rightarrow B \in \mathcal{L}_Y$ es sintácticamente derivada de (o inferida por) una teoría $\Sigma \subseteq \mathcal{L}_Y$, denotado por $\Sigma \vdash A \Rightarrow B$, si existe una secuencia $\sigma_1, \ldots, \sigma_n \in \mathcal{L}_Y$ tal que σ_n es $A \Rightarrow B$ y, para todo $1 \leq i \leq n$, una de las siguientes condiciones se cumple:

- $\sigma_i \in \Sigma$;
- σ_i es un axioma (Reflexividad);
- σ_i se obtiene aplicando reglas de inferencia (Composición, Simplificación o Extensión) a implicaciones de $\{\sigma_j \mid 1 \leq j < i\}$.

El siguiente teorema asegura que ambos pilares de la lógica, las derivaciones semánticas y sintácticas, coinciden.

Teorema 1 (Corrección y completitud): Para cualquier implicación $A \Rightarrow B \in \mathcal{L}_Y$ y cualquier teoría $\Sigma \subseteq \mathcal{L}_Y$, las siguientes afirmaciones se cumplen:

1. $\Sigma \vdash A \Rightarrow B$ implica $\Sigma \models A \Rightarrow B$.
2. Si L^Y es finito, $\Sigma \models A \Rightarrow B$ implica $\Sigma \vdash A \Rightarrow B$.

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