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Fuzzy information and contexts for designing Automatic Decision-making Systems*

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***Abstract*—The replacement of people by Automatic Decision-making Systems (ADS) has become a threat today. However, it seems that this replacement is unstoppable. Thus, the need for future and current ADS to perform their tasks as perfectly as possible is, more than a necessity an obligation. Hence, the design of these ADS must be carried out in accordance with the theoretical models on which they are to be built. From this point of view, this paper considers the classic definition of General Decision Making Problem and introduces two new key elements for building ADS: the nature of the information available and the context in which the problem is being solved. The new definition allows to cover different models and decision and optimization problems, some of which are presented for illustrative purposes.**

***Index Terms*—Fuzzy Information, Decision Making problems, Contexts**

OWA OPERATORS IN THE CALCULATION OF THE AVERAGE GREEN-HOUSE GASES EMISSIONS

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Abstract

This study proposes, through weighted averages and ordered weighted averaging operators, a new aggregation system for the analysis of average gases emissions. We introduce the ordered weighted averaging operators gases emissions, the induced ordered weighted averaging operators gases emissions, the weighted ordered weighted averaging operators gases emissions and the induced probabilistic weighted ordered weighted averaging operators gases emissions. These operators represent a new way of analyzing the average gases emissions of different variables like countries or regions. The work presents further generalizations by using generalized and quasi-arithmetic means. The article also presents an illustrative example regarding the calculations of the average gases emissions in the European region.

Keywords: Green-house gases emission, aggregation operators, decision making, ordered weighted average

1. Introduction

In the very recent decades, because of an enormous growth of the population and the necessity to provide food for them from one hand and the other hand an immethodical consumption of fossil fuel, our planet is experiencing an unexampled growth in terms of green-house gases (GHG) emission such as CO_2, CH_4 and N_2O in its atmosphere that cause an ascending amount of global warming year by year and a drastic climate change [6,20].

There are many works that study the ways that can lead the GHG emission toward the minimization. [19], evaluate the potential influence of vehicle electrification on grid infrastructure and road-traffic green-house emission. [5] study the impact of electrical power generation on GHG emission in Europe.

Besides, although these works exist but it seems vital to present a comprehensive forecast about the future of countries based on the experts' opinions to provide a clear plan and make a suitable decision to decrease this emission in any of the studied sectors and under various conditions.

Aggregation operators in the related literature with the aim of decision making are diverse and each of them can be used to collect the information [1,2,9-14, 21,23,25]. These techniques give importance to the variables according to certain available subjective or objective findings [17].

A very popular aggregation operator is the weighted average. This aggregation operator is flexible to use in a wide range of problems. Another popular aggregation operator is the ordered weighted average (OWA) [24]. The OWA operator provides a parametrized family of aggregation operators between the minimum and the maximum, weighting the data according to the attitudinal character of the decision-maker. Based on this operator and with the purpose of expanding it, many authors expand and generalize it [3,7,18,22]. There are several types for the concept of expanding and generalizing and the most important item is the form of integrating OWA operator with some key concepts such as, using the induced variables, the probability and the weighted average. [23] propose some new aggregation operators such as the induced ordered weighted geometric averaging (IOWGA) operator, generalized induced ordered weighted averaging (GIOWA) operator, hybrid weighted averaging (HWA) operator.

The purpose of this work is to concentrate on the analysis of the use of the aggregation operators in the calculation of green-house gases (GHG) emission with the aim of developing better decision-making techniques. To this end, the paper studies several aggregation operators including the WA [1], OWA [24], OWAWA and IOWAWA [11], IOWA [27], POWAWA and IPOWAWA operator [12]. With the use of each operator, a new operator for GHG emission is produced including the OWA GHG emission (OWAGE), induced OWA GHG emission (IOWAGE), ordered weighted averaging-weighted average GHG emission (OWAWAGE), induced OWAWA GHG emission (IOWAWAGE), probabilistic OWAWA GHG emission (POWAWAGE) and induced probabilistic OWAWA GHG emission (IPOWAWAGE).

The work also presents further generalizations by using generalized and quasi-arithmetic means obtaining the generalized OWAGE (GOWAGE). The aim of this approach is to show a more general framework in the analysis of averages by using complex aggregations including with geometric and quadratic averages. The study presents a wide range of particular types of aggregations under this approach.

The work presents an application regarding the calculation of the average gases emissions in Europe. For doing so, the paper considers a multi-expert aggregation problem where four experts analyze the expected average emissions of each European country for the next period. From, the analysis develops several aggregation methods based on the tools developed in the paper including the OWAGE, IOWG and OWAWAGE operators. The main advantage of the OWA operator is the possibility of under or overestimate the information according to the attitudinal character of the decision maker. Thus, depending on the degree of optimism or pessimism of the decision maker, the results may lead to different decisions and interpretations of the information.

2. Preliminaries

2.1. The weighted average (WA)

The WA [1] is one of the most common aggregation operators found in the literature. It has been used in a wide range of applications. It can be defined as follows.

Definition 1. A WA operator of dimension n is a mapping $WA: R^n \rightarrow R$ that has an associated weighting vector V, with $v_j \in [0,1]$ and $\sum_{i=1}^{n} v_i = 1$, such that:

$$WA(a_1,\ldots,a_n) = \sum_{i=1}^{n} v_i a_i \quad (1)$$

where a_i represents the argument variable.

2.2. The OWA operators

The OWA operator [24] is an aggregation operator that provides a parametrized family of aggregation operators that include the maximum, the minimum and the average criteria as especial cases and can be defined as follows.

Definition 2. An OWA operator of dimension n is a mapping $OWA: R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that sum of the weight is 1 and $w_j \in [0,1]$, then:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (2)$$

where b_j is the jth largest of the a_i.

2.3. The Induced OWA operator (IOWA)

The IOWA operator [27] is an extension of the OWA operator. The main difference between OWA and IOWA is that the reordering step is not developed with the values of the arguments a_i. In this case, the reordering step is carried out with order inducing variables. The IOWA operator also includes as particular cases the maximum, the minimum and the average criteria. It can be defined as follows.

Definition 3. An IOWA operator of dimension n is a mapping $IOWA: R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$IOWA(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j, \quad (3)$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i. u_i is the order-ranking variable and a_i is the argument variable.

2.4. The probabilistic ordered weighted averaging-weighted average (POWAWA)

The POWAWA [17] operator uses probabilities, weighted average and OWA in the same formulation. It unifies these three concepts by considering the degree of importance that each concept has in the aggregation, depending on the situation considered. The POWAWA operator is defined as follows.

Definition 4. A POWAWA operator of dimension n is a mapping $POWAWA: R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$POWAWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{4}$$

where b_j is the jth largest of the a_i, each argument a_i has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, a probability p_i with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0,1]$, $\hat{v}_j = C_1 w_j + C_2 v_j + C_3 p_j$, with C_1, C_2 and $C_3 \in [0,1]$, $C_1 + C_2 + C_3 = 1$ and v_j p_j are the weights v_i and p_i ordered according to b_j, that is to say, according to the jth largest of the a_i.

2.5. The Induced probabilistic OWAWA operator

The IPOWAWA [12] is an aggregation operator that extends POWAWA operator that uses order-inducing variables that represent complex reordering processes of an aggregation. Thus, it is an aggregation operator that uses induced variables, the probability, the weighted average and the OWA operator. Moreover, it can assess complex reordering processes by using order-inducing variables. Its main advantage is that it provides a more robust formulation than the POWAWA operator because it includes a wide range of cases. It can be defined as follows.

Definition 5. The IPOWAWA operator of dimension n is a mapping $IPOWAWA: R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$IPOWAWA(\langle u_1, e_1 \rangle, \langle u_2, e_2 \rangle, \ldots, \langle u_n, e_n \rangle) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{5}$$

where b_j is the a_i value of the IPOWAWA pair $\langle u_i, e_i \rangle$ having the jth largest u_i, u_i is the order-inducing variable, each argument a_i has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, a probability p_i with $\sum_{i=1}^{n} p_i$ and $p_i \in [0,1]$, $\hat{v}_j = C_1 w_j + C_2 v_j + C_3 p_j$, with C_1, C_2 and $C_3 \in [0,1]$, $C_1 + C_2 + C_3 = 1$, v_j and p_j are the weights v_i and p_i ordered according to b_j, that is to say according to the jth largest of the e_i.

3. Calculation of the average green-house gases (GHG) emission with OWA operators

The purpose of this paper is to calculate the average GHG emission. The average GHG emission represents a numerical value that reports the information of the GHG emission. To calculate this item, using many aggregation operators is possible likewise normal arithmetic mean. These possible aggregation operators could be WA, OWA, IOWA or a combination of them such as OWAWA, IOWAWA, etc. Through using them we prepare some possibilities for the future of GHG emission in different scenarios in a spectrum from the worst case to the best case based on experts' opinions.

The basic operator for analyzing a set of GHG emission is OWAGE. The OWAGE operator is an aggregation operator that analyses an average GHG emission under uncertainty situation. It can be defined as follows for the set of GHG emission $A = \{e_1, e_2, \ldots, e_n\}$:

$$OWAGE(e_1, e_2, \ldots, e_n) = \sum_{j=1}^{n} w_j f_j \tag{6}$$

where f_j is the jth largest of the e_i.

The other significant aggregation operator is the induced OWA (IOWA) that its reordering step is developed with order including variables. So, by using the IOWA operator we obtain IOWA GHG emission (IOWAGE) that can be defined as follows:

$$IOWAGE(\langle u_1, e_1 \rangle, \langle u_2, e_2 \rangle, \ldots, \langle u_n, e_n \rangle) = \sum_{j=1}^{n} w_j f_j \tag{7}$$

where f_j is the e_i value of the IOWA pair $\langle u_i, e_i \rangle$ having the jth largest u_i. u_i is the order-ranking variable and e_i is the argument variable.

It is important to mention that this operator is based on considering no extra information. One of the very important aspects of the average GHG emission is the importance of each of them and in other words, their weights in comparison with each other. To this end it is better to use some approaches of information aggregation that combine OWA operators and WA. In the literature there are some aggregation operators with this structure like, the WOWA operator [21], the hybrid average [9] and the OWAWA operators [8]. In this work we apply OWAWA to obtain the OWAWA GHG emission (OWAWAGE) and it is defined as follows for a set of GHG emission $A=\{e_1,e_2,\ldots,e_n\}$:

$$OWAWAGE(e_1,e_2,\ldots,e_n)=\sum_{j=1}^{n}\hat{v}_j f_j \tag{8}$$

where f_j is the *j*th largest of the e_i, each argument e_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i=1$ and $v_i\in[0,1]$, $\hat{v}_j=\beta w_j+(1-\beta)v_j$ with $\beta\in[0,1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the *j*th largest of the e_i.

To focus more deeply on our contributions, we implement IOWAWA which is a combination of IOWA operators and WA in the same formulation. By using the IOWAWA operator we obtain IOWAWA GHG emission (IOWAWAGE) that can be defined as follows:

$$IOWAWAGE(\langle u_1,e_1\rangle,\langle u_2,e_2\rangle,\ldots,\langle u_n,e_n\rangle)=\sum_{j=1}^{n}\hat{v}_j f_j \tag{9}$$

where f_j is the e_i value of the IOWAWA pair $\langle u_i,e_i\rangle$ having the *j*th largest u_i, u_i is the order including variable and e_i is the argument variable, each argument e_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i=1$ and $v_i\in[0,1]$, $\hat{v}_j=\beta w_j+(1-\beta)v_i$ with $\beta\in[0,1]$ and v_j is the weight (WA) v_i ordered according to f_j, that is, according to the *j*th largest u_i.

Besides, the other aspect that can be considered and leads results to a better form is probabilities in the attitudinal character of the decision-maker. For this reason, we apply POWAWA operator. By applying the Eq. (7) we could obtain the probabilistic OWAWA GHG emission (POWAWAGE). It can be defined as follows:

$$POWAWAGE(e_1,e_2,\ldots,e_n)=\sum_{j=1}^{n}\hat{v}_j f_j \tag{10}$$

where f_j is the *j*th largest of the e_i, each argument e_i has an associated weight v_i with $\sum_{i=1}^{n} v_i=1$ and $v_i\in[0,1]$, a probability p_i with $\sum_{i=1}^{n} p_i=1$ and $p_i\in[0,1]$, $\hat{v}_j=C_1w_j+C_2v_j+C_3p_j$, with C_1,C_2 and $C_3\in[0,1]$, $C_1+C_2+C_3=1$ and v_j p_j are the weights v_i and p_i ordered according to f_j, that is to say, according to the *j*th largest of the e_i.

Let us analyze the different families of IOWAWAGE and POWAWAGE in the following paragraphs

First, we are considering the two main cases of the IOWAWAGE operator that are found by analyzing the coefficient β. Basically:

- If $\beta=0$, we get the WA.
- If $\beta=1$, the IOWA operator.
- If $\beta=1$ and the ordered position of u_i is the same than the ordered position of f_i such that f_j is the *j*th largest of e_i, the OWA operator.
- Note that when β increases, we are giving more importance to the IOWAGE operator and when β decreases, we give more importance to the WA.

Another group of interesting families are the maximum-WAGE, the minimum-WAGE, the step-IOWAWAGE operator and the usual average.

- The maximum-WAGE is found when $w_p=1$ and $w_j=0,$ for all $j\neq p,$ and $u_p=Max\{e_i\}$.
- The minimum-WAGE is formed when $w_p=1$ and $w_j=0,$ for all $j\neq p,$ and $u_p=Min\{e_i\}$.

The arithmetic-WAGE is obtained when $w_j=1/n$ for all *j*, and the weighted average is equal to the OWA when the ordered position of *i* is the same as the ordered position of *j*. The arithmetic-WAGE (A-WAGE) can be formulated as follows:

$$A-WAGE(\langle u_1,e_1\rangle,\langle u_2,e_2\rangle,\ldots,\langle u_n,e_n\rangle)=\frac{1}{n}\beta a_i+(1-\beta)\sum_{i=1}^{n}v_ie_i, \tag{11}$$

Note that if $v_i=1/n,$ for all *i*, then, we get the unification between the arithmetic mean (or simple average) and the IOWAGE operator, that is, the arithmetic-IOWAGE (A-IOWAGE). The A-IOWAGE operator can be formulated as follows:

$$A-IOWAGE(\langle u_1,e_1\rangle,\langle u_2,e_2\rangle,\ldots,\langle u_n,e_n\rangle)=\beta\sum_{j=1}^{n}w_jb_j+(1-\beta)\frac{1}{n}e_i. \tag{12}$$

Following the OWA literature [13, 25, 27], we can develop many other families of IOWAWA operators such as:

- The olympic-IOWAWAGE operator ($w_1=w_n=0,$ and $w_j=1/(n-2)$ for all others).

- The general olympic-IOWAWAGE operator ($w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others $w_{j*} = 1/(n-2k)$, where $k < n/2$).
- The centered-IOWAWAGE (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).

Many other particular cases can be studied by looking at different expressions of the weighting vectors and the coefficients C_1, C_2 and C_3. For example:

- If $C_1 = 1$, we obtain the OWAGE operator.
- If $C_2 = 1$, the weighted GHG emission (WGE).
- If $C_3 = 1$, the probabilistic GHG emission (PGE).
- If $C_1 = 0$, the probabilistic weighted averaging GHG emission (PWAGE).
- If $C_2 = 0$, the probabilistic OWA GHG emission (POWAGE).
- If $C_3 = 0$, the OWAWA GHG emission (OWAWAGE) [14].

4. Generalizations with generalized and quasi-arithmetic means

Generalization of the OWA operators is possible to do by generalized and quasi-arithmetic averaging aggregation operators that as the most common one generalized OWA (GOWA) [26] and then quasi-arithmetic OWA (Quasi-OWA) [4] are formed. These functions apply a general framework including particular cases. The GOWA operator applied to the analysis of gases emissions is called GOWA gases emissions (GOWAGE) and is defined as follows.

Definition 6. A GOWAGE operator of dimension n is a mapping $GOWA: R^n \to R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, such that:

$$GOWAGE(e_1, e_2, ..., e_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \quad (13)$$

where b_j is the jth largest of the e_i, and λ is a parameter such that $\lambda \in (-\infty, \infty) - \{0\}$.

Like the section 3, this operator also has the particular cases of the maximum, the minimum and the generalized mean (GM). Besides, there are some special cases that can be obtained by maneuvering on the values of λ, such as:

- If $\lambda = 1$, the usual OWAGE operator.
- If $\lambda \to 0$, the ordered weighted geometric average gases emissions (OWGAGE).
- If $\lambda = 2$, the ordered weighted quadratic average gases emissions (OWQAGE).

Quasi-arithmetic OWA gases emissions (Quasi-OWAGE) operator is the other generalization that uses the quasi-arithmetic means instead of the generalized means. So, it replaces the parameter λ by a strictly continuous monotonic function g.

Definition 7. A Quasi-OWAGE operator of dimension n is a mapping Quasi-OWAGE: $R^n \to R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, then:

$$Quasi-OWAGE(e_1, e_2, ..., e_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_{(j)}\right) \right) \quad (14)$$

where b_j is the jth largest of the e_i and g is a strictly continuous monotonic function.

5. Conclusions

The purpose of this study is to concentrate on the analysis of the use of the aggregation operators in the calculation of GHG emission with the aim of developing better decision-making techniques. In this study we reviewed some of the important operators of the family of OWA. This review started with simple WA and continued with OWA operator. Moreover, we also analyzed some operators that form by combination of two or more aggregation operators. So, these operators are, IOWAGE, OWAWAGE, IOWAWAGE, POWAWAGE and IPOWAWAGE.

In addition, through these formulations, we found some particular cases in either IOWAWAGE or POWAWAGE operators such as, olympic-IOWAWAGE, S-IOWAWAGE, centered-IOWAWAGE, maximum, minimum and arithmetic probabilistic weighted average, and arithmetic probabilistic ordered weighted average. Furthermore, some other generalizations are developed by using generalized and quasi-arithmetic means obtaining the GOWAGE and the Quasi-OWAGE operators.

The study provides a simple example to review the function of two simple aggregations operators of average green-house gases emission. During this example we review weighted average gases emission (WAGE) and ordered weighted average gases emission (OWAGE) to represent the difference between the result of the calculation based on these operators.

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Fuzzy linguistic ranking model for Web Accessibility Test tools

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***Abstract*—There are dozens of tools to automatically evaluate web accessibility. Some are online, and some are toolbars to complement web browsers. In order to select the best Web Accessibility Test Tool, various aspects should be considered. Among the various aspects, the evaluation environment has an important role to assume in the evaluation criteria of the website. The ability to evaluate websites that require user permissions or they are freely accessible could affect the accessibility outcome due to limited or no access to the tool. In addition, the interpretability of the results differs from one tool to another, and it can be difficult to identify the areas of opportunity for improvement of the website evaluated. To select the best tool that matches experts' needs, it is important to have a group of experts in the area. These experts will give their opinions on the criteria according to which the tools will be evaluated. Each Web Accessibility Test Tool is an alternative in a decision making problem (DM). A DM which is evaluated by a group of experts is called a Multi-Expert Multi-Criteria (MEMC). Contrary to studies where the assessments are quantitative, this research uses Computing with Words (CW) processes. Because experts may have uncertainty at the time of issuing their evaluation, Intuitionistic Fuzzy Sets (IFS) are used to work with that uncertainty. Finally, a ranking of the evaluated tools is carried out by TOPSIS.**

***Keywords*—Intuitionistic Fuzzy Set (IFS), TOPSIS, Multi-Expert Multi-Criteria Linguistic Decision Making (MEMCLDM), Web Accesibility Tools Test, Ranking.**

I. Introduction

The World Wide Web Consortium (W3C) is an international community working towards international standards for the web. W3C dictates a series of standards –WCAG 1.0 [1] in 1999 and WCAG 2.0 [2] in 2008– to make web information accessible to everyone regardless of hardware, software, network infrastructure, language, culture, geographic location, or physical or mental ability. Currently there are several tools that evaluate the accessibility of websites automatically. The tools contain different features that may or may not facilitate the evaluation of the site depending on the context in which are applied. Choosing the right tool for the expert's needs is a decision making problem (DM). A DM is a typical problem that has different alternatives to choose from valued by experts in the topic.

In this document, a fuzzy linguistic model is proposed for evaluating accessibility tools through nine criteria: (1) Learnability, (2) Scope of application, (3) Displays element evaluation,ig (4) Accessibility level, (5) Accuracy, (6) CSS evaluation, (7) Reports, (8) Intuitivity, (9) Standardized output. Commonly, the valuation process is done using numerical scales. This research makes use of enhanced linguistic terms in order to take advantage of the knowledge and experience of the experts in a better way, since the evaluation is performed using natural language. The decision-making process is carried out by intuitionist linguistic representation using linguistic sets in the expert opinions on the criteria to be evaluated of the accessibility tools. Finally, using TOPSIS, the results are aggregated to rank the tools evaluated.

This document is structured as follows: Section II provides a descriptive summary of the preliminaries relating to the Intuitionist linguistic model as well as TOPSIS as a technique for the ranking of alternatives. In Section III, we present a Fuzzy linguistic ranking model. In Section IV, we apply a ranking model for Web Accessibility Test tools. Finally, in Section V, the conclusions are presented.

II. Preliminaries

This section describes the methodologies applied to the Multi-Expert Multi-Criteria Linguistic Decision Making (MEMCLDM) problem for ranking in the alternative selection.

A. *Multi Criteria Decision Making*

The Multi-Criteria Decision Making (MCDM) was introduced in the early 1970s. It is a tool used for problem assessment and decision making with multiple alternatives that are evaluated considering multiple criteria [1], [2].

MCDM often deal with different types of problems such as selection, ranking and classification problems. The aim on each kind of problem is different: (1) selection problems is expected to find the best alternative; (2) the ranking problems are aimed at determining the suitability of all alternatives, which is presented as a hierarchy from the best to the worst and (3) in the classification problems we want to know which alternatives belong to which class of a predefined set of ordered classes.

There are several methods of solving MCDM problems that are used to form a ranking of alternatives. The TOPSIS method [3] is one of them. TOPSIS is based on an aggregation function

[1] https://www.w3.org/TR/WAI-WEBCONTENT/
[2] https://www.w3.org/TR/WCAG20/

of the experts' evaluation scores and determines the best alternative by calculating the distances between the positive and negative ideal solution.

MCDM problems can be evaluated by various experts to be approached as MEMCLDM problems. These experts are usually people with experience in the subject to be assessed. Alternatives can be assessed quantitatively or qualitatively. To evaluate qualitative information, the use of the Fuzzy Set Theory (FST) [4], proposed by Zadeh in 1965, has been very successful. In order to achieve an efficient evaluation of the perception of the experts, the use of linguistic variables[5] and the procedures of Computation with Words (CW) [6], [7] are used in intelligent computer systems [8], [9], [10], [11].

B. Intuitionistic fuzzy representation model

Intuitionistic Fuzzy Set (IFS) was proposed by Atanassov in 1986. IFS [12] is characterized by having simultaneously a membership and a non-membership with a degree of hesitance. The IFS are models of information representation used to support decision making and are very useful because of the ability to express imprecise or uncertain information more flexibly than the traditional fuzzy sets [2], [13].

According to Atanassov [12] an IFS A, in the universe $X = \{x_1, x_2, \ldots, x_n\}$, it is represented as:

$$\widetilde{A} = \langle x_j, \mu_{\widetilde{A}}(x_j), \nu_{\widetilde{A}}(x_j)\rangle \mid x_j \in X$$

where $\mu_{\widetilde{A}}(x_j) \in [0,1]$ and $\nu_{\widetilde{A}}(x_j) \in [0,1]$ represents respectively the membership and the non-membership degrees of the element x_j. Then an IFS has the following requirement:

$$0 \leq \mu_{\widetilde{A}}(x_j) + \nu_{\widetilde{A}}(x_j) \leq 1$$

The function $\pi_{\widetilde{A}}(x_j)$ represents the degree of hesitancy of x_j and it is defined as:

$$\pi_{\widetilde{A}}(x_j) = 1 - \mu_{\widetilde{A}}(x_j) - \nu_{\widetilde{A}}(x_j)$$

Let α and β be two intuitionistic fuzzy sets, λ be a number. Hence, the main algebraic operations of any two intuitionistic fuzzy sets $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\beta = (\mu_\beta, \nu_\beta)$ can be defined in the following way [14] and [15]:

1) *Addition* $\oplus$*:*

$$\alpha \oplus \beta = (\mu_\alpha + \mu_\beta - \mu_\alpha\mu_\beta, \nu_\alpha\nu_\beta); \quad (1)$$

2) *Product* $\otimes$*:*

$$\alpha \otimes \beta = (\mu_\alpha\mu_\beta, \nu_\alpha + \nu_\beta - \nu_\alpha\nu_\beta); \quad (2)$$

3) *Scalar product:*

$$\lambda\alpha = (1 - (1 - \mu_\alpha)^\lambda, \nu_\alpha^\lambda), \lambda > 0; \quad (3)$$

4) *Scalar power:*

$$\alpha^\lambda = (\mu_{\alpha^\lambda}, 1 - (1 - \nu_\alpha)^\lambda), \lambda > 0. \quad (4)$$

The Intuitionistic Fuzzy Weighted Average (IFWA) aggregation operator was initially proposed by [15], it has been used to aggregate the individual opinions of decision makers [16], [17]. Let $R^{(k)} = (r_{ij}^{(k)})m \times n$ be an intuitionist decision-making matrix with the evaluations of each alternative A_i with $(i = 1, \ldots, m)$ and criterion C_j with $(j = 1, \ldots, n)$ by each decision maker DM_k with $(k = 1, \ldots, d)$. Let w_k be the weight of each decision maker DM_k where $\Sigma_{k=1}^n w_k = 1$. Then the final result of applying the $IFWA_w$ aggregation operatio operator is an IFS value given by Eq.(5):

$$\begin{aligned} IFWA_w &= IFWA_w(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(d)}) \\ &= w_1 r_{ij}^{(1)} \oplus w_2 r_{ij}^{(2)} \oplus w_3 r_{ij}^{(3)} \oplus \cdots \oplus w_k r_{ij}^{(d)} \\ &= [\chi, \psi, \delta] \end{aligned} \quad (5)$$

where

$$\begin{aligned} \chi &= 1 - \Pi_{k=1}^d (1 - \mu_{ij}^{(k)})^{w_k} \\ \psi &= \Pi_{k=1}^d (\nu_{ij}^{(k)})^{w_k} \\ \delta &= \Pi_{k=1}^d (1 - \mu_{ij}^{(k)})^{w_k} - \Pi_{k=1}^d (\nu_{ij}^{(k)})^{w_k}. \end{aligned}$$

Intuitionistic fuzzy representation has been widely used with multicriteria decision-making techniques [18] such as Intuitionistic fuzzy TOPSIS [17], Intuitionistic fuzzy AHP [19], Intuitionistic fuzzy VIKOR [20], Intuitionistic fuzzy ELECTRE [21], among others, with the purpose of order the alternatives.

C. Intuitionistic fuzzy TOPSIS

There are different techniques for the ranking of alternatives in MEMCLDM. TOPSIS is a technique which uses the order by similarity with an ideal solution [3]. TOPSIS is based on the fact that the alternative selected must be the one that contains the closest distance from the ideal solution and the furthest distance from the negative solution. The Fuzzy TOPSIS technique is widely applied in decision making [22] and it is considered as one of the best MCDM methods to solve problems. Due to its simplicity of application avoiding it application in alternatives ranking when a new alternative is inserted [23]. Boran et al. [16] proposes the Intuitionistic fuzzy TOPSIS to be applied following these steps:

1) Let $W_j = (\mu_j, \nu_j)$ be the intuitionistic fuzzy weight of each criteria C_j according to alternative A_i. Let $R' = (r'_{ij})_{m \times n}$ be the matrix of the aggregated intuitionistic fuzz sets with m alternatives and n criteria. Then the weighted normalized matrix is calculated by Eq.(6) and Eq.(7).

$$R' \otimes W_i = \{(\mu_{r'_{ij}}\mu_{W_j}, \nu_{r'_{ij}} + \nu_{W_j} - \nu_{r'_{ij}}\nu_{W_j}) \mid r'_{ij} \in R'\} \quad (6)$$

$$\pi_{r'_{ij}} = (1 - \nu_{r'_{ij}} - \nu_{W_j} - \mu_{r'_{ij}}\mu_{W_j} + \nu_{r'_{ij}}\nu_{W_j}) \quad (7)$$

2) The positive intuitionistic fuzzy ideal solution vector A^+ can be determined as:

$$A^+ = (\mu_{A^-}W_j, \nu_{A^+}W_j) \quad (8)$$

where

$$\mu_{A^+}W_j = \max_i \mu_{r'_{ij}} W_j$$
$$\nu_{A^+}W_j = \min_i \nu_{r'_{ij}} W_j.$$

3) The negative intuitionistic fuzzy ideal solution vector A^- can be determined as:

$$A^- = (\mu_{A^-}W_j, \nu_{A^-}W_j) \tag{9}$$

where

$$\mu_{A^-}W_j = min_i \mu_{r'_{ij}W_j}$$
$$\nu_{A^-}W_j = max_i \nu_{r'_{ij}W_j}.$$

4) Calculate the distance measurement, using the Euclidean distance. The separation of each alternative from the ideal solution is given as:

$$S_i^+ = \sqrt{\tfrac{1}{2n}\Sigma_{j=1}^{n}[(\mu_{r'_{ij}}W_j - \mu_{A^+}W_j)^2 + (\nu_{r'_{ij}}W_j - \nu_{A^+}W_j)^2 + (\pi_{r'_{ij}}W_j - \pi_{A^+}W_j)^2]}. \tag{10}$$

5) Similarly, the separation of the negative solution is given as:

$$S_i^- = \sqrt{\tfrac{1}{2n}\Sigma_{j=1}^{n}[(\mu_{r'_{ij}}W_j - \mu_{A^-}W_j)^2 + (\nu_{r'_{ij}}W_j - \nu_{A^-}W_j)^2 + (\pi_{r'_{ij}}W_j - \pi_{A^-}W_j)^2]}. \tag{11}$$

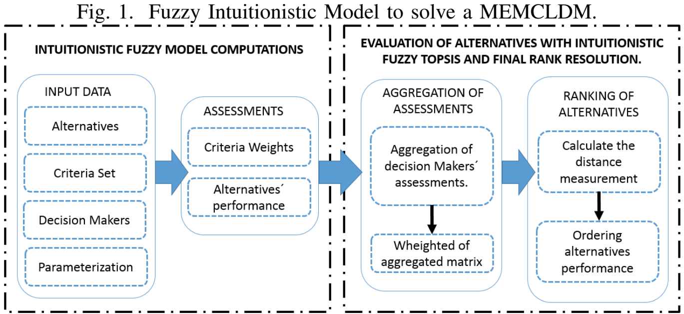
6) Then rank the order of preference by the relative proximity coefficient as:

$$RP_{A_i} = \frac{S_i^-}{S_i^- + S_i^+}. \tag{12}$$

III. A Fuzzy linguistic ranking model

The proposed model for the ranking problem of the list of tools for evaluating Web Accessibility Test tools consists of three basic stages: (1) Representation phase, (2) Aggregation phase and (3) Exploitation phase. Figure (1) presents the proposed phases:

Fig. 1. Fuzzy Intuitionistic Model to solve a MEMCLDM.



A. Representation phase

The first phase consist of the definition of Linguistic Terms Set (LTS) that will be used for decision making. Decision makers denoted as DM_k with $(k = 1, \ldots, d)$. Should be select a set of diverse DM that have experience and knowledge directly related to the problem to evaluate the available alternatives. The set of alternatives A_i with $(i = 1, \ldots, m)$ are evaluated using a set of criteria C_j with $(j = 1, \ldots, n)$. The weight W_j of each criterion C_j assigned by a coordinator using an Intuitionistic Fuzzy Linguistic Set (IFLS). It would be desirable to represent the input linguistic information with a representation able to express uncertainty and subjectivity in judgments.

B. Aggregation phase

The second phase consists of aggregating the decision makers judgments. The IFWA operator presented in Eq. (5) can be used to aggregate them into a matrix of group decision making $R' = (r'_{ij})_{m\times n}$, where m and n denotes the number of evaluated alternatives and criteria respectively. Next, a weighting process of the group decision making matrix R' is performed using the weight vector W_j and applying Eq. (6) and Eq. (7).

C. Exploitation phase

In the last step of the decision-making process, the performance of each alternative should be calculated using the distance measurement. The ideal intuitionistic fuzzy positive and negative solutions are obtained as Eq. (8) and Eq. (9). The distance between each aggregated weighted evaluation of alternatives' performances and the ideal intuitionistic fuzzy positive and negative solution is calculated by Eq. (10) and Eq. (11). The final score of the alternatives performance is calculated by relative proximity as in Eq. (12). Finally, the alternatives are ordered.

IV. A ranking model for Web Accessibility Test tools

It is desired to evaluate the level of accessibility of a web system. There are several tools supported by the World Wide Web Consortium (W3C) with different features. The DM set limits the main set to the six tools commonly used among them. The set of Web Accessibility Test (WAT) tools are evaluated in relation to a set of nine criteria. It must be decided which of this set of six tools is best suited to your needs. The tools considered are the following:

1) Wave[3] (c_1),
2) Achecher[4] (c_2),
3) eXaminator[5] (c_3),
4) AccessMonitor[6] (c_4),
5) Accessibility Check[7] (c_5),
6) TAW[8] (c_6).

A. Representation phase

For this case-study there are $d = 7$ decision makers who have knowledge in web development. The set of $m = 6$ WAT tools are assessed according to a set of $n = 9$ criteria described in Table (I). The linguistic intuitionistic variables in

[3] http://wave.webaim.org/
[4] https://achecker.ca/checker/index.php
[5] http://examinator.ws/
[6] http://www.acessibilidade.gov.pt/accessmonitor/
[7] http://www.etre.com/tools/accessibilitycheck/
[8] https://www.tawdis.net/

TABLE I
DESCRIPTION OF THE CRITERIA USED TO EVALUATE WEB ACCESSIBILITY TOOLS.

Criteria for evaluating accessibility assessment tools.		
Criteria C_j		Definition
c_1	Learnability	Enables simple and efficient learning.
c_2	Scope of application	Allows evaluation on sites with user permissions from an external site.
c_3	Displays element evaluation	Reveals the evaluations of each element to resolve any errors efficiently.
c_4	Accessibility level	Calculates the overall level of accessibility (A, AA, AAA).
c_5	Accuracy	Describes the assessments: (1) failed, 2 (warning) and (3) passed in a similar form as a manual assessment.
c_6	CSS evaluation	Evaluates accessibility of CSS content.
c_7	Reports	Generates reports that are easily interpreted by experts as ordinary users.
c_8	Intuitivity	Indicates inspected items for easy identification of errors, warnings and approvals.
c_9	Standardized output	Provides a report in standardized format: XML, JSON and YAML.

TABLE II
LINGUISTIC VARIABLES FOR THE IMPORTANCE WEIGHT OF EACH CRITERIA EXPRESSED AS IFS

Label	Short	μ	ν
Very High	VH	0.90	0.05
High	H	0.75	0.20
Medium	M	0.50	0.45
Low	L	0.35	0.60
Very Low	VL	0.10	0.90

Table (II) were used by a coordinator to define the weight W_j of each criteria C_j, as presented in Table (III). Also, using Table (II), each DM_k evaluates the performance of each alternative in each criterion. Table (IV) presents decision makers assessments.

B. Aggregation phase

Once the individual matrix of intuitionistic evaluations has been obtained, the matrix must be aggregated using the IFWA operator presented in Eq. (5). In this study-case, the weight of the decision makers were considered the same. The IFWA operator is used to aggregate them into a group decision making matrix. The aggregation results are presented in Table (V). Due to the space, only two assessments are displayed. Next, a weighting process of the group decision matrix R' is performed using the weight vector W_j and applying the Eq. (6) and Eq. (7). The results of the weighted aggregation for the criteria c_1 and c_2 are displayed in Table (VI).

TABLE III
INTUITIONISTIC FUZZY NUMBER (IFN) FOR EACH LINGUISTIC LABEL IN EACH CRITERIA

Intuitionistic Fuzzy Sets				
C_j	Criteria	Assessment	μ	ν
c_1	Learnability	Very High	0.90	0.05
c_2	Scope of application	Very High	0.90	0.05
c_3	Displays element evaluation	High	0.75	0.20
c_4	Accessibility level	Medium	0.50	0.45
c_5	Accurancy	High	0.75	0.20
c_6	Css evaluation	Medium	0.50	0.45
c_7	Reports	High	0.75	0.20
c_8	Intuition	High	0.75	0.20
c_9	Standardized output	Very High	0.90	0.05

C. Exploitation phase

In this step, the performance of each alternative should be calculated using the distance measurement from Fuzzy TOPSIS technique. The intuitionistic fuzzy ideal positive and negative solution are founded using Eq. (8) and Eq. (9). The distance between each aggregate weighted evaluation of the alternatives' results and the intuitionistic fuzzy ideal positive and negative solution is calculated using the Eq. (10) and Eq. (11). The results are presented by Table (VIII) and Table (IX) respectively.

Finally, the alternatives are ordered by relative proximity as in Eq. (12). The resulting order is presented in Table (X) with $a_1 > a_2 > a_6 > a_5 > a_3 > a_4$. Alternative a_1 is selected as the best tool with the best scores in 6 of 9 criteria evaluated. The selected tool can be highlight as very intuitive, since it marks errors and warnings in the html label, due this, it is easy to identify where are the errors.

V. CONCLUSION

This study proposed and tested a Multi-Expert Multi-Criteria Decision model in order to evaluate and select Web Accessibility Test tools that combines the Intuitionistic fuzzy representation and the TOPSIS technique. It fulfills several important characteristics for a decision making process:

- The use of linguistic variables instead of numerical scales enhances the assessment of alternatives in decision-making problems because the cognitive processes of human beings accept words rather than numbers;
- The use of Intuitionistic Fuzzy Set (ISF) is used due to the imprecision found in the parameterization since there may be a degree of hesitation. IFS takes into account the degree of membership, degree of non-membership and hesitancy;
- Finally, being able to rank the alternatives with fuzzy information allows a better interpretability of results for decision makers.

The Wave tool resulted best valuated in comparison with five tools. Wave is considered an easy learning tool, the plug-in installation to the browser is very simple. Wave has a high degree of intuitiveness and as a toolbar, it allows the evaluation of websites with the users' permissions. The assessment is displayed on each element evaluated, enabling rapid identification of errors and warnings. Experts rank the Wave tool first, considering the set of criteria. As a further research, it is suggested to explore other techniques in combination with linguistic fuzzy representations and compare their results with the proposal presented in this study.

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TABLE IV
Valuations of $d = 7$ Decision makers for $m = 6$ alternatives according to $n = 9$ criteria

Criteria	Alternatives	dm_1			dm_2			dm_3			dm_4			dm_5			dm_6			dm_7		
		Linguistic label	μ	ν	Linguistic label	μ	ν	Linguistic label	μ	ν	Linguistic label	μ	ν	Linguistic label	μ	ν	Linguistic label	μ	ν	Linguistic label	μ	ν
c_1	a_1	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20
	a_2	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	VH	0.90	0.05	L	0.35	0.60	H	0.75	0.20
	a_3	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45
	a_4	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60
	a_5	L	0.35	0.60	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45	L	0.35	0.60
	a_6	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20
c_2	a_1	VH	0.90	0.05	VH	0.90	0.05	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	VH	0.90	0.05
	a_2	H	0.75	0.20	VL	0.10	0.90	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	M	0.50	0.45	VL	0.10	0.90
	a_3	H	0.75	0.20	VH	0.90	0.05	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20	VL	0.10	0.90	H	0.75	0.20
	a_4	H	0.75	0.20	L	0.35	0.60	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20	H	0.75	0.20	VL	0.10	0.90
	a_5	M	0.50	0.45	H	0.75	0.20	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45
	a_6	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20
c_3	a_1	H	0.75	0.20	VH	0.90	0.05	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	VH	0.90	0.05
	a_2	VH	0.90	0.05	M	0.50	0.45	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	M	0.50	0.45	H	0.75	0.20
	a_3	H	0.75	0.20	VL	0.10	0.90	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20
	a_4	L	0.35	0.60	VL	0.10	0.90	H	0.75	0.20	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	VL	0.10	0.90
	a_5	M	0.50	0.45	VL	0.10	0.90	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	VL	0.10	0.90	M	0.50	0.45
	a_6	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20
c_4	a_1	H	0.75	0.20	VL	0.10	0.90	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	L	0.35	0.60	M	0.50	0.45
	a_2	H	0.75	0.20	H	0.75	0.20	VL	0.10	0.90	VH	0.90	0.05	H	0.75	0.20	VH	0.90	0.05	VL	0.10	0.90
	a_3	H	0.75	0.20	H	0.75	0.20	VL	0.10	0.90	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20
	a_4	L	0.35	0.60	VL	0.10	0.90	VL	0.10	0.90	VL	0.10	0.90	VL	0.10	0.90	VL	0.10	0.90	VL	0.10	0.90
	a_5	VL	0.10	0.90	VH	0.90	0.05	VL	0.10	0.90	VH	0.90	0.05	VH	0.90	0.05	VH	0.90	0.05	VL	0.10	0.90
	a_6	VH	0.90	0.05	H	0.75	0.20	VL	0.10	0.90	VH	0.90	0.05	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20
c_5	a_1	H	0.75	0.20	VH	0.90	0.05	M	0.50	0.45	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20
	a_2	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	M	0.50	0.45
	a_3	H	0.75	0.20	VL	0.10	0.90	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	L	0.35	0.60	M	0.50	0.45
	a_4	L	0.35	0.60	L	0.35	0.60	H	0.75	0.20	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60
	a_5	M	0.50	0.45	L	0.35	0.60	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	L	0.35	0.60	M	0.50	0.45
	a_6	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20
c_6	a_1	H	0.75	0.20	VL	0.10	0.90	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	VL	0.10	0.90	H	0.75	0.20
	a_2	H	0.75	0.20	M	0.50	0.45	VL	0.10	0.90	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	L	0.35	0.60
	a_3	H	0.75	0.20	L	0.35	0.60	VL	0.10	0.90	VH	0.90	0.05	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45
	a_4	VL	0.10	0.90	VL	0.10	0.90	M	0.50	0.45	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60
	a_5	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	M	0.50	0.45
	a_6	VH	0.90	0.05	M	0.50	0.45	VL	0.10	0.90	VH	0.90	0.05	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45
c_7	a_1	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45
	a_2	VH	0.90	0.05	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	M	0.50	0.45
	a_3	L	0.35	0.60	VH	0.90	0.05	M	0.50	0.45	H	0.75	0.20	VH	0.90	0.05	L	0.35	0.60	VL	0.10	0.90
	a_4	VL	0.10	0.90	VL	0.10	0.90	M	0.50	0.45	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	M	0.50	0.45
	a_5	L	0.35	0.60	L	0.35	0.60	M	0.50	0.45	VH	0.90	0.05	H	0.75	0.20	L	0.35	0.60	L	0.35	0.60
	a_6	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45	H	0.75	0.20	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45
c_8	a_1	VH	0.90	0.05	VH	0.90	0.05	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20
	a_2	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20	H	0.75	0.20	H	0.75	0.20	M	0.50	0.45	M	0.50	0.45
	a_3	H	0.75	0.20	VL	0.10	0.90	M	0.50	0.45	H	0.75	0.20	VH	0.90	0.05	L	0.35	0.60	M	0.50	0.45
	a_4	L	0.35	0.60	L	0.35	0.60	M	0.50	0.45	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60
	a_5	L	0.35	0.60	L	0.35	0.60	M	0.50	0.45	VH	0.90	0.05	VH	0.90	0.05	L	0.35	0.60	M	0.50	0.45
	a_6	H	0.75	0.20	H	0.75	0.20	M	0.50	0.45	H	0.75	0.20	VH	0.90	0.05	M	0.50	0.45	H	0.75	0.20
c_9	a_1	H	0.75	0.20	VL	0.10	0.90	VH	0.90	0.05	VL	0.10	0.90	VL	0.10	0.90	H	0.75	0.20	VL	0.10	0.90
	a_2	H	0.75	0.20	VH	0.90	0.05	L	0.35	0.60	VH	0.90	0.05	H	0.75	0.20	M	0.50	0.45	VL	0.10	0.90
	a_3	M	0.50	0.45	VL	0.10	0.90	L	0.35	0.60	VL	0.10	0.90	VL	0.10	0.90	L	0.35	0.60	VL	0.10	0.90
	a_4	L	0.35	0.60	VL	0.10	0.90	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60
	a_5	L	0.35	0.60	L	0.35	0.60	L	0.35	0.60	VL	0.10	0.90	M	0.50	0.45	L	0.35	0.60	VL	0.10	0.90
	a_6	VL	0.10	0.90	M	0.50	0.45	M	0.50	0.45	L	0.35	0.60	M	0.50	0.45	M	0.50	0.45	M	0.50	0.45

TABLE V
Aggregation Matrix of a_1 and a_2

	a_1			a_2		
	μ	ν	π	μ	ν	π
c_1	0.85	0.09	0.06	0.81	0.13	0.06
c_2	0.86	0.08	0.06	0.51	0.44	0.05
c_3	0.84	0.10	0.06	0.77	0.16	0.07
c_4	0.67	0.27	0.07	0.72	0.21	0.07
c_5	0.79	0.15	0.06	0.76	0.18	0.06
c_6	0.69	0.23	0.07	0.67	0.27	0.07
c_7	0.63	0.32	0.05	0.64	0.29	0.07
c_8	0.81	0.13	0.06	0.66	0.28	0.05
c_9	0.54	0.39	0.07	0.71	0.22	0.07

TABLE VI
Weighted Aggregation Matrix for a_1 and a_2

	a_1			a_2		
	μ	ν	π	μ	ν	π
c_1	0.77	0.14	0.10	0.73	0.17	0.10
c_2	0.77	0.13	0.10	0.46	0.46	0.07
c_3	0.63	0.28	0.09	0.58	0.33	0.10
c_4	0.33	0.60	0.07	0.36	0.56	0.07
c_5	0.59	0.32	0.09	0.57	0.35	0.08
c_6	0.35	0.58	0.08	0.33	0.60	0.07
c_7	0.47	0.45	0.07	0.48	0.43	0.09
c_8	0.61	0.31	0.09	0.50	0.43	0.08
c_9	0.49	0.42	0.09	0.64	0.26	0.10

TABLE VII
Ideal intuitionistic fuzzy positive and negative solutions.

	c_1			c_2			$\cdots$	c_9		
	μ	ν	π	μ	ν	π	$\cdots$	μ	ν	π
A+	0.77	0.14	0.10	0.77	0.13	0.10	$\cdots$	0.64	0.26	0.10
A-	0.57	0.32	0.11	0.46	0.46	0.07	$\cdots$	0.22	0.74	0.04

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TABLE VIII
POSITIVE DISTANCE OF a_1 AND a_2

	a_1			a_2		
	μ	ν	π	μ	ν	π
c_1	0.00	0.00	0.00	0.00	0.00	0.00
c_2	0.00	0.00	0.00	0.09	0.11	0.00
c_3	0.00	0.00	0.00	0.00	0.00	0.00
c_4	0.01	0.01	0.00	0.00	0.00	0.00
c_5	0.00	0.00	0.00	0.00	0.00	0.00
c_6	0.00	0.00	0.00	0.00	0.00	0.00
c_7	0.00	0.00	0.00	0.00	0.00	0.00
c_8	0.00	0.00	0.00	0.01	0.01	0.00
c_9	0.02	0.03	0.00	0.00	0.00	0.00

TABLE IX
NEGATIVE DISTANCE a_1 AND a_2

	a_1			a_2		
	μ	ν	π	μ	ν	π
c_1	0.04	0.03	0.00	0.02	0.02	0.00
c_2	0.09	0.11	0.00	0.00	0.00	0.00
c_3	0.12	0.14	0.00	0.09	0.11	0.00
c_4	0.07	0.10	0.00	0.08	0.12	0.00
c_5	0.07	0.08	0.00	0.06	0.07	0.00
c_6	0.04	0.05	0.00	0.03	0.04	0.00
c_7	0.05	0.06	0.00	0.05	0.07	0.00
c_8	0.11	0.12	0.00	0.05	0.05	0.00
c_9	0.07	0.10	0.00	0.17	0.23	0.00

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TABLE X
RANKING OF WEB ACCESSIBILITY TEST TOOLS

Solution	S^+	S^-	RP_{A_i}	Ranking
a_1	0.062	0.287	0.822	1
a_2	0.118	0.269	0.694	2
a_3	0.185	0.197	0.516	5
a_4	0.303	0.042	0.122	6
a_5	0.175	0.192	0.523	4
a_6	0.115	0.253	0.689	3

Estabilidad de las decisiones en el tiempo, ¿cómo medirla?

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***Resumen*—En esta contribución se propone un enfoque no tradicional en la medición de la estabilidad de las decisiones tomadas por los agentes a lo largo del tiempo. Esta contribución se centra en medir la estabilidad de las decisiones de los agentes en un contexto intertemporal bajo el supuesto de considerar las decisiones temporales como preordenes completos sobre las alternativas. Con este objetivo principal, se introduce el concepto general de *medida de la estabilidad de las decisiones* intertemporales, así como dos medidas particulares.**

***Index Terms*—Estabilidad de las decisiones en el tiempo, preferencias intertemporales, preordenes, efecto pérdida de memoria**

I. Introducción

El comportamiento humano implica decisiones intertemporales. En estas decisiones la persona debe evaluar los costes y los beneficios de decidir sobre un conjunto de alternativas en diferentes momentos del tiempo. Todos los días, los seres humanos tomamos decisiones intertemporales, por ejemplo cuando seleccionamos entre comer algo a media mañana o comer una comida completa a medio día o entre ir de vacaciones o aumentar la contribución al fondo de pensiones, etc.

El estudio de la elección intertemporal ha recibido atención desde diversas áreas de investigación como Economía, Psicología, Análisis de Decisión, Neurociencia, etc.

En términos generales, en la literatura especializada existen dos enfoques diferentes para tratar la medición de la estabilidad temporal de las decisiones de los individuos. El primero se centra en explorar las causas del comportamiento mediante procesos de optimización general (ver [1], [2] y [3], entre otros). En esta línea, es posible enmarcar los modelos económicos contemporáneos en los que los humanos toman decisiones intertemporales maximizando una función de utilidad de descuento exponencial. El segundo enfoque, proporcionado por los psicólogos, es el empírico. El comportamiento humano en las elecciones intertemporales se estudia mediante datos empíricos recopilados de los laboratorios (ver [4], [5], [6] y [7]).

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El objetivo de esta contribución es aportar un nuevo enfoque a la medición de la estabilidad de las decisiones intertemporales. En particular, se pretende desarrollar una nueva herramienta capaz de analizar el comportamiento humano en la toma de decisiones intertemporales y de medir la estabilidad de esas decisiones. Este trabajo está inspirado en la metodologa propuesta por González-Arteaga, de Andrés Calle y Peral [8], [9] donde la noción de estabilidad de las decisiones intertemporales se considera en la misma línea que la noción de cohesión. En [8] los agentes establecen sus preferencias sobre una única alternativa de manera dicotómica. La suposición de opiniones dicotómicas y de considerar una única alternativa en este contexto particular podría limitar y perturbar los resultados del análisis del comportamiento debido a que la evidencia sugiere que los humanos pueden experimentar dificultades para expresar el conocimiento incierto de forma dicotómica [10].

En consecuencia, esta investigación se centra en un problema intertemporal de toma de decisiones bajo un marco general, es decir, los agentes expresan sus decisiones temporales sobre un conjunto finito de alternativas mediante preordenes completos en diferentes momentos de tiempo. Por tanto, el objetivo de esta aportación es determinar cuánto de estables son las decisiones de los agentes a lo largo del tiempo y para ello se define una nuevo concepto general, la *medida de estabilidad de decisiones* en el tiempo. Por otra parte, se proponen dos medidas específicas de estabilidad de las decisiones, la *medida local de estabilidad* y la *medida global de estabilidad*.

La estructura general de esta contribución se divide en tres secciones. La Sección II introduce la notación empleada así como los conceptos básicos utilizados. En la Sección III se define el concepto general de medida de estabilidad de las decisiones intertemporales. Además, en esta sección se introducen las dos clases específicas consideradas. En la Sección IV se incluye un ejemplo real sobre la estabilidad de las decisiones temporales en educación de algunos países basado en los datos procedentes del informe PISA. Finalmente se proporcionan algunas conclusiones e ideas sobre investigaciones futuras.

II. Notación

Sea $\mathbf{N} = \{1, \ldots, n\}$, $n > 1$ un conjunto de individuos, expertos o miembros de una sociedad. Sea $\mathbf{X} = \{x_1, \ldots, x_k\}$ un conjunto finito de alternativas, $\mid \mathbf{X} \mid \geq 2$. Por simplicidad, en ocasiones la alternativa x_s será denotada por s.

Los miembros de la sociedad establecen sobre el conjunto de alternativas rankings mediante preordenes completos, $\mathcal{R}$. $\mathbf{W}(\mathbf{X})$ denota el conjunto de todos los preordenes completos sobre $\mathbf{X}$.

Sea $\mathcal{R} \in \mathbf{W}(\mathbf{X})$ un preorden completo, entonces $x_i \succ_{\mathcal{R}} x_j$ significa que la alternativa x_i es estrictamente preferida a la alternativa x_j, $x_i \sim_{\mathcal{R}} x_j$ significa que la alternativas son igualmente preferidas y por último $x_i \succcurlyeq_{\mathcal{R}} x_j$ significa que la alternativa x_i es al menos tan preferida como la alternativa x_j.

Denotaremos por $\mathcal{R}_i \in \mathbf{W}(\mathbf{X})$ a la *decisión temporal* que la sociedad toma sobre el conjunto de alternativas $\mathbf{X}$ en el momento del tiempo $t_i \in \mathbf{T}$, $\mathbf{T} = \{t_0, \ldots, t_T\}$. Una permutación π sobre las alternativas $\{x_1, \ldots, x_k\}$ determina otro preorden ${}^{\pi}\mathcal{R}$ tal que: $x_i \succcurlyeq_{{}^{\pi}\mathcal{R}} x_j \iff x_{\pi^{-1}(i)} \succcurlyeq_{\mathcal{R}} x_{\pi^{-1}(j)}$ para $i, j \in \{1, \ldots, k\}$.

Sea $\mathcal{P} = (\mathcal{R}_0, \ldots, \mathcal{R}_T) \in \mathbf{W}(\mathbf{X}) \times \ldots \times \mathbf{W}(\mathbf{X}) = \mathbf{W}(\mathbf{X})^{T+1}$ un *perfil de decisión temporal* sobre el conjunto de alternativas $\mathbf{X}$. El elemento $\mathcal{R}_i \in \mathcal{P}$ representa la decisión temporal de la sociedad sobre el conjunto de alternativas $\mathbf{X}$ en el momento del tiempo t_i, $i \in \{0, \ldots, T\}$. El reverso del perfil de decisión temporal, denotado por P^{-1}, es el perfil $(\mathcal{R}_0^{-1}, \ldots, \mathcal{R}_T^{-1})$ donde $x_i \succcurlyeq_{\mathcal{R}^{-1}} x_j \iff x_j \succcurlyeq_{\mathcal{R}} x_i$.

Una permutación π de las alternativas $\{x_1, \ldots, x_k\}$ determina un perfil de decisión temporal ${}^{\pi}\mathcal{P}$ donde cada decisión temporal es permutada de acuerdo a π. Una permutación σ de los momentos del tiempo $\{t_0, \ldots, t_T\}$ determina una permutación sobre el perfil de decisión temporal $\mathcal{P}^{\sigma}$ mediante la permutación de las decisiones temporales de acuerdo a σ.

Tratar con información ordinal implica necesariamente establecer de manera precisa la forma en que se representa. La primera discusión y análisis formal sobre la transformación de información ordinal en valores numéricos surgió con el trabajo de Borda [11]. Posteriormente, se han propuesto diversos procedimientos para este fin como [12], [13] y [14], entre otros.

La elección de un procedimiento de codificación robusto es un aspecto esencial para trabajar con información ordinal y obtener resultados consistentes. En esta contribución se utiliza el método de codificación propuesto en [15], el cuál está caracterizado.

Dada una decisión temporal $\mathcal{R}_i \in \mathbf{W}(\mathbf{X})$, su *vector de codificación canónica* es

$$\mathbf{c}_{\mathcal{R}_i} = (c_1^{\mathcal{R}_i}, \ldots, c_k^{\mathcal{R}_i}) \in (\{1, \ldots, k\})^k$$

donde $c_j^{\mathcal{R}_i}$ es el número de alternativas que son clasificadas al menos tan buenas como x_j en el momento del tiempo t_i, esto es $c_j^{\mathcal{R}_i} = \mid \{q : x_j \succcurlyeq_{\mathcal{R}_i} x_q\}$.

El conjunto de todas las posibles codificaciones canónicas asociadas a $\mathbf{W}(\mathbf{X})$ es denotado por $\mathbf{F} = \mathbf{F}(\mathbf{W}(\mathbf{X}))$.

Dado un perfil de decisión temporal $\mathcal{P} = (\mathcal{R}_0, \ldots, \mathcal{R}_T) \in \mathbf{W}(\mathbf{X})^{T+1}$, su *perfil de codificación canónica* es una matriz $k \times (T+1)$, $\mathcal{M}_{\mathcal{P}} = (\mathbf{c}_{\mathcal{R}_0}, \ldots, \mathbf{c}_{\mathcal{R}_T}) \in \mathbb{M}_{k \times (T+1)}$ donde la columna i-ésima, denotada por $\mathbf{c}_{\mathcal{R}_i}$ representa la codificación canónica de la decisión temporal $\mathcal{R}_i$ en el momento $t_i \in \mathbf{T}$.

Una permutación π de las alternativas $\{x_1, \ldots, x_k\}$ determina un perfil de codificación canonica ${}^{\pi}M_{\mathcal{P}}$ mediante la permutación de las filas de $M_{\mathcal{P}}$: la fila i del perfil ${}^{\pi}M_{\mathcal{P}}$ es la fila $\pi(i)$ del perfil $M_{\mathcal{P}}$. Una permutación σ de los momentos del tiempo $\{t_0, \ldots, t_T\}$ determina una permutación sobre el perfil de de codificación canonica $M_{\mathcal{P}}^{\sigma}$ mediante la permutación de las columnas de $M_{\mathcal{P}}$: la columna i del perfil $M_{\mathcal{P}}^{\sigma}$ es la columna $\sigma(i)$ del perfil $M_{\mathcal{P}}$.

III. Medición de la estabilidad de las decisiones en el tiempo

Esta sección está dedicada a introducir nuestra propuesta general para la medición de estabilidad de las decisiones a lo largo del tiempo, así como dos medias particulares.

Definición 1: Sea $\mathcal{P} \in \mathbf{W}(\mathbf{X})^{T+1}$ un perfil de decisión temporal. Una *medida de estabilidad de las decisiones* en el tiempo sobre $\mathcal{P} \in \mathbf{W}(\mathbf{X})^{T+1}$ es una función $\mu : \mathbf{W}(\mathbf{X})^{T+1} \longrightarrow [0, 1]$ que asigna a cada perfil de decisión temporal un valor en el intervalo unidad $\mu(\mathcal{P})$ con las siguientes propiedades:

i. Estabilidad máxima de la decisión:

$$\mu(\mathcal{P}) = 1 \iff \mathcal{R}_0 = \ldots = \mathcal{R}_T$$

ii. Neutralidad de la decisión sobre las alternativas:

$$\mu({}^{\pi}\mathcal{P}) = \mu(\mathcal{P})$$

para cada permutación π sobre el conjunto de las alternativas.

Definición 2: Sea $\mathcal{P} = (\mathcal{R}_0, \ldots, \mathcal{R}_T) \in \mathbf{W}(\mathbf{X})^{T+1}$ un perfil de decisión temporal y $\mathcal{M}_{\mathcal{P}} = (\mathbf{c}_{\mathcal{R}_0}, \ldots, \mathbf{c}_{\mathcal{R}_T}) \in \mathbb{M}_{k \times (T+1)}$ su correspondiente matriz de codificación canónica. La *medida local de estabilidad* entre las decisiones temporales $\mathcal{R}_{i-1}$ y $\mathcal{R}_i$ es una aplicación $\theta_i : \mathbf{W}(\mathbf{X})^2 \longrightarrow [0, 1]$ tal que

$$\theta_i(\mathcal{P}) = \theta_{[i-1,i]}\ (\mathcal{P}) = 1 - \frac{\| \mathbf{c}_{\mathcal{R}_{i-1}} - \mathbf{c}_{\mathcal{R}_i} \|_1}{r}$$

donde $\mathbf{c}_{\mathcal{R}_{i-1}}$ y $\mathbf{c}_{\mathcal{R}_i}$ son los vectores de codificación canónica asociados a las decisiones temporales tomadas en t_{i-1} y t_i, respectivamente; $\| \cdot \|_1$ denota la norma l_1, y por tanto $\| \mathbf{c}_{\mathcal{R}_{i-1}} - \mathbf{c}_{\mathcal{R}_i} \|_1 = \sum_{h=1}^{k} \mid c_h^{\mathcal{R}_{i-1}} - c_h^{\mathcal{R}_i} \mid$, finalmente $r = \max_{\mathbf{c}, \mathbf{c}' \in \mathbf{F}} \| \mathbf{c} - \mathbf{c}' \|_1$.

Por consiguiente:

$$\theta_i(\mathcal{P}) = \theta_{[i-1,i]}\ (\mathcal{P}) = 1 - \frac{\sum_{h=1}^{k} \mid c_h^{\mathcal{R}_{i-1}} - c_h^{\mathcal{R}_i} \mid}{\max\limits_{\mathbf{c},\mathbf{c}' \in \mathbf{F}} \parallel \mathbf{c} - \mathbf{c}' \parallel_1}$$

La medida local de estabilidad entre dos decisiones es una medida de estabilidad dado que verifica las propiedades:

i. Estabilidad máxima de la decisión:

$$\theta_i(\mathcal{P}) = \theta_{[i-1,i]}\ (\mathcal{P}) = 1 \Longleftrightarrow \mathcal{R}_{i-1} = \mathcal{R}_i$$

ii. Neutralidad de la decisión sobre las alternativas:

$$\theta_i(^{\pi}\mathcal{P}) = \theta_{[i-1,i]}\ (^{\pi}\mathcal{P}) = \theta_{i-1,i}\ (\mathcal{P}) = \theta_i(\mathcal{P})$$

para cada permutación π sobre las alternativas $\{1, \ldots, k\}$.

Proposición 1: La máxima inestabilidad entre dos decisiones temporales viene dada por el diámetro de la envolvente convexa del conjunto de todas las codificaciones canónicas asociadas a $\mathbf{W}(\mathbf{X})$, i.e., $\mathbf{F} = \mathbf{F}(\mathbf{X})$ para la norma l_1:

$$r = \max_{u,v \in Conv(\mathbf{F})} \parallel u - v \parallel_1$$

donde

$$Conv(\mathbf{F}) = \left\{ \sum_{j=1}^{|\mathbf{F}|} \alpha_j \cdot \mathbf{c}_j :\ 0 \leq \alpha_j \leq 1,\ \sum_{j=1}^{|\mathbf{F}|} \alpha_j = 1 \right\}.$$

Proposición 2: La máxima inestabilidad entre dos decisiones temporales esta determinada por las siguientes expresiones:

- Para $\mathbf{X} = \{x_1, \ldots, x_k\}$ con k par:

$$r = \frac{3}{4}k^2 - \frac{k}{2}$$

- Para $\mathbf{X} = \{x_1, \ldots, x_k\}$ con k impar:

$$r = \frac{3}{4}k^2 - \frac{k}{2} - \frac{1}{4}$$

Con el objetivo de mejorar la comprensión de la notación y de las definiciones introducidas, se presenta el siguiente ejemplo ilustrativo.

Ejemplo 1: Sea $\mathbf{X} = \{x_1, x_2, x_3\}$ un conjunto de tres alternativas ($k = 3$). Todos los posibles preordenes, $\mathcal{R} \in \mathbf{W}(\mathbf{X})$, para esas tres alternativas están recogidos en la primera columna de la Tabla 1. La segunda columna incluye la codificación canónica de estos preordenes, $\mathbf{c}_{\mathcal{R}}$.

Supongamos ahora dos momentos en el tiempo t_1 y t_2 donde se han tomado las decisiones temporales $\mathcal{R}_1$ y $\mathcal{R}_2$, respectivamente.

$$\mathcal{R}_1 : x_1 \sim x_2 \succ x_3$$
$$\mathcal{R}_2 : x_1 \succ x_2 \succ x_3$$

$\mathcal{R}$	$\mathbf{c}_{\mathcal{R}}$
$x_1 \sim x_2 \sim x_3$	$(3,3,3)$
$x_1 \sim x_2 \succ x_3$	$(3,3,1)$
$x_1 \succ x_2 \sim x_3$	$(3,2,2)$
$x_2 \succ x_1 \sim x_3$	$(2,3,2)$
$x_3 \succ x_1 \sim x_2$	$(2,2,3)$
$x_1 \succ x_2 \succ x_3$	$(3,2,1)$
$x_1 \succ x_3 \succ x_2$	$(3,1,2)$
$x_2 \succ x_1 \succ x_3$	$(2,3,1)$
$x_2 \succ x_3 \succ x_1$	$(1,3,2)$
$x_3 \succ x_1 \succ x_2$	$(2,1,3)$
$x_3 \succ x_2 \succ x_1$	$(1,2,3)$

Cuadro I: Preordenes y codificación canónica para $k = 3$.

Para analizar el nivel de estabilidad de las decisiones tomadas en t_1 y t_2 se utiliza la medida local de estabilidad proporcionada por la Definición 2:

$$\theta_2(\mathcal{P}) = \theta_{[1,2]}\ (\mathcal{P}) = 1 - \frac{\sum_{h=1}^{3} \mid c_h^{\mathcal{R}_1} - c_h^{\mathcal{R}_2} \mid}{\max\limits_{\mathbf{c},\mathbf{c}' \in \mathbf{F}} \parallel \mathbf{c} - \mathbf{c}' \parallel_1}$$

En este caso, la envolvente convexa del conjunto de todas las codificaciones canónicas está representada desde diferentes perspectivas en las Figuras 1 y 2.

Aplicando los resultados de las Proposiciones 1 y 2, el diámetro de esta envolvente convexa viene dado por la expresión:

$$r = \frac{3}{4}k^2 - \frac{k}{2} - \frac{1}{4} = \frac{3}{4} \cdot 3^2 - \frac{3}{2} - \frac{1}{4} = 5$$

Por tanto la estabilidad local de las decisiones $\mathcal{R}_1$ y $\mathcal{R}_2$ es:

$$\theta_2(\mathcal{P}) = 1 - \frac{\mid 3-3 \mid + \mid 3-2 \mid + \mid 1-1 \mid}{5} = 0.8$$

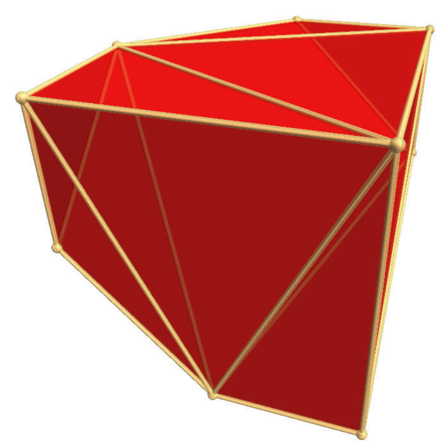

Figura 1: Representación de la envolvente convexa para $k = 3$

Definición 3: Sea $\mathcal{P} = (\mathcal{R}_0, \ldots, \mathcal{R}_T) \in \mathbf{W}(\mathbf{X})^T$ un perfil de decisión temporal y $\mathcal{M}_{\mathcal{P}} = (\mathbf{c}_{\mathcal{R}_0}, \ldots, \mathbf{c}_{\mathcal{R}_T}) \in \mathbb{M}_{k\times(T+1)}$

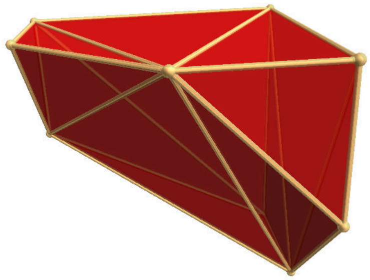

Figura 2: Representación de la envolvente convexa para $k = 3$

su correspondiente matriz de codificación canónica. La *medida global de estabilidad de las decisiones* en el tiempo sobre $\mathcal{P} \in \mathbf{W}(\mathbf{X})^{T+1}$ es una función $\Theta : W(x)^{T+1} \times R^{+} \longrightarrow [0,1]$ definida por:

$$\Theta(\mathcal{P},\lambda) = \sum_{i=1}^{T} w_i \cdot \theta_i(\mathcal{P}) = \sum_{i=1}^{T} w_{[i-1,i]} \cdot \theta_{[i-1,i]}\ (\mathcal{P})$$

donde $w_i = w_{[i-1,i]} = A \cdot e^{-\lambda(T-i+1)}$, $A = \dfrac{e^{\lambda}-1}{1-e^{-\lambda T}}$ con $\sum_{i=1}^{T} w_i = 1$ y $\lambda \geq 0$.

El parámetro λ recoge el efecto pérdida de memoria. Cuando $\lambda = 0$ los agentes recuerdan las decisiones tomadas en todos los momentos del tiempo con la misma intensidad. Mientras que cuando $\lambda > 0$, los agentes tienen un efecto pérdida de memoria positivo, esto es, los agentes recuerdan con mayor intensidad las últimas decisiones.

La medida globlal de estabilidad de las decisiones es una medida de estabilidad de decisiones dado que verifica las propiedades:

i. Estabilidad máxima de la decisión:

$$\Theta(\mathcal{P},\lambda) = 1 \Longleftrightarrow \mathcal{R}_0 = \ldots = \mathcal{R}_T$$

ii. Neutralidad de la decisión sobre las alternativas:

$$\Theta(^{\pi}\mathcal{P},\lambda) = \Theta(\mathcal{P},\lambda)$$

para cada permutación π sobre las alternativas $\{1,\ldots,k\}$.

El cumplimiento de estas propiedades por parte de la medida global es consecuencia directa del cumplimiento de las mismas por parte de la medida local.

IV. Caso de estudio: Estabilidad de decisiones en Educación

Con el objetivo de poner en relieve la aplicabilidad de la metodología propuesta, en esta sección se presenta un caso real de estudio. En concreto, se analiza y se mide la estabilidad a lo largo del tiempo de los rankings sobre rendimiento académico proporcionados por el informe PISA[1] cada tres años a nivel mundial.

El informe PISA es un estudio realizado por la OCDE desde el año 2000 cuyo objetivo principal es evaluar el rendimiento académico de estudiantes en Matemáticas, Ciencias y comprensión lectora[2]. Mediante la realización de este informe la OCDE proporciona datos comparables que permiten a los diversos países mejorar sus políticas de educación.

Para realizar el estudio de caso sobre la estabilidad de las decisiones temporales de los paises en Educación se han utilizado los datos proporcionados por el informe PISA para los años 2000, 2003, 2006, 2009, 2012 y 2015 para 25 países[3].

En las Tablas II, IV y III se muestran los rankings sobre los países analizados en las tres habilidades incluidas en el informe: Matemáticas, comprensión lectora y Ciencias, respectivamente. Estos rankings constituyen los distintos perfiles de decisión temporal teniendo en cuenta la totalidad de estudiantes.

Además de considerar estos rankings, en esta contribución se ha analizado la estabilidad intertemporal de los rankings proporcionados por el informe PISA según el sexo de los estudiantes aunque los perfiles correspondientes no se detallan en esta contribución.

Por otra parte y para analizar como el efecto pérdida de memoria afecta a la estabilidad de las decisiones intertemporales se han considerado cinco valores diferentes del parámetro λ: $0, 0.25, 0.5, 0.75$ y 1. Los resultados se muestran en la Tabla V y en las Figuras IV, IV y IV.

Tal y como se puede comprobar en la Tabla V y en las Figuras IV, IV y IV, la medida de estabilidad toma valores bastante altos en todos los países situándose entorno al $(0.8 - 0.9)$ independientemente del valor de λ. La independencia de los resultados con respecto al parámetro puede deberse a que los cambios que afectan al nivel educativo de los países se perciben de forma lenta y no drástica, lo que es un claro indicativo de la estabilidad de las mediciones.

V. Conclusiones y trabajos futuros

La investigación sobre la estabilidad de las decisiones intertemporales se ha realizado principalmente en Economía. El objetivo del presente trabajo es proporcionar una nueva metodología desde una perspectiva no tradicional al problema de medir la estabilidad de las decisiones a lo largo del tiempo. La contribución se centra en un marco de evaluación donde los agentes expresan sus opiniones sobre diferentes alternativas en diferentes momentos del tiempo mediante preordenes completos. Se introduce la noción general de medida de

[1]Programa Internacional para la Evaluación de Estudiantes.

[2]El estudio se basa en el análisis del rendimiento de estudiantes de 15 años a partir exámenes estandarizados.

[3]AUS= Australia, BEL = Bélgica, BRA = Brasil, CAN = Canada, CHE = Suiza, CZE = Repblica Checa, DEU = Alemania, DNK = Dinamarca, ESP = España, FIN = Finalancia, FRA = Francia, GRC = Grecia, HUN = Hungría, IDN = Indonesia, IRL = Irlanda, ISL = Islandia, ITA = Italia, JPN = Japón, KOR = Corea del Sur, MEX = México, NOR = Noruega, NZL = Nueva Zelanda, POL = Polonia, PRT = Portugal, SWE = Suecia.

2003	2006	2009	2012	2015
FIN	FIN	KOR	KOR	JPN
KOR	KOR	FIN	JPN	KOR
JPN	CHE	CHE	CHE	CHE
CAN	CAN	JPN	FIN	CAN
BEL	JPN	CAN	CAN, POL	DNK, FIN
CHE	NZL	NZL	BEL	BEL
AUS	AUS, BEL	BEL	DEU	DEU
NZL	DNK	AUS	AUS	IRL, POL
CZE	CZE	DEU	IRL	NOR
ISL	ISL	ISL	DNK, NZL	NZL
DNK	DEU	DNK	CZE	AUS, SWE
FRA	SWE	NOR	FRA	FRA
SWE	IRL	FRA	ISL	CZE, PRT
DEU, IRL	FRA	POL	NOR	ITA
NOR	POL	SWE	PRT	ISL
HUN, HUN	POL	CZE	ITA	ESP
ESP	NOR	HUN	ESP	HUN
ITA, PRT	ESP	IRL, PRT	SWE	GRC
GRC	PRT	ESP, ITA	HUN	MEX
MEX	ITA	GRC	GRC	IDN
IDN	GRC	MEX	MEX	BRA
BRA	MEX	BRA	BRA	
	IDN	MEX	IDN	
	BRA			

Cuadro II: Rankings de los países en distintos años en Matemáticas

2006	2009	2012	2015
FIN	FIN	JPN	JPN
CAN	JPN	FIN	FIN
JPN	KOR	KOR	CAN
NZL	NZL	POL	KOR
AUS	CAN	CAN	NZL
KOR	AUS	DEU	AUS
DEU	DEU	IRL	DEU
CZE	CHE	AUS	CHE
CHE	IRL, POL	NZL	IRL
BEL	BEL	CHE	BEL, DNK
IRL	HUN	CZE	POL, PRT
HUN	CZE	BEL	NOR
SWE	NOR	FRA	FRA
POL	DNK	DNK	CZE, ESP, SWE
DNK	FRA	ESP	ITA
FRA	ISL	NOR	HUN
ISL	SWE	HUN, ITA	ISL
ESP	PRT	PRT	GRC
NOR	ITA	SWE	MEX
ITA	ESP	ISL	IDN
PRT	GRC	GRC	BRA
GRC	MEX	MEX	
MEX	BRA	BRA	
IDN	IDN	IDN	
BRA			

Cuadro III: Rankings de los países en distintos años en Ciencias

estabilidad de las decisiones intertemporales así como dos formulaciones específicas, una de ellas prestando especial atención a cualesquiera dos sucesivos momentos de tiempo. Finalmente la metodología propuesta se aplica al caso de la estabilidad de las decisiones de los países respecto a la Educación según informes de la OCDE.

En cuanto a las futuras líneas de investigación, existen diversos aspectos que podrían ser analizados. Por una parte, la línea de actuación más inmediata sería la aplicación de la novedosa propuesta de esta contribución a una amplia gama de campos tales como la estabilidad de las decisiones de los

2000	2003	2006	2009	2012	2015
FIN	FIN	KOR	KOR	JPN	CAN
CAN	KOR	FIN	FIN	KOR	FIN
NZL	CAN	CAN	CAN	FIN	IRL
AUS	AUS	NZL	NZL	CAN, IRL	KOR
IRL	NZL	IRL	JPN	pol	JPN
KOR	IRL	AUS	AUS	AUS, NZL	NOR
JPN	SWE	POL	BEL	BEL, CHE	DEU, NZL
SWE	BEL	SWE	NOR	DEU	POL
BEL, ISL	NOR	BEL	CHE	FRA	AUS
FRA, NOR	CHE	CHE	ISL, POL	NOR	DNK, SWE
DNK	JPN	JPN	DEU, SWE	DNK	BEL, FRA
CHE	POL	DEU	FRA, IRL	CZE	PRT
ESP	FRA	DNK	DNK	ITA	ESP
CZE	DNK, ISL	FRA	HUN	ESP, HUN, PRT	CHE
ITA	DEU	ISL, NOR	PRT	ISL, SWE	CZE
DEU	CZE	CZE	ITA	GRC	ITA
HUN	HUN	HUN	GRC	MEX	ISL
POL	ESP	PRT	ESP	BRA	HUN
GRC	PRT	ITA	CZE	IDN	GRC
PRT	ITA	ESP	MEX		MEX
MEX	GRC	GRC	BRA		BRA
BRA	BRA	MEX	IDN		IDN
IDN	MEX	BRA			
	IDN	IDN			

Cuadro IV: Rankings de los países en distintos años en comprensión lectora

$\lambda = 0$	Total	Chicas	Chicos
Matemáticas	0.912829	0.893640	0.887610
Ciencias	0.894006	0.879386	0.886696
Lectura	0.879386	0.867105	0.868860

$\lambda = 0.25$	Total	Chicas	Chicos
Matemáticas	0.907834	0.890512	0.884263
Ciencias	0.891127	0.875995	0.883674
Lectura	0.872121	0.864099	0.859995

$\lambda = 0.5$	Total	Niñas	Niños
Matemáticas	0.904190	0.888388	0.882652
Ciencias	0.888446	0.872551	0.880929
Lectura	0.865605	0.861989	0.852755

$\lambda = 0.75$	Total	Chicas	Chicos
Matemáticas	0.901939	0.887243	0.882537
Ciencias	0.886087	0.869262	0.878575
Lectura	0.860759	0.860896	0.847942

$\lambda = 1$	Total	Chicas	Chicos
Matemáticas	0.900816	0.886851	0.883424
Ciencias	0.884109	0.866284	0.876654
Lectura	0.857641	0.860587	0.845295

Cuadro V: Valores de la medida global de estabilidad de las decisiones intertemporales

consumidores, decisiones de inversión, etc. Por otra parte, la metodología propuesta podría ser extendida a un horizonte temporal infinito.

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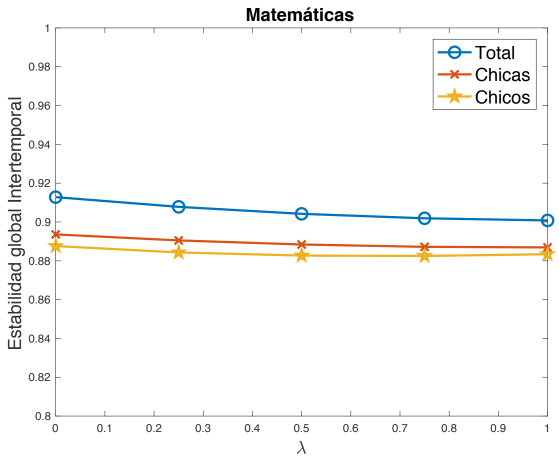

Figura 3: Estabilidad global intertemporal de las decisiones en Matemáticas para diferentes valores de λ

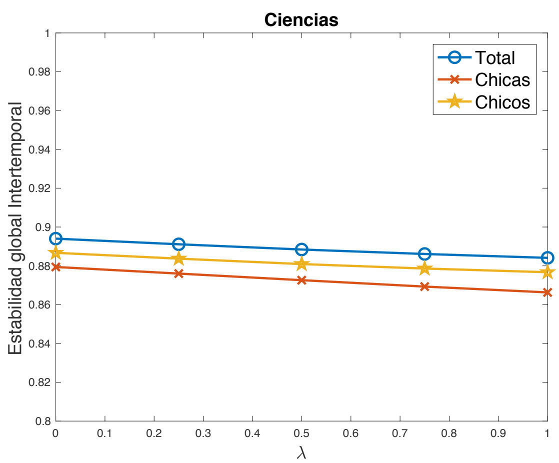

Figura 4: Estabilidad global intertemporal de las decisiones en Ciencias para diferentes valores de λ

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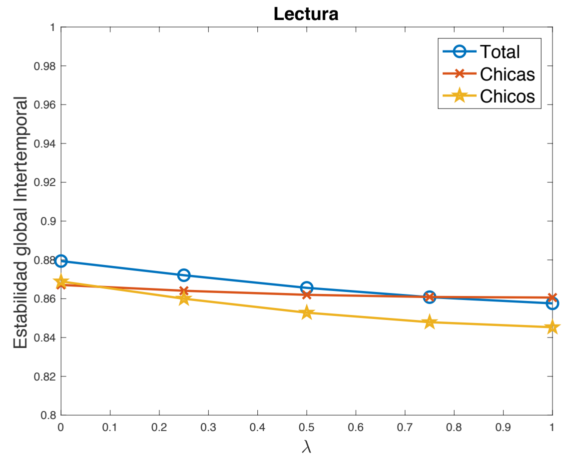

Figura 5: Estabilidad global intertemporal de las decisiones en comprensión lectora para diferentes valores de λ

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Some remarks on "Preference stability over time: The time cohesiveness measure"

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***Abstract*—This work introduces a non-traditional approach about the problem of measuring the stability of agents' preferences over time under the assumption of considering dichotomous opinions. The general concept of *time cohesiveness measure* is introduced as well as a particular formulation based on the consideration of any two successive moments of time, *the sequential time cohesiveness measure*. Moreover, some properties of the novel measure are also provided. Finally, a case of study is presented. This essay presents the main contributions of the paper entitled "*Preference stability along time: The time cohesiveness measure*" published in the journal *Progress in Artificial Intelligence*.**

***Index Terms*—Time cohesiveness measure; Dichotomous opinions; Preference stability; Patients' preferences**

I. Introduction

Intertemporal decision making is an important research area and it has been obtaining attention from several research fields such as Economics, Health Economics, Social Choice, Psychology, Marketing, Decision Analysis, Neuroscience, and so on. One of the main topics of this area is the study of preference stability over time. Traditionally, intertemporal preferences have ussually been considered permanent by theoretical and empirical studies (see [1], [2] and [3], among other) and the research to date has tend to explore preference stability over time by means of statistical methods.

In order to enhance the preference stability topic, the aim of this contribution is to develop a new tool capable of measuring preference stability from a non-traditional perspective. For this purpose, the notion of preference stability is considered in the same vein that the notion of cohesiveness. This seems natural because the measurement of preference stability resembles the notion of measurement of cohesiveness over time in the sense that the maximum value captures the notion of full stability, that is, unanimity along time, while the minimum value captures the notion of total lack of stability, that is, total disagreement along time.

Taking into account the previous contributions on preference stability and cohesiveness measure, this paper is focused on an intertemporal decision-making problem where a set of agents express their opinions on an alternative along different moments of time. To be precise, agents have to approve or disapprove the alternative under study at diverse points of time. Thus, the paper objective is to determine how much stability or cohesiveness agents opinions conveys to the group on the alternative along time. In order to measure such stability, a new general approach is defined, the *time cohesiveness measure*. Moreover, an specific formulation of the time cohesiveness measure is introduced, the *sequential time cohesiveness measure* as well as a study of its analytic properties. Under this approach, the stability of preferences is understood like the probability that for a randomly chosen moment of time, two randomly chosen agents have the same opinion at such a time and its consecutive.

Furthermore, the measurement proposed is put in practice in a real case of study to emphasize its applicability. In particular, the stability of preferences for life-sustaining treatments in terminally cancer patients' last year of life is analysed.

This contribution is structured as follows. Section 2 introduces the notation and the novel proposals to measure preference stability. Section 3 includes a brief description of the paper application. Finally, some closing comments are provided.

II. The time cohesiveness measure: Notation and definitions

Let $\mathbf{N} = \{1, 2, ..., N\}$ a set of agents or experts. Agents express their opinions on an alternative, x, at different time moments $\mathbf{T} = \{t_1, \ldots, t_T\}$ by means of dichotomous opinions.

A *time preference profile* of a set of agents $\mathbf{N}$ on an alternative x at T different time moments is a matrix $\mathbf{P} = \left(P_{it_j}\right)_{N\times T}$ where P_{it_j} is the opinion of the agent i over alternative x at t_j moment, in the sense

$$P_{it_j} = \begin{cases} 1 & \text{if agent } i \text{ approves } x \text{ at the } t_j \text{ time,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{P}_{N\times T}$ denote the set of all such $N \times T$ matrices.

A time preference profile $\mathbf{P}$ is *unanimous* if alternative x is approved (resp. disapproved) over $\mathbf{T}$ by all agents. In matrix terms, if the time preference profile $\mathbf{P} \in \mathbb{P}_{N\times T}$ is constant, $\mathbf{P} = (1)_{N\times T}$ (resp. $\mathbf{P} =$

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$(0)_{N\times T}$). Any permutation σ of the agents $\{1, 2, ..., N\}$ determines a time preference profile $\mathbf{P}^\sigma$ by permutation of the rows of $\mathbf{P}$, that is, row i of the profile $\mathbf{P}^\sigma$ is row $\sigma(i)$ of the profile $\mathbf{P}$.

Definition 1: A *time cohesiveness measure* for a group of agents $\mathbf{N} = \{1, ..., N\}$ on an alternative x is a mapping $\tau : \mathbb{P}_{N\times T} \to [0, 1]$ that assigns a number $\tau(\mathbf{P}) \in [0, 1]$ to each time preference profile $\mathbf{P}$, with the properties:

i) $\tau(\mathbf{P}) = 1$ if and only if $\mathbf{P}$ is unanimous (full stability).
ii) $\tau(\mathbf{P}^\sigma) = \tau(\mathbf{P})$ for each permutation σ of the agents and $\mathbf{P} \in \mathbb{P}_{N\times T}$ (anonymity).

Definition 2: The *sequential time cohesiveness measure* for a group of agents $\mathbf{N} = \{1, ..., N\}$ on an alternative x is the mapping $\tau_S : \mathbb{P}_{N\times T} \to [0, 1]$ given by

$$\tau_S(\mathbf{P}) = \sum_{b\in\{0,1\}} \frac{1}{T-1} \cdot \frac{\sum_{j=1}^{j=T-1} n_{b,b}^{t_j,t_{j+1}} \cdot (n_{b,b}^{t_j,t_{j+1}} - 1)}{N(N-1)}$$

where $n_{0,0}^{t_j,t_{j+1}}$ denotes the number of agents that disapprove alternative x at t_j and keep their opinion at the following point of time t_{j+1}. Similarly, $n_{1,1}^{t_j,t_{j+1}}$ denotes the number of agents that approve alternative x at t_j and keep their opinion at the following point of time t_{j+1}.

Intuitively, it measures the probability that for a randomly chosen moment of time, two randomly chosen agents of a group have the same opinion upon an alternative at the moment of time selected and its consecutive.

The sequential cohesiveness measure verifies the following meaningful properties: reversal invariance, time-reducibility, replication monotonicity, minimum time stability, leaving minimum time stability, time monotonicity and convergence to full stability.

III. A case study of preference stability in Clinical Decision Making

So as to implement our proposal for measuring the stability of preferences over time of a group of agents, this contribution is inspired and motivated by the study of Tang et al. [4]. In [4], the authors examined the stability of life-sustaining treatment preferences at end of life of cancer patient's last year by means of an statistical approach. Authors collected patients' preferences about life support choices by the *Life Support Preferences Questionnaire* (LSPQ) [5].

Based on this study, a finite set of 257 patients is considered in this contribution. Theses patients expressed their opinions by dichotomous opinions on a finite set of 3 treatments for life-sustaining at end of life being: cardiopulmonary resuscitation (CPR), dying in an intensive care unit (ICU) and mechanical ventilation support (MSV). Patients expressed their preferences about approving o disapproving the aforementioned treatments at four different time moments along their illness. Thus, patients' opinions can be formalized by means of a time preference profile for each treatment $\mathbf{P}^{CPR}$, $\mathbf{P}^{ICU}$ and $\mathbf{P}^{MSV}$. The information provided by the three previous time preference profiles can be group in Table I.

Treatment	$n_{0,0}^{t_1,t_2}$	$n_{1,1}^{t_1,t_2}$	$n_{0,0}^{t_2,t_3}$	$n_{1,1}^{t_2,t_3}$	$n_{0,0}^{t_3,t_4}$	$n_{1,1}^{t_3,t_4}$
CPR	190	34	210	24	228	15
ICU	142	79	156	63	184	26
MSV	170	44	187	38	209	25

Table I: Number of patients that approve and disapprove different treatments at different moments of time

Using Definition 2, the sequential time cohesiveness measure for each profile, that is, for each treatment can be computed. Table II shows such values including all moments of time and all patients.

Treatment	Profile	$\tau_S(\mathbf{P})$
CPR	$\mathbf{P}^{CPR}$	0.676
ICU	$\mathbf{P}^{ICU}$	0.449
MVS	$\mathbf{P}^{MVS}$	0.562

Table II: Values of the sequential time cohesiveness measure for each treatment

Moreover, these results were explored in depth in [6]. The set of patients was partitioned, differentiating between patients with and without metastases.

IV. Closing comments

In this work, a non-traditional perspective on preference stability topic is set out. The problem of measuring the degree of cohesiveness in a setting where agents express their opinions on an alternative at different times by means of an approval or disapproval evaluation is explored. A general concept of *time cohesiveness measure* is introduced and a particular formulation based on the consideration of any two successive times is proposed, namely *the sequential time cohesiveness measure*. Some properties which make our proposal appealing are also provided. The applicability of our proposal to real situations is emphasized by means of adapting a factual problem in Clinical Decision Making. Concretely, the case of terminally cancer patients' last year of life is studied using the new sequential time cohesiveness measure.

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