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Short communication

Monotonicity and the Hirsch index

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ABSTRACT

The Hirsch index is a number that synthesizes a researcher's output. It is the maximum number h such that the researcher has h papers with at least h citations each. Woeginger [Woeginger, G. J. (2008a). An axiomatic characterization of the Hirsch-index. *Mathematical Social Sciences*, *56*(2), 224–232; Woeginger, G. J. (2008b). A symmetry axiom for scientific impact indices. *Journal of Informetrics*, *2*(3), 298–303] characterizes the Hirsch index when indices are assumed to be integer-valued. In this note, the Hirsch index is characterized, when indices are allowed to be real-valued, by adding to Woeginger's monotonicity two axioms in a way related to the concept of monotonicity.

1. Introduction

Woeginger (2008a, 2008b) characterizes the Hirsch (2005) index for indices taking values in the set of non-negative integers. This note suggests another characterization of the Hirsch index but for indices taking values in the set of non-negative real values.

One of the axioms postulated in Woeginger's characterizations is monotonicity: more citations or papers do not lower the index. This note axiomatizes the Hirsch index in terms of monotonicity and another two axioms. To a certain extent, both axioms also express some form of monotonicity. The first axiom establishes upper and lower bounds to the index. The lower bound grows monotonically with the number of cited papers and the smallest number of citations. The upper bound grows monotonically with the number of papers and the largest number of citations. The second axiom states that, for two outputs with the same index, a certain change should generate the same qualitative effect on both outputs.

2. Definitions and axioms

Let $\mathbb{N}$ be the set of non-negative integers and R the set of non-negative real numbers. Members of $\mathbb{N}$ represent both the number of papers of a given researcher and the number of citations that a paper can receive. Define X to be the set of all vectors $x=(x_1, x_2, \ldots, x_n)$ such that $n \in \mathbb{N}\backslash\{0\}$ and $x_1 \geq x_2 \geq \ldots \geq x_n$. For $x \in X$: (i) d_x is the number of components of vector x (the dimension of x); (ii) c_x is the number of components of vector x different from 0; and (iii) for $i \in \{1, \ldots, d_x\}$, x_i is the ith component of vector x and stands for the total number of citations of paper i. With $\emptyset$ designating the empty vector, a researcher's output will be represented by a member of $D = X \cup \{\emptyset\}$. For $x = \emptyset$ the convention is that $c_x = d_x = min\{x_1, \ldots, x_{d_x}\} = max\{x_1, \ldots, x_{d_x}\} = 0$.

Definition 2.1. An index is a mapping f: $D \rightarrow R$.

Woeginger (2008a, p. 225) defines an index as a mapping f: $D \rightarrow \mathbb{N}$. This definition appears restrictive because there is no obvious reason to exclude average indices.

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Definition 2.2. The Hirsch index is the index h such that $h(\emptyset)=0$ and, for all $x\in X$, $h(x)=max\{n\in\{0, 1, \ldots, d_x\}: x_n\geq n\}$.

A1. For all $x\in D$, $min\{min\{x_1, \ldots, x_{d_x}\}, c_x\}\leq f(x)\leq min\{max\{x_1, \ldots, x_{d_x}\}, d_x\}$.

A1 sets upper and lower bounds to the index. Since $min\{x_1, \ldots, x_{d_x}\}=x_{d_x}$ and $max\{x_1, \ldots, x_{d_x}\}=x_1$, A1 is equivalent to $min\{x_{d_x}, c_x\}\leq f(x)\leq min\{x_1, d_x\}$. Hence, the index is limited above by the smallest of two numbers: the number of citations of the most cited paper and the number of papers. And it is limited below by the smallest of two numbers: the number of citations of the least cited paper and the number of cited papers.

The fact that the inputs of f are citations and papers suggests considering the possibility of measuring the output of f in citations or papers. If one interprets that $f(x)$ defines a representative or average number of citations in x then $min\{x_1, \ldots, x_{d_x}\}\leq f(x)\leq max\{x_1, \ldots, x_{d_x}\}$ is a reasonable requirement. On the other hand, if one interprets that $f(x)$ defines the number of "quality" papers in x then $f(x)\leq d_x$ also appears reasonable. A1 can be viewed as the result of adopting an ambiguous position with respect to the question of the units of measure of the index by adapting the condition $min\{x_1, \ldots, x_{d_x}\}\leq f(x)\leq max\{x_1, \ldots, x_{d_x}\}$ to take into account the effect of the number of papers (or of cited papers). The Hirsch index is characterized by this ambiguity: $h(x)$ is not only the maximum number h of papers in x having h citations each but also the maximum number h of citations obtained by h papers in x.

Specifically, $f(x)\geq min\{min\{x_1, \ldots, x_{d_x}\}, c_x\}$ means that neither a large number of cited papers nor a large number of citations of the least cited paper are enough for the index to be large: both must be large enough and the smallest of the two determines the exact representative value. Hence, if f is measured in citations then it is good to increase the number of citations of the least cited paper, but up to a point, represented by the number of cited papers. Conversely, if f is measured in papers then it is good to increase the number of cited papers, but up to a point, given by the number of citations of the least cited paper.

The condition $f(x)\leq min\{max\{x_1, \ldots, x_{d_x}\}, d_x\}$ can be interpreted analogously. If f is measured in citations then it is good to increase the number of citations of the most cited paper, but up to a point, defined by the number of papers. And if f is measured in papers then it is good to increase the number of papers, but up to a point, which corresponds to the number of citations of the most cited paper. Recapitulating, by A1, it can be interpreted that f is measured in representative citations, but facing a paper constraint, or measured in representative papers, but facing a citation constraint.

For all $x\in X$ and $y\in X$, $x\geq y$ holds if, and only if, $d_x\geq d_y$ and, for all $i\in\{1, \ldots, d_y\}$, $x_i\geq y_i$. For $x\in D$ and $a\in\mathbb{N}$, $(x\oplus a)$ is the vector $y\in X$ such that $d_y=d_x+1$, $y_{d_y}=a$ and, for all $i\in\{1, \ldots, d_x\}$, $y_i=x_i$.

A2. For all $x\in X$ and $y\in X$, $x\geq y$ implies $f(x)\geq f(y)$.

A2 is Woeginger's (2008a, p. 225) monotonicity axiom: more citations or papers do not reduce the index.

A3. There is $^*\in\{>, =\}$ such that, for all $x\in X$, $y\in X$ and $a\in\mathbb{N}$ such that $(x\oplus a)\in X$ and $(y\oplus a)\in X$, if $f(x)=f(y)$ and $f(x\oplus a)\ ^*\ f(x)$ then $f(y\oplus a)\ ^*\ f(y)$.

To motivate A3 when "*" is "=", suppose that the index does not distinguish between two outputs x and y, in the sense that both x and y are assigned the same value. Suppose as well that another paper with $a\leq min\{x_{d_x}, y_{d_y}\}$ citations is added to x. Suppose finally that this paper produces no effect on x; that is, $f(x\oplus a)=f(x)$. Then consistency seems to demand that the addition of that paper to y should cause no effect: $f(y\oplus a)=f(y)$. For the case in which "*" is ">", A3 can be viewed as a monotonicity property: if the index is strictly monotonic in passing from x to $(x\oplus a)$, it should also be strictly monotonic in passing from y to $(y\oplus a)$, provided that the new outputs are members of X and the index attributes the same value to outputs x and y. This property expresses some sort of consistent monotonicity.

3. Result

Theorem 3.1. An index f satisfies A1, A2 and A3 if, and only if, f is the Hirsch index.

Proof. "$\Leftarrow$" Step 1: h satisfies A1. Let $x\in X$. If $x=\emptyset$ then, by definition of the Hirsch index, $h(\emptyset)=0$, so $min\{min\{x_1, \ldots, x_{d_x}\}, c_x\}\leq h(x)\leq min\{max\{x_1, \ldots, x_{d_x}\}, d_x\}$. If $x\neq\emptyset$ then, by definition of the Hirsch index, $h(x)\leq d_x$ and $h(x)\leq x_1$. Therefore, $h(x)\leq min\{max\{x_1, \ldots, x_{d_x}\}, d_x\}$. To prove that $h(x)\geq min\{min\{x_1, \ldots, x_{d_x}\}, c_x\}$, observe that $min\{x_1, \ldots, x_{d_x}\}=x_{d_x}$. If $min\{x_{d_x}, c_x\}=x_{d_x}$ then there are x_{d_x} papers in x with at least x_{d_x} citations each, so $h(x)\geq x_{d_x}$. If $min\{x_{d_x}, c_x\}=c_x$ then there are c_x papers in x with at least $x_{d_x}\geq c_x$ citations each and, accordingly, $h(x)\geq c_x$. Step 2: h satisfies A2. If $x\geq y$ then y has at least $h(x)$ papers with $h(x)$ citations each. This implies $h(y)\geq h(x)$. Step 3: h satisfies A3. Let $(x\oplus a)\in X$, $(y\oplus a)\in X$ and $h(x)=h(y)$. Case 1: $h(x\oplus a)>h(x)$. Then $a>h(x)$. Since $(x\oplus a)\in X$, $x_{d_x}>h(x)$. This implies $h(x)=d_x$. Hence, $h(y)=d_x$. Therefore, $a>h(y)$. Given this and $(y\oplus a)\in X$, $h(y\oplus a)>h(y)$. Case 2: $h(x\oplus a)=h(x)$. In this case, $a\leq h(x)$ and, accordingly, $a\leq h(y)$, which implies $h(y\oplus a)=h(y)$.

"$\Rightarrow$" Let f be an index satisfying A1, A2 and A3. Choose $x\in D$. It must be shown that $f(x)=h(x)$. Case 1: $h(x)=0$. If $x=\emptyset$ then, since $min\{x_1, \ldots, x_{d_x}\}=c_x=max\{x_1, \ldots, x_{d_x}\}=d_x=0$, by A1, $f(x)=0$. Otherwise, $c_x=max\{x_1, \ldots, x_{d_x}\}=0$ and, by A1, $0\leq f(x)\leq 0$.

Case 2: $h(x)\geq 1$. Let $h=h(x)$. For $k\in\{1, \ldots, d_x\}$, define x^k to be the vector consisting of the k first components of x. Formally, x^k is the member z of X such that: (i) $d_z=k$; and (ii) for all $i\in\{1, \ldots, k\}$, $z_i=x_i$. Observe that, for $k\in\{2, \ldots, d_x\}$, $x^k=(x^{k-1}\oplus x_k)$. For $k\in\{h, \ldots, d_x\}$, let $y^k\in X$ satisfy: (i) y^k has k components; (ii) for all $i\in\{1, \ldots, h\}$, $y_i^k=h$; and (iii) if $k>h$ then, for all $i\in\{h+1, \ldots, d_x\}$, $y_i^k=x_i$. The fact that $h(x)=h$ guarantees that each y^k is a member of X, because, for all $i\in\{h+1, \ldots, d_x\}$, $x_i\leq h$. Observe as well that, for $k\in\{h+1, \ldots, d_x\}$, $y^k=(y^{k-1}\oplus x_k)$.

By A1, $min\{h, h\} \le f(y^h) \le min\{h, h\}$, so $f(y^h) = h$. By A1, $min\{min\{x_1, \ldots, x_h\}, h\} \le f(x^h) \le min\{max\{x_1, \ldots, x_h\}, h\}$. Since $h(x) = h \ge 1$, $min\{x_1, \ldots, x_h\} \ge h$ and $max\{x_1, \ldots, x_h\} \ge h$. Therefore, $f(x^h) = h = f(y^h)$. The proof is complete if $d_x = h$. Otherwise, consider x^{h+1}. By A1, $f(y^{h+1}) \le h$. By A2, $f(y^{h+1}) \ge f(y^h) = h$. As a consequence, $f(y^{h+1}) = h$. If "*" is "=" in A3 then, by A3, $f(y^{h+1}) = f(y^h)$ implies $f(x^{h+1}) = f(x^h) = h$. If "*" is ">" in A3 then, by A3, $f(x^{h+1}) > f(x^h)$ would imply $f(y^{h+1}) > f(y^h)$: contradiction. Thus, $f(x^{h+1}) \le f(x^h)$. By A2, $f(x^{h+1}) \ge f(x^h)$. Hence, in both cases, $f(x^{h+1}) = f(x^h) = f(y^h) = f(y^{h+1}) = h$.

The proof is complete if $d_x = h + 1$. Otherwise, taking $f(x^{h+1}) = f(y^{h+1}) = h$ as the base of an induction argument, choose $k \in \{h+2, \ldots, d_x\}$ and assume that, for all $i \in \{h+1, \ldots, k-1\}$, $f(x^i) = f(y^i) = h$. It must be shown that $f(x^k) = f(y^k) = h$. By A1, $f(y^k) \le h$. By A2, $f(y^k) \ge f(y^{k-1})$. By the induction hypothesis, $f(y^{k-1}) = h$. Accordingly, $f(y^k) = h$. If "*" is "=" in A3 then, by A3, $f(y^k) = f(y^{k-1})$ implies $f(x^k) = f(x^{k-1}) = h$. If "*" is ">" in A3 then, by A3, $f(x^k) > f(x^{k-1})$ would imply $f(y^k) > f(y^{k-1})$: contradiction. Hence, $f(x^k) \le f(x^{k-1})$. By A2, $f(x^k) \ge f(x^{k-1})$. As a result, $f(x^k) = f(x^{k-1}) = h = f(y^k)$. □

The following examples show that no axiom in Theorem 3.1 is redundant.

Example 3.2. With $n \in \mathbb{N}$, let f be the index such that, for all $x \in D$, $f(x) = n$. Then f: (i) satisfies A2 and A3; (ii) does not satisfy A1; and (iii) is not the Hirsch index.

Example 3.3. Let f be the index such that, for all $x \in D$, $f(x) = min\{min\{x_1, \ldots, x_{d_x}\}, c_x\}$. Then f: (i) satisfies A1 and A3; (ii) does not satisfy A2 (because $f(2) = 1$ and $f(2, 0) = 0$); and (iii) is not the Hirsch index.

Example 3.4. Let f be the index such that, for all $x \in D$, $f(x) = min\{max\{x_1, \ldots, x_{d_x}\}, d_x\}$. Then f: (i) satisfies A1 and A2; (ii) does not satisfy A3 (because $f(8, 8, 8) = f(3, 3, 3) = 3$, $4 = f(8, 8, 8, 1) > f(8, 8, 8)$ and $f(3, 3, 3, 1) = f(3, 3, 3)$); and (iii) is not the Hirsch index.

4. Comments

Two of the reviewers pointed out the close relationship between Theorem 3.1 and the characterizations of the Hirsch index in Quesada (in press). One of those characterizations is based on axioms B1, B2 and B3 below, where: (i) $D_0 = \{\emptyset\}$ and, for $n \in \mathbb{N}\backslash\{0\}$, $D_n = \{x \in D: d_x = n\}$; (ii) for $x \in X$, $x^\Sigma = x_1 + x_2 + \ldots + x_{dx}$; (iii) for $x \in X$ and $y \in X$, $\delta(x, y) = max\{x^\Sigma, y^\Sigma\} - min\{x^\Sigma, y^\Sigma\}$ is the distance between x and y; and (iv) for $n \in \mathbb{N}\backslash\{0\}$ and $x \in D_n$, $x_{-n} = (x_1, \ldots, x_{n-1})$ is the member of D_{n-1} obtained from x by deleting the last component x_n of x.

B1. For all $x \in X$, if $c_x = d_x$ then $min\{min\{x_1, \ldots, x_{d_x}\}, d_x\} \le f(x) \le d_x$.

B2. For all $n \in \mathbb{N}$, $x \in D_n$ and $y \in D_{n+1}$, if $y \ge x$ and $\max\{f(z)\}_{z \in D_{n+1}} = f(y) > f(x)$ then $\delta(x, y) > c_x$.

B3. For all $n \in \mathbb{N}\backslash\{0\}$ and $x \in D_n$, if $f(x) \ne \max\{f(y)\}_{y \in D_n}$ then $f(x) = f(x_{-n})$.

B1 is weaker than A1. On the one hand, the condition $f(x) \le d_x$ is less restrictive than the condition $f(x) \le min\{x_1, d_x\}$. On the other hand, the constraint $min\{min\{x_1, \ldots, x_{d_x}\}, d_x\} \le f(x) \le d_x$ only applies when all the papers have citations, whereas in A1 the upper and lower bounds always apply.

B2 deals with the following situation. Let output x be given. Suppose that output y is obtained from x by adding another paper not having more citations that the least cited paper in x and by, possibly, increasing the number of citations of existing papers. Suppose also that y is assigned a higher index than x and, moreover, that y achieves the maximum index within the set of outputs having the same dimension as y. Then the distance between x and y, measured in citations, must be larger than the number of cited papers in x. This number can be viewed as a measure of the effort necessary to increase the index of x. Neither B2 implies A2 nor vice versa: $f(x) = 1/(1 + h(x))$ satisfies B2 (because h satisfies B2) but not A2; and $f(x) = min\{x^\Sigma, 4d_x\}$ satisfies A2 but not B2. Neither B2 implies A3 nor vice versa: $f(x) = x^\Sigma + d_x c_x$ satisfies B2 (since, for $n \ge 1$, f has no maximum on D_n) but not A3 (because $f(8, 0) = f(3, 3) = 10$ but $11 = f(8, 0, 0) < f(3, 3, 0) = 12$); and $f(x) = c_x$ satisfies A3 but not B2.

B3 states that if output x does not achieve the maximum index within the set of outputs having the same dimension as x then the output obtained by removing the least cited paper in x is attributed the same index as x. B3 does not imply, nor is implied, by A2: the average citation index satisfies A2 but not B3 (since $2 = f(3, 1) \ne f(3) = 3$); and $f(x) = 1/(1 + h(x))$ satisfies B3 (because h satisfies B3) but not A2. B3 does not imply, nor is implied, by A3: an index in which no two outputs have the same value satisfies A3 but not B3; and $f(x) = x^\Sigma + d_x c_x$ satisfies B3 (for $n \ge 1$, f does not reach a maximum on D_n) but not A3.

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