Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 160 (2009) 802–815
www.elsevier.com/locate/fss

Multidimensional generalized fuzzy integral

Yasuo Narukawa[a], Vicenç Torra[b,*]

[a]Toho Gakuen, 3-1-10 Naka, Kunitachi, Tokyo 186-0004, Japan
[b]IIIA-CSIC, Institut d'Investigació en Intel.ligència Artificial, Campus de Bellaterra, 08193 Bellaterra (Catalonia), Spain

Available online 21 October 2008

Abstract

In this paper, we study multidimensional integrals. We introduce a generalized fuzzy integral and we present a Fubini-like theorem for this generalized fuzzy integral. As this research was partly motivated by the definition of citation indices, we also describe how such multidimensional integrals can be used to define such indices.

Keywords: Fuzzy measure; Fuzzy integral; Choquet integral; Sugeno integral; Multidimensional integrals

1. Introduction

Citation indices are a topic of practical interest due to the fact that they are used to evaluate the impact of researchers' activities. Our interest for such indices, permitted us to prove that two of the most used indices for authors of papers can be represented as fuzzy integrals. In particular, in [13,9,8] we proved that the Hirsch index [4] and the number of citations correspond, respectively, to the Sugeno integral [11] and the Choquet integral [2] of the same function. The function corresponds to the number of citations for a given paper. Another interesting result from [13] is that both integrals are calculated with respect to the same fuzzy measure: the measure defined as the cardinality of the set.

These results have turned our attention, again, into the field of multidimensional integrals. The Hirsch index and the number of citations correspond to the integral of the function *f*, where $f(x)$ corresponds to the number of citations of paper x at the time of study. Nevertheless, we can consider the number of citations of x in a particular year y. If $g_y(x)$ represents this value, $f(x) = \sum_{y \leqslant \gamma} g_y(x)$, and it is meaningful to compute the integral of *f*. In this paper we consider generalized fuzzy (GF) integrals for this purpose. Besides, as this integral is a multidimensional integral, it is meaningful to study whether Fubini-like theorems apply. Indeed, in this paper we present a Fubini-like theorem.

Another relevant issue when dealing with multidimensional integrals is the set where they are measurable. In [9], we considered measurable functions on the set:

$$\mathscr{X} \times \mathscr{Y} := \{A \times B | A \in \mathscr{X}, B \in \mathscr{Y}\}.$$

Note that for $\mathscr{X} := 2^X$ and $\mathscr{Y} := 2^Y$, $\mathscr{X} \times \mathscr{Y} \neq 2^{X \times Y}$, the set of $\mathscr{X} \times \mathscr{Y}$-measurable functions is not equivalent to the set of $2^{X \times Y}$-measurable functions. This is a problem when directly computing the integral in the product space. Due to this, we considered the extension of fuzzy measures from $\mathscr{X} \times \mathscr{Y}$ into $2^{X \times Y}$.

* Corresponding author. Tel.: +34 93 580 9570; fax: +34 93 580 9661.
E-mail addresses: narukawa@d4.dion.ne.jp (Y. Narukawa), vtorra@iiia.csic.es (V. Torra).

0165-0114/$ - see front matter
doi:10.1016/j.fss.2008.10.006

In this paper we consider another extension, based on pseudo-addition, that permits us to construct a measure in the extended measure space $2^{X\times Y}$ that corresponds to the cardinality of a set. So, with this new extension we can define a measure that corresponds to the one used for both Sugeno and Choquet integrals when used to define the Hirsch index and the number of citations. Besides of that, we also consider here the case where the universal set X is infinite.

The structure of this paper is as follows. First, we review in Section 2 some concepts that are needed later on in this paper. In Section 3, we define GF integrals. Then, in Section 4 we will consider the multidimensional integrals. Section 6 describes how the results presented here can be applied in the problem of defining citation indices. The paper finishes with some conclusions.

2. Preliminaries

This section reviews some basic definitions on fuzzy measures that are used in the rest of this paper. In particular, we present the definition of a fuzzy measure and the one of Choquet and Sugeno integrals.

Definition 1. Let X be a universal set and $\mathcal{X}$ be a subset of 2^X with $\emptyset \in \mathcal{X}$ and $X \in \mathcal{X}$. Then, $(X, \mathcal{X})$ is called a fuzzy measurable space. We say that a function $f : X \to \mathbb{R}^+$ is $\mathcal{X}$-measurable if $\{x|f(x)\geqslant a\} \in \mathcal{X}$ for all a.

Definition 2 (*Dellacherie [3]*). Let f and g be $\mathcal{X}$-measurable functions on X; then, we say that f and g are comonotonic if

$$f(x) < f(y) \;\Rightarrow\; g(x)\leqslant g(y)$$

for $x, y \in X$.

Definition 3 (*Sugeno [11]*). Let $(X, \mathcal{X})$ be a fuzzy measurable space; then, a fuzzy measure μ on $(X, \mathcal{X})$ is a real valued set function, $\mu : \mathcal{X} \longrightarrow \mathbb{R}^+$ with the following properties.

(1) $\mu(\emptyset) = 0$, $\mu(X) = k$ where $k \in (0, \infty)$.
(2) $\mu(A)\leqslant\mu(B)$ whenever $A \subset B$, $A, B \in \mathcal{X}$.
(3) $A_n \uparrow A$ implies $\mu(A_n) \uparrow \mu(A)$ if $A_n \in \mathcal{X}$ and $A \in \mathcal{X}$.

A triplet $(X, \mathcal{X}, \mu)$ is said to be a fuzzy measure space.

Definition 4 (*Choquet [2], Murofushi and Sugeno [7]*). Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and let f be a $\mathcal{X}$-measurable function; then, the Choquet integral of f with respect to μ is defined by

$$(C)\int f\,\mathrm{d}\mu := \int_0^\infty \mu_f(r)\,\mathrm{d}r,$$

where $\mu_f(r) = \mu(\{x|f(x)\geqslant r\})$.

Definition 5 (*Benvenuti et al. [1]*). For any $r > 0$ and $A \in \mathcal{X}$, the basic simple function $b(r, A)$ is defined by $b(r, A)(x) = r$ if $x \in A$ and $b(r, A)(x) = 0$ if $x \notin A$.

A function f is a simple function if it can be expressed as $f := \sum_{i=1}^n b(a_i, A_i)$ for $a_i > 0$ and $f := \bigvee_{i=1}^n b(a'_i, A_i)$ for $a'_1 > \cdots > a'_n > 0$, where $A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_n$, $A_i \in \mathcal{X}$.

Then, when an $\mathcal{X}$-measurable function f is a simple function, we have

$$(C)\int f\,\mathrm{d}\mu = \sum_{i=1}^n a_i\mu(A_i).$$

Definition 6 (*Ralescu and Adams [10], Sugeno [11]*). Let $(X, \mathcal{X}, \mu)$ be a fuzzy measure space and let $f : X \to [0, \infty)$ be an $\mathcal{X}$-measurable function; then, the Sugeno integral of f with respect to μ is defined by

$$(S)\int f\,\mathrm{d}\mu := \sup_{r\in[0,\infty)} [r \wedge \mu_f(r)].$$

When f is a simple function, the Sugeno integral is written as

$$(S)\int f\,\mathrm{d}\mu = \bigvee_{i=1}^{n}(a_i' \wedge \mu(A_i)).$$

3. GF integral

In this section, we define a GF integral in terms of a pseudo-addition $\oplus$ and a pseudo-multiplication $\boxdot$. Formally, $\oplus$ and $\boxdot$ are binary operators that generalize addition and multiplication, and also max and min. We want to recall that GF integrals have been investigated by Benvenuti et al. [1].

Note that we will use $k \in (0, \infty)$ in the rest of this paper.

Definition 7. A pseudo-addition $\oplus$ is a binary operation on $[0, k]$ or $[0, \infty)$ fulfilling the following conditions:

(A1) $x \oplus 0 = 0 \oplus x = x$.
(A2) $x \oplus y \leqslant u \oplus v$ whenever $x \leqslant u$ and $y \leqslant v$.
(A3) $x \oplus y = y \oplus x$.
(A4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
(A5) $x_n \to x,\ y_n \to y$ implies $x_n \oplus y_n \to x \oplus y$.

A pseudo-addition $\oplus$ is said to be strict if and only if $x \oplus y < x \oplus z$ whenever $x > 0$ and $y < z$, for $x, y, z \in (0, k)$; and it is said to be Archimedean if and only if $x \oplus x > x$ for all $x \in (0, k)$.

Definition 8. A pseudo-multiplication $\boxdot$ is a binary operation on $[0, k]$ or $[0, \infty)$ fulfilling the conditions:

(M1) There exists a unit element $e \in (0, k]$ such that $x \boxdot e = e \boxdot x = x$.
(M2) $x \boxdot y \leqslant u \boxdot v$ whenever $x \leqslant u$ and $y \leqslant v$.
(M3) $x \boxdot y = y \boxdot x$.
(M4) $(x \boxdot y) \boxdot z = x \boxdot (y \boxdot z)$.
(M5) $x_n \uparrow x$, $y_n \uparrow y$ implies $x_n \boxdot y_n \uparrow x \boxdot y$.

Example 9.

(1) The maximum operator $x \vee y$ is a non-Archimedean pseudo-addition on $[0, k]$.
(2) The sum $x + y$ is an Archimedean pseudo-addition on $[0, \infty)$.
(3) The Sugeno operator $x +_\lambda y := 1 \wedge (x + y + \lambda xy)$ $(-1 < \lambda < \infty)$ is an Archimedean pseudo-addition on $[0, 1]$.

Proposition 10 (*Ling [5]*). *If a pseudo-addition $\oplus$ is Archimedean, then there exists a continuous and strictly increasing function $g : [0, k] \to [0, \infty]$ such that $x \oplus y = g^{(-1)}(g(x) + g(y))$, where $g^{(-1)}$ is the pseudo-inverse of g defined by*

$$g^{(-1)}(u) := \begin{cases} g^{(-1)}(u) & \text{if } u \leqslant g(k), \\ k & \text{if } u > g(k). \end{cases}$$

The function g is called an additive generator of $\oplus$.

Definition 11. Let μ be a fuzzy measure on a fuzzy measurable space $(X, \mathscr{X})$; then, we say that μ is a $\oplus$-measure or a $\oplus$-decomposable fuzzy measure if $\mu(A \cup B) = \mu(A) \oplus \mu(B)$ whenever $A \cap B = \emptyset$ for $A, B \in \mathscr{X}$.

A $\oplus$-measure μ is called normal when either $\oplus = \vee$ or $\oplus$ is Archimedean and $g \circ \mu$ is an additive measure. Here, g corresponds to an additive generator of $\oplus$.

Definition 12. Let $k \in (0, \infty)$, let $\oplus$ be a pseudo-addition on $[0, k]$ or $[0, \infty)$ and let $\boxdot$ be a pseudo-multiplication on $[0, k]$ or $[0, \infty)$; then, we say that $\boxdot$ is $\oplus$-fitting if

(F1) $a \boxdot x = 0$ implies $a = 0$ or $x = 0$,
(F2) $a \boxdot (x \oplus y) = (a \boxdot x) \oplus (a \boxdot y)$.

Under these conditions, we say that $(\oplus, \boxdot)$ is a pseudo-fitting system.

Let $\oplus$ be a pseudo-addition; then, we define its pseudo-inverse $-_{\oplus}$ as

$$a -_{\oplus} b := \inf\{c | b \oplus c \geqslant a\}$$

for all $(a, b) \in [0, k]^2$.

Definition 13 (*Sugeno and Murofushi [12]*). Let μ be a fuzzy measure on a fuzzy measurable space $(X, \mathscr{X})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, when μ is a normal $\oplus$-measure, we define the pseudo-decomposable integral of a measurable simple function f on X such that $f = \bigoplus_{1i=1}^{n} b(r_i, D_i)$ where $D_i \cap D_j \neq \emptyset$ for $i \neq j$, as follows:

$$(D) \int f \, \mathrm{d}\mu := \bigoplus_{i=1}^{n} r_i \boxdot \mu(D_i).$$

For a measurable function f, there exists a sequence f_n such that $f_n \uparrow f$. Then, its pseudo-decomposable integral is defined as follows:

$$(D) \int f \, \mathrm{d}\mu := \sup_n (D) \int f_n \, \mathrm{d}\mu.$$

Since μ is a $\oplus$-measure, it is obvious that the integral is well defined.

Definition 14. Let μ be a fuzzy measure on a measurable space $(X, \mathscr{X})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, the GF-integral of a measurable simple function $f := \bigoplus_{i=1}^{n} b(a_i, A_i)$, with $a_i > 0$ and $A_1 \supsetneq A_2 \supsetneq \cdots A_n$, $A_i \in \mathscr{X}$, is defined as follows:

$$(GF) \int f \, \mathrm{d}\mu := \bigoplus_{i=1}^{n} a_i \boxdot \mu(A_i).$$

The GF-integral of a simple function is well defined [1].

The following proposition can be proven. It will permit us to define the GF-integral of a measurable function.

Proposition 15. *Let f be a measurable function, $\{f_n\}$ be a sequence of simple functions such that $f_n \uparrow f$ and g be a simple function such that $g \leqslant f$. Then we have*

$$(GF) \int g \, \mathrm{d}\mu \leqslant \sup_n (GF) \int f_n \, \mathrm{d}\mu.$$

Proof. Let $g := \bigoplus_{i=1}^{m} b(b_i, B_i)$, $A := \{x | g(x) > 0\}$ and $A_n := \{x | 0 < g(x) < f_n(x) \oplus b(\varepsilon, X)\}$ for an arbitrary ε. since we assume $\mu(X) = k < \infty$, we have

$$\begin{aligned}
(GF) \int (f_n \oplus \varepsilon 1_X) \, \mathrm{d}\mu &= (GF) \int f_n \, \mathrm{d}\mu \oplus \varepsilon \boxdot \mu(X) \\
&\geqslant (GF) \int (f_n \oplus b(\varepsilon, X)) \boxdot b(e, A \cap A_n) \, \mathrm{d}\mu \\
&\geqslant (GF) \int g \boxdot b(e, A \cap A_n) \, \mathrm{d}\mu \\
&= (GF) \int \bigoplus_{i=1}^{m} b(b_i, B_i \cap A \cap A_n) \, \mathrm{d}\mu \\
&= \bigoplus_{i=1}^{m} b_i \boxdot \mu(B_i \cap A \cap A_n) = \bigoplus_{i=1}^{m} b_i \boxdot \mu(B_i \cap A_n).
\end{aligned}$$

Since $f_n \uparrow f$ as $n \to \infty$, we have $A_n \uparrow A$. Therefore we have

$$\sup_n (GF) \int f_n \, \mathrm{d}\mu \oplus \varepsilon \boxdot \mu(X) \geqslant \bigoplus_{i=1}^{m} b_i \boxdot \mu(B_i) = (GF) \int g \, \mathrm{d}\mu.$$

Since ε is arbitrary and $\mu(X) < \infty$, we have

$$\sup_n (GF)\int f_n \,\mathrm{d}\mu \geqslant (GF)\int g \,\mathrm{d}\mu. \qquad \square$$

From this proposition, it follows that the GF-integral of a measurable function f is well defined.

Definition 16. When f is not a simple function, we define the integral as follows:

$$(GF)\int f \,\mathrm{d}\mu := \sup_n (GF)\int f_n \,\mathrm{d}\mu,$$

where $\{f_n\}$ is a non-decreasing sequence of simple functions with pointwise convergence to f.

It is obvious from the definition above that a GF-integral is monotone, that is,

$$f \leqslant g \;\Rightarrow\; (GF)\int f \,\mathrm{d}\mu \leqslant (GF)\int g \,\mathrm{d}\mu.$$

The next proposition follows from the definition of the pseudo-inverse $-_{\oplus}$, the generalized t-conorm integral (Definition 14), and the t-conorm integral (Definition 13).

Proposition 17. *Let μ be a fuzzy measure on a fuzzy measurable space $(X, \mathscr{X})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, if μ is a normal $\oplus$-measure, the GF integral coincides with the pseudo-decomposable integral.*

Example 18.

(1) When $\oplus = +$ and $\boxdot = \cdot$, the GF integral is a Choquet integral.
(2) When $\oplus = \vee$ and $\boxdot = \wedge$, the GF integral is a Sugeno integral.

Since we assume the continuity from below for fuzzy measures, we can prove the next monotone convergence theorem.

Theorem 19 (*Monotone convergence theorem*). *Let $(X, \mathscr{X}, \mu)$ be a fuzzy measure space and let $(\oplus, \boxdot)$ be a pseudo-fitting system. If a non-decreasing sequence $\{f_n\}$ of the measurable functions converges to a measurable function f, that is, $f_n \uparrow f$, then we have*

$$(GF)\int f \,\mathrm{d}\mu = \lim_{n\to\infty} (GF)\int f_n \,\mathrm{d}\mu.$$

Proof. Let f_n and f be measurable functions such that $f_n \uparrow f$. Let $\{f_{n,k}\}$ be a non-decreasing sequence of simple functions such that $f_{n,k} \uparrow f_n$ as $k \to \infty$. Define $g_k := \sup\{f_{n,k} | n \leqslant k\}$. Then $\{g_k\}$ is a non-decreasing sequence of simple functions. Let $g := \lim_{n\to\infty} g_k$. Then, $f_{n,k} \leqslant g_k \leqslant f_k \leqslant f$ if $n \leqslant k$. Then we have $f_n \leqslant g \leqslant f$ as $k \to \infty$. Let $n \to \infty$, we have $f \leqslant g \leqslant f$. Therefore $f = g$. We have

$$\lim_{k\to\infty} (GF)\int g_k \,\mathrm{d}\mu = (GF)\int f \,\mathrm{d}\mu.$$

Since $g_k \leqslant f_k$, we have

$$(GF)\int g_k \,\mathrm{d}\mu \leqslant (GF)\int f_k \,\mathrm{d}\mu.$$

Then we have

$$(GF)\int f \,\mathrm{d}\mu \leqslant \lim_{k\to\infty} (GF)\int f_k \,\mathrm{d}\mu.$$

On the other hand, since $f_n \leqslant f$ for all n, we have

$$\lim_{n\to\infty}(GF)\int f_n\,\mathrm{d}\mu \leqslant (GF)\int f\,\mathrm{d}\mu. \qquad \square$$

Let f, g be comonotonic measurable functions. Then, since for all $a, b > 0$ either $\{x|f(x)\geqslant a\} \subset \{x|g(x)\geqslant b\}$ or $\{x|f(x)\geqslant a\} \supset \{x|g(x)\geqslant b\}$, the following theorem can be proved.

Theorem 20. *Let $(X, \mathscr{X}, \mu)$ be a fuzzy measure space and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, for comonotonic measurable functions f, and g, we have*

$$(GF)\int (f \oplus g)\,\mathrm{d}\mu = (GF)\int f\,\mathrm{d}\mu \oplus (GF)\int g\,\mathrm{d}\mu.$$

Proof. Suppose f and g are comonotonic. Let f, g be measurable simple functions. Since we have

$$\begin{aligned}(f \oplus g)(x) &= (a+b)\boxdot b(e, A)(x)\\ &= a\boxdot b(e, A)(x) \oplus b\boxdot b(e, A)(x)\end{aligned}$$

if $f = a\boxdot b(e, A)$ and $g = b\boxdot b(e, A)$. Since $\{x|f(x)\geqslant a\} \subset \{x|g(x)\geqslant b\}$ or $\{x|f(x)\geqslant a\} \supset \{x|g(x)\geqslant b\}$ for every $a, b > 0$, we have

$$(f \oplus g)(x) := \bigoplus_{i=1}^{n} a_i \boxdot b(e, A_i)(x),$$

where $a_i > 0$, and $A_1 \supsetneq A_2 \supsetneq \cdots A_n$, $A_i \in \mathscr{X}$. Therefore we have

$$(GF)\int (f \oplus g)\,\mathrm{d}\mu = (GF)\int f\,\mathrm{d}\mu \oplus (GF)\int g\,\mathrm{d}\mu.$$

If f and g are not simple functions, there exists a sequence of simple functions $\{f_n\}$ and $\{g_n\}$ such that $f_n \uparrow f$, and $g_n \uparrow g$, and where f_n and g_n are comonotonic. Therefore we have

$$\begin{aligned}(GF)\int (f \oplus g)\,\mathrm{d}\mu &= \lim_{n\to\infty}(GF)\int (f_n \oplus g_n)\,\mathrm{d}\mu\\ &= \lim_{n\to\infty}\left((GF)\int f_n\,\mathrm{d}\mu \oplus (GF)\int g_n\,\mathrm{d}\mu\right)\\ &= \lim_{n\to\infty}(GF)\int f_n\,\mathrm{d}\mu \oplus \lim_{n\to\infty}(GF)\int g_n\,\mathrm{d}\mu\\ &= (GF)\int f\,\mathrm{d}\mu \oplus (GF)\int g\,\mathrm{d}\mu. \qquad \square\end{aligned}$$

We call this property the $\oplus$-additivity of a GF integral.

4. Multidimensional integrals

In this section we consider the case of multidimensional integrals, extending the GF integral discussed in the previous section to the multidimensional case. We start considering the concept of measurable function.

We consider first the case of the product of two fuzzy measurable spaces. Let X and Y be two universal sets and $X \times Y$ be the direct product of X and Y, let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be two fuzzy measurable spaces; then, we define the following class of sets:

$$\mathscr{X} \times \mathscr{Y} := \{A \times B | A \in \mathscr{X}, B \in \mathscr{Y}\}.$$

Now, let us consider the measurable space $(X \times Y, \mathscr{X} \times \mathscr{Y})$. Suppose that $\mathscr{X} := 2^X$ and $\mathscr{Y} := 2^Y$. Note that $\mathscr{X} \times \mathscr{Y} \neq 2^{X\times Y}$ if $|X| > 1$ and $|Y| > 1$.

Therefore, the class of $\mathscr{X} \times \mathscr{Y}$-measurable functions is smaller than the class of $2^{X\times Y}$-measurable functions.

Example 21. Let $X := \{x_1, x_2\}$ and $Y := \{y_1, y_2\}$; then, we have

$$2^X \times 2^Y := \{ \emptyset, \{(x_1, y_1)\}, \{(x_1, y_2)\}, \{(x_2, y_1)\}, \{(x_2, y_2)\}, \{(x_1, y_1), (x_2, y_1)\}, \{(x_1, y_2), (x_2, y_2)\}, \{(x_1, y_1), (x_1, y_2)\}, \{(x_2, y_1), (x_2, y_2)\}, \{(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_2)\}\}.$$

Hence, $\{(x_1, y_1), (x_2, y_2)\} \notin 2^X \times 2^Y$.

The next proposition follows from the definition of a $\mathscr{X} \times \mathscr{Y}$-measurable function.

Proposition 22. *Let $f : X \times Y \to [0, k]$ be a $\mathscr{X} \times \mathscr{Y}$-measurable function; then,*

(1) *for fixed $y \in Y$, $f(\cdot, y)$ is $\mathscr{X}$-measurable, and*
(2) *for fixed $x \in X$, $f(x, \cdot)$ is $\mathscr{Y}$-measurable.*

Proof. Let $f : X \times Y \to [0, k]$ be a $\mathscr{X} \times \mathscr{Y}$-measurable function. Then, for every $a \geqslant 0$,

$$\{(x, y) | f(x, y) \geqslant a\} \in \{A \times B | A \in \mathscr{X}, B \in \mathscr{Y}\}.$$

That is, there exist $A \in \mathscr{X}$ and $B \in \mathscr{Y}$ such that $A \times B = \{(x, y) | f(x, y) \geqslant a\}$.

To prove (1), we consider a fixed $y \in Y$. Then, $A \times \{y\} = \{(x, y) | f(x, y) \geqslant a\} = \{x | f(x, y) \geqslant a\} \times \{y\}$. Therefore $f(\cdot, y)$ is $\mathscr{X}$-measurable.

As the proof of (2) is similar but considering a fixed $x \in X$, the proposition is proven. □

Example 23. Let $X := \{x_1, x_2\}$ and $Y := \{y_1, y_2\}$. Let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be the two measurable spaces defined with $\mathscr{X} = 2^X$ and $\mathscr{Y} = 2^Y$. Under these conditions, we consider the two functions below and study whether they are $\mathscr{X} \times \mathscr{Y}$-measurable functions.

(1) Let us define $f : X \times Y \to [0, 1]$ by

$$f(x_1, y_1) = f(x_1, y_2) = 0.2,$$
$$f(x_2, y_1) = 0.6,$$
$$f(x_2, y_2) = 1.$$

Then, we have

$$\begin{aligned}\{(x, y) | f(x, y) \geqslant 1\} &= \{(x_2, y_2)\} \\ &= \{x_2\} \times \{y_2\}, \\ \{(x, y) | f(x, y) \geqslant 0.6\} &= \{(x_2, y_1), (x_2, y_2)\} \\ &= \{x_2\} \times \{x_1, y_2\}, \\ \{(x, y) | f(x, y) \geqslant 0.2\} &= \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\} \\ &= \{x_1, x_2\} \times \{y_1, y_2\}.\end{aligned}$$

Therefore, f is $\mathscr{X} \times \mathscr{Y}$-measurable.

(2) Let us define $g : X \times Y \to [0, 1]$ by

$$g(x_1, y_1) = 0.2,$$
$$g(x_1, y_2) = 0.4,$$
$$g(x_2, y_1) = 0.6,$$
$$g(x_2, y_2) = 1.$$

Then, we have

$$\{(x, y) | g(x, y) \geqslant 0.4\} = \{(x_1, x_2), (x_2, y_1), (x_2, y_2)\} \notin \mathscr{X} \times \mathscr{Y}.$$

Therefore, g is not a $\mathscr{X} \times \mathscr{Y}$-measurable function.

In fact, if $A \in \mathscr{X} \times \mathscr{Y}$, we have $|A| = 0, 1, 2$, or 4.

Next we will consider the multidimensional GF integral. The next theorem is the main result of this paper.

Theorem 24. *Let* $(X, \mathscr{X}, \mu)$ *and* $(Y, \mathscr{Y}, v)$ *be two fuzzy measure spaces*, $(\oplus, \boxdot)$ *be a pseudo-fitting system and let* $f : X \times Y \to [0, k]$ *be an* $\mathscr{X} \times \mathscr{Y}$*-measurable function. Then, there exists a fuzzy measure m on* $\mathscr{X} \times \mathscr{Y}$ *such that*

$$(GF)\int\left((GF)\int f\,\mathrm{d}\mu\right)\mathrm{d}v = (GF)\int f\,\mathrm{d}m$$
$$= (GF)\int\left((GF)\int f\,\mathrm{d}v\right)\mathrm{d}\mu.$$

Proof. If *f* is a simple function, it can be represented as

$$f(x) := \bigoplus_{i=1}^{n} b(a_i, A_i \times B_i),$$

where $a_i > 0$, $A_1 \supset A_2 \supset \cdots A_n$, $A_i \in \mathscr{X}$, and $B_1 \supset B_2 \supset \cdots B_n$, $B_i \in \mathscr{Y}$.

Now, on the one hand, as $b(e, A\times B) = b(e, A)\boxdot b(e, B)$, we have $f(x) := \bigoplus_{i=1}^{n} a_i \boxdot b(e, A_i)\boxdot\boxdot(e, B_i)$. Therefore, we have

$$(GF)\int f\,\mathrm{d}v = \bigoplus_{i=1}^{n} a_i\boxdot b(e, A_i)\boxdot v(B_i)$$
$$= \bigoplus_{i=1}^{n} a_i\boxdot v(B_i)\boxdot b(e, A_i).$$

Since each $b(e, A_i)$ is comonotonic for all *i*, it follows from comonotonic $\oplus$-additivity that

$$(GF)\int\left((GF)\int f\,\mathrm{d}v\right)\mathrm{d}\mu = (GF)\int\left(\bigoplus_{i=1}^{n} a_i\boxdot v(B_i)\boxdot b(e, A_i)\right)\mathrm{d}\mu$$
$$= \bigoplus_{i=1}^{n}(GF)\int (a_i\boxdot v(B_i)\boxdot b(e, A_i))\,\mathrm{d}\mu$$
$$= \bigoplus_{i=1}^{n} a_i\boxdot v(B_i)\boxdot \mu(A_i)$$
$$= \bigoplus_{i=1}^{n} a_i\boxdot \mu(A_i)\boxdot v(B_i).$$

On the other hand, since $f(x) := \sum_{i=1}^{n} a_i\boxdot b(e, \boxdot b(e, B_i)\boxdot b(e, A_i)$, we have

$$(GF)\int f\,\mathrm{d}\mu = \bigoplus_{i=1}^{n} a_i\boxdot b(e, B_i)\boxdot \mu(A_i)$$
$$= \bigoplus_{i=1}^{n} a_i\boxdot \mu(A_i)\boxdot b(e, B_i).$$

Since each 1_{B_i} is comonotonic for all *i*, it follows from comonotonic $\oplus$-additivity that

$$(GF)\int\left((GF)\int f\,\mathrm{d}\mu\right)\mathrm{d}v = (GF)\int\left(\bigoplus_{i=1}^{n} a_i\boxdot \mu(A_i)\boxdot b(e, B_i)\right)\mathrm{d}v$$
$$= \bigoplus_{i=1}^{n}(GF)\int (a_i\boxdot \mu(A_i)\boxdot b(e, B_i))\,\mathrm{d}v$$
$$= \bigoplus_{i=1}^{n} a_i\boxdot \mu(A_i)\boxdot v(B_i).$$

Let us define a fuzzy measure m on $\mathscr{X} \times \mathscr{Y}$ by

$$m(A \times B) := \mu(A) \boxdot \nu(B) \quad \text{for } A \times B \in \mathscr{X} \times \mathscr{Y}.$$

Then we have

$$(GF)\int \left((GF)\int f\,\mathrm{d}\mu\right)\mathrm{d}\nu = (GF)\int f\,\mathrm{d}m = (GF)\int \left((GF)\int f\,\mathrm{d}\nu\right)\mathrm{d}\mu.$$

Let f be an arbitrary measurable function. Then, there exists a sequence of simple functions f_n such that $f_n \uparrow f$. Therefore, from the monotone convergence theorem (Theorem 19) of the GF-integral, we have

$$(GF)\int f_n\,\mathrm{d}\mu \uparrow (GF)\int f\,\mathrm{d}\mu$$

and

$$(GF)\int f_n\,\mathrm{d}\nu \uparrow (GF)\int f\,\mathrm{d}\nu. \qquad \square$$

Since both Choquet and Sugeno integrals are generalizations of the GF-integral, and considering $\oplus = +$ and $\boxdot = \cdot$, we have the next equality, which was proven by Machida [6]:

$$\begin{aligned}(C)\int \left((C)\int f(x,y)\,\mathrm{d}\mu\right)\mathrm{d}\nu &= (C)\int \left((C)\int f(x,y)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\ &= (C)\int f(x,y)\,\mathrm{d}m.\end{aligned}$$

Considering the characteristic function of $A \times B \in \mathscr{X} \times \mathscr{Y}$, we have $m = \mu\nu$.

In the case of the Sugeno integral, let $\oplus = \vee$ and $\boxdot = \wedge$; then, we have a similar equality:

$$\begin{aligned}(S)\int \left((S)\int f(x,y)\,\mathrm{d}\mu\right)\mathrm{d}\nu &= (S)\int \left((S)\int f(x,y)\,\mathrm{d}\nu\right)\mathrm{d}\mu \\ &= (S)\int f(x,y)\,\mathrm{d}m.\end{aligned}$$

Considering the characteristic function of $A \times B \in \mathscr{X} \times \mathscr{Y}$, we have $m = \mu \wedge \nu$.

5. Extension of the domain

Let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be two fuzzy measurable spaces. In this section we assume that $\mathscr{X}$ and $\mathscr{Y}$ are algebras. Even if $|X|, |Y|$ are finite and such that $\mathscr{X} := 2^X$ and $\mathscr{Y} := 2^Y$, the class of $\mathscr{X} \times \mathscr{Y}$ is smaller than the class of $2^{X \times Y}$, as we have shown in Example 21. In general, the class of $\mathscr{X} \times \mathscr{Y}$-measurable functions is too small as shown in Example 23.

In this section we consider an extension of the domain of the measure. In general, unless there are additional constraints or conditions on the fuzzy measures, it is impossible to extend the domain. However, in our case, we assume that μ on $(X, \mathscr{X})$ and ν on $(Y, \mathscr{Y})$ are normal $\oplus$-measures. In this case, an extension is possible. We define the extension below.

Definition 25. Let us define the class $\overline{\mathscr{X} \times \mathscr{Y}}$ of sets $A \in 2^{X \times Y}$ by

$$\overline{\mathscr{X} \times \mathscr{Y}} := \left\{ A \in 2^{X \times Y} | A = \bigcup_{i \in I} A_i,\, A_i \in \mathscr{X} \times \mathscr{Y},\, I : \text{finite} \right\}.$$

We say that $(X \times Y, \overline{\mathscr{X} \times \mathscr{Y}})$ is an extended fuzzy measurable space.

The next proposition follows immediately from this definition.

Proposition 26. *Let $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$ be two fuzzy measurable spaces. Let $(X \times Y, \overline{\mathscr{X} \times \mathscr{Y}})$ be an extended fuzzy measurable space.*

(1) *$\overline{\mathscr{X} \times \mathscr{Y}}$ is an algebra.*
(2) *Let f on $X \times Y$ be $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable*
 (a) *$f(x, \cdot)$ is $\mathscr{X}$-measurable.*
 (b) *$f(\cdot, y)$ is $\mathscr{Y}$-measurable.*

We consider now a special class of fuzzy measures.

Definition 27. Let $(X, \mathscr{X})$ be a measurable space. We call a fuzzy measure μ on X $\oplus$-submodular if

$$\mu(A \cap B) \oplus \mu(A \cup B) \leqslant \mu(A) \oplus \mu(B)$$

for $A, B \in \mathscr{X}$, and $\oplus$-supermodular if

$$\mu(A \cap B) \oplus \mu(A \cup B) \geqslant \mu(A) \oplus \mu(B)$$

for $A, B \in \mathscr{X}$, $\oplus$-subadditive if

$$\mu(A \cup B) \leqslant \mu(A) \oplus \mu(B)$$

for $A, B \in \mathscr{X}$ and $A \cap B = \emptyset$, and $\oplus$-superadditive if

$$\mu(A \cup B) \geqslant \mu(A) \oplus \mu(B)$$

for $A, B \in \mathscr{X}$ and $A \cap B = \emptyset$.

It is obvious from this definition that a belief function is $+$ superadditive, a plausibility measure is $+$ subadditive, a possibility measure is $\vee$ superadditive, and a necessity measure is subadditive.

Let us suppose that $|X|, |Y|$ are finite. Since $\{(x, y)\} = \{x\} \times \{y\} \in \mathscr{X} \times \mathscr{Y}$ for every $x \in X$ and $y \in Y$, we have the next corollary.

Corollary 28. *Let $(X \times Y, \overline{\mathscr{X} \times \mathscr{Y}})$ be an extended fuzzy measurable space. Let us suppose that $|X|, |Y|$ are finite. If $\mathscr{X} = 2^X$ and $\mathscr{Y} = 2^Y$, then $\overline{\mathscr{X} \times \mathscr{Y}} = 2^{X \times Y}$.*

It follows from Corollary 28 that every function $f : X \times Y \to [0, k]$ is $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable, if $|X|, |Y|$ are finite.

Definition 29. Let $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ be two fuzzy measure spaces, and $(X \times Y, \overline{\mathscr{X} \times \mathscr{Y}})$ be an extended fuzzy measurable space. We define the fuzzy measures $\overline{m}$ and $\underline{m}$ on $\overline{\mathscr{X} \times \mathscr{Y}}$ induced by μ and ν by

$$\overline{m}(C) := \sup \left\{ \bigoplus_{i \in I} \mu(A_i) \boxdot \nu(B_i) | C = \bigcup_{i \in I} (A_i \times B_i),\ A_i \times B_i \in \overline{\mathscr{X} \times \mathscr{Y}},\ I : \text{finite} \right\}$$

and

$$\underline{m}(C) := \inf \left\{ \bigoplus_{i \in I} \mu(A_i) \boxdot \nu(B_i) | C = \bigcup_{i \in I} (A_i \times B_i),\ A_i \times B_i \in \overline{\mathscr{X} \times \mathscr{Y}},\ I : \text{finite} \right\},$$

where each $A_i \times B_i$ and $A_j \times B_j$ are disjoint. We call $\overline{m}(C)$ the upper $\oplus$-fuzzy measure induced by μ and ν, and $\underline{m}(C)$ the lower $\oplus$-fuzzy measure induced by μ and ν.

Consider a simple function $f = b(c_1, C_1) \oplus b(c_2, C_2)$, with $C_1 \supset C_2$ and $C_i \in \overline{\mathscr{X} \times \mathscr{Y}}$, defined in such a way that $C_1 := \bigcup_{i \in I} (A_{1i} \times B_{1i})$ for disjoint pairs $(A_{1i} \times B_{1i})$ with $A_{1i} \in \mathscr{X}$, $B_{1i} \in \mathscr{Y}$, and $C_2 := \bigcup_{i \in I} (A_{2i} \times B_{2i})$ for disjoint pairs $(A_{2i} \times B_{2i})$ with $A_{2i} \in \mathscr{X}$, $B_{2i} \in \mathscr{Y}$, and satisfying $A_{1i} \supset A_{2i}$, $B_{1i} \supset B_{2i}$.

Then, in relation to the integral of f with respect to $\overline{m}$, we have

$$c_1 \boxdot m(C_1) \oplus c_2 \boxdot m(C_2) \geqslant \left(c_1 \boxdot \left(\bigoplus_{i\in I} \mu(A_{1i}) \boxdot \nu(B_{1i})\right)\right) \oplus \left(c_2 \boxdot \left(\bigoplus_{i\in I} \mu(A_{1i}) \boxdot \nu(B_{2i})\right)\right).$$

The next proposition is the generalization of this inequality.

Proposition 30. *Let $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ be two fuzzy measure spaces, $(X \times Y, \overline{\mathscr{X} \times \mathscr{Y}})$ be an extended fuzzy measurable space, $\overline{m}$ the upper $\oplus$-fuzzy measure induced by μ and ν, and $\underline{m}$ the lower $\oplus$-fuzzy measure induced by μ and ν. Let f on $X \times Y$ be an $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable function. Then, there exists a partition of $X \times Y$, that is, $X \times Y := \bigcup_{i\in I}(A_i \times B_i)$, $A_i \in \mathscr{X}$, $B_i \in \mathscr{Y}$, such that*

(1)

$$\bigoplus_{i\in I}(GF)\int_{B_i}\left((GF)\int_{A_i} f\,\mathrm{d}\mu\right)\mathrm{d}\nu = \bigoplus_{i\in I}(GF)\int_{A_i}\left((GF)\int_{B_i} f\,\mathrm{d}\nu\right)\mathrm{d}\mu.$$

(2)

$$(GF)\int f\,\mathrm{d}\overline{m} \geqslant \bigoplus_{i\in I}(GF)\int_{Bi}\left((GF)\int_{A_i} f\,\mathrm{d}\mu\right)\mathrm{d}\nu.$$

(3)

$$\bigoplus_{i\in I}(GF)\int_{Bi}\left((GF)\int_{A_i} f\,\mathrm{d}\mu\right)\mathrm{d}\nu \geqslant (GF)\int f\,\mathrm{d}\underline{m}.$$

In the following we suppose that $|X|, |Y|$ are finite. Let $A \in \overline{\mathscr{X} \times \mathscr{Y}}$; then, A can be represented as

$$A = \bigcup_{(i,j)\in I} \{x_i\} \times \{y_j\}$$

for a finite set I. Let μ on $(X, \mathscr{X})$ and ν on $(Y, \mathscr{Y})$ be both $\oplus$-subadditive measures. Then we have

$$\overline{m}(A) := \bigoplus_{(i,j)\in I} \mu(\{x_i\}) \boxdot \nu(\{y_j\}).$$

Therefore we have the next proposition.

Proposition 31. *Let $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ be two fuzzy measure spaces, $(X \times Y, \overline{\mathscr{X} \times \mathscr{Y}})$ be an extended fuzzy measurable space, $\overline{m}$ the upper $\oplus$-fuzzy measure induced by μ and ν, and $\underline{m}$ the lower $\oplus$-fuzzy measure induced by μ and ν. Then, let us define the normal decomposable measures $\overline{\mu}$ and $\overline{\nu}$ by $\overline{\mu}(A_1) = \bigoplus_i \mu(\{x_i\});\ x_i \in A_1,\ A_1 \in \mathscr{X}$ and $\overline{\nu}(A_2) = \bigoplus_i \nu(\{x_i\});\ x_i \in A_2,\ A_2 \in \mathscr{Y}$.*

Finally, let f on $X \times Y$ be an $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable function.

(1) *If μ and ν are $\oplus$-submodular, then we have*

$$(GF)\int f\,\mathrm{d}\overline{m} = (GF)\int\left((GF)\int f\,\mathrm{d}\overline{\mu}\right)\mathrm{d}\overline{\nu} = (GF)\int\left((GF)\int f\,\mathrm{d}\overline{\nu}\right)\mathrm{d}\overline{\mu}.$$

(2) *If μ and ν are $\oplus$-supermodular, then we have*

$$(GF)\int f\,\mathrm{d}\underline{m} = (GF)\int\left((GF)\int f\,\mathrm{d}\overline{\mu}\right)\mathrm{d}\overline{\nu} = (GF)\int\left((GF)\int f\,\mathrm{d}\overline{\nu}\right)\mathrm{d}\overline{\mu}.$$

If μ and v are both $\oplus$-measures, then μ and v are both $\oplus$-sub- and $\oplus$-superadditive. Therefore we have $\overline{\mu} = \mu$ and $\overline{v} = v$. It follows from this latter proposition (Proposition 31) that we can define a fuzzy measure m on $\overline{\mathscr{X} \times \mathscr{Y}}$, as shown above.

Definition 32. Let $A \in \overline{\mathscr{X} \times \mathscr{Y}}$; then, A can be represented as

$$A = \bigcup_{(i,j)\in I} \{x_i\} \times \{y_j\}$$

for a finite set I. Let μ on $(X, \mathscr{X})$ and v on $(Y, \mathscr{Y})$ be $\oplus$-measures.

Then, we define a fuzzy measure m on $\overline{\mathscr{X} \times \mathscr{Y}}$ generated by μ and v by

$$m(A) := \bigoplus_{(i,j)\in I} \mu(\{x_i\}) \boxdot v(\{y_j\}).$$

This definition permits the measure $m(A)$ to be the cardinality of A. The next proposition follows immediately from these definitions.

Proposition 33. *Let μ on $(X, \mathscr{X})$ and v on $(Y, \mathscr{Y})$ be normal $\oplus$-measures. The fuzzy measure m generated by μ and v on $\overline{\mathscr{X} \times \mathscr{Y}}$ is a normal $\oplus$-measure.*

The next theorem follows from the definition of the pseudo-decomposable integral (Definition 13).

Theorem 34. *Let μ on $(X, \mathscr{X})$ and v on $(Y, \mathscr{Y})$ be normal $\oplus$-measures and let f on $X \times Y$ be an $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable function. Then, there exists a fuzzy measure m on $\overline{\mathscr{X} \times \mathscr{Y}}$ such that*

$$(D) \int \left((D) \int f \, \mathrm{d}\mu \right) \mathrm{d}v = (D) \int f \, \mathrm{d}m$$
$$= (D) \int \left((D) \int f \, \mathrm{d}v \right) \mathrm{d}\mu.$$

Considering $\oplus = +$ and $\boxdot = \cdot$, this integral corresponds to the standard Lebesgue integral. Then, Theorem 34 is a part of the standard Fubini's Theorem.

In contrast, considering $\oplus = \vee$ and $\boxdot = \wedge$, we have the next corollary.

Corollary 35. *Let $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, v)$ be two fuzzy measure spaces and let f be an $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable function. Then, there exists a fuzzy measure m on $\overline{\mathscr{X} \times \mathscr{Y}}$ such that*

$$(S) \int \left((S) \int f(x, y) \, \mathrm{d}\mu \right) \mathrm{d}v = (S) \int \left((S) \int f(x, y) \, \mathrm{d}v \right) \mathrm{d}\mu$$
$$= (S) \int f(x, y) \, \mathrm{d}m.$$

Considering the characteristic function of $A \times B \in \mathscr{X} \times \mathscr{Y}$, we have $m(A) = \mu(A) \wedge v(B)$ and m is a possibility measure.

6. On the use of multidimensional integrals for citation analysis

The definition of multidimensional integrals was partly motivated by our interest on citation indices. We consider here, again, the functions f and g already introduced in Section 1. That is, $f(x)$ corresponds to the number of citations of a paper x at the time of study and $g_\gamma(x)$ the number of citations obtained by the paper x in year γ. Then, naturally, for a fixed year γ, $f(x) = \sum_{y \leqslant \gamma} g_y(x)$.

Table 1
Set of functions corresponding to the new number of citations for each year in the period 2002–2006 and for the papers $X = \{p_1, p_2, p_3, p_4, p_5\}$

	p_1	p_2	p_3	p_4	p_5	p_6
g_{2006}	60	30	5	4	2	1
g_{2005}	30	0	2	1	0	0
g_{2004}	8	20	0	0	0	0
g_{2003}	2	0	0	0	0	0
g_{2002}	0	0	0	0	0	0

Example 36. Let $X = \{p_1, p_2, p_3, p_4, p_5\}$ and $Y = \{2002, 2003, 2004, 2005, 2006\}$. Then, we define the functions g_y for $2002 \leqslant y \leqslant 2006$ from Table 1. From these functions g_y, we define the function $f : X \times Y \to [0, 100]$ as $f(x) = \sum_{y \leqslant 2006} g_y(x)$.

Now, we consider the multidimensional integral of g in Example 36. First note that the function g is not $\mathscr{X} \times \mathscr{Y}$-measurable but it is $\overline{\mathscr{X} \times \mathscr{Y}}$-measurable. For example $\{(p_1, 2006), (p_2, 2006), (p_3, 2005)\}$ is not in $\mathscr{X} \times \mathscr{Y}$ but it is in $\overline{\mathscr{X} \times \mathscr{Y}} = 2^{X \times Y}$. Therefore, we cannot apply Theorem 24 in this case. In contrast, Theorem 34 is applicable.

Note that the direct integration of the function $g_y(x)$ needs the measure m defined on $\{(p_1, 2006), (p_2, 2006), (p_1, 2005), (p_2, 2004)\}$. While this is well defined using the extension used in Theorem 34, this is not the case in Theorem 24.

The application of these theorems requires measures μ and v. Let us consider $\mu(A) = |A|$ for $A \subseteq X$ and $v(B) = |B|$ for $B \subseteq Y$. Let us also consider $\oplus = +$ and $\boxdot = \cdot$. It is clear that both μ and v are $\oplus$-measures. Therefore, Theorem 34 can be applied. This theorem states that the order in which we integrate the function g is not relevant. Moreover, if we consider Definition 32 and we define

$$m(A) := \bigoplus_{(i,j)\in I} \mu(\{x_i\}) \boxdot v(\{y_j\}) = \bigoplus_{(i,j)\in I} 1 \boxdot 1,$$

the results of the integrals are also equivalent to the integral of g with respect to m. As we have that $\oplus = +$ and $\boxdot = \cdot$, $m(A)$ corresponds to the cardinality of the set A. I.e., $m(A) = |A|$.

In the particular case of the set $\{(p_1, 2006), (p_2, 2006), (p_1, 2005), (p_2, 2004)\}$, the measure is, of course, 4.

So, taking all this into account, we have that the three equivalent expressions in Theorem 34 correspond in our case to the number of citations (i.e., $\sum_y \sum_x g_y(x)$). This also corresponds to the Choquet integral of the function $f(x) = \sum_y g_y(x)$ with respect to a measure equal to the cardinality of the sets [13].

In contrast, note that for these measures μ and v we cannot apply Theorem 34 with $\oplus = \vee$ because in this case μ and v are not $\oplus$-measures.

In general, the definition of the different measures μ and v will result into different indices. In the particular case where the measures are $\oplus$-measures, Theorem 34 can be applied and, thus, the result of the integral is independent of the order in which the variables are considered. In the particular case of $\oplus = \vee$ and $\boxdot = \wedge$, other fuzzy measures than the cardinality should be considered because, as shown above, this measure is not a $\vee$-measure.

7. Conclusions and future work

We have introduced in this paper an extension of fuzzy measures in $2^{\mathscr{X} \times \mathscr{Y}}$ using pseudo-additions, which permits us to construct fuzzy measures that go beyond the [0, 1] interval. Such measures are analogous to the ones in [13] that permitted us to show that some indices correspond to fuzzy integrals. We plan to work further on this direction defining new indices using multidimensional integrals.

Acknowledgements

Partial support by the Spanish MEC (projects ARES—CONSOLIDER INGENIO 2010 CSD2007-00004—and eAEGIS—TSI2007-65406-C03-02) is acknowledged. The comments of the referees are gratefully acknowledged.

References

[1] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, Amsterdam, 2002, pp. 1329–1379.

[2] G. Choquet, Theory of capacities, Ann. Inst. Fourier 5 (1953/54) 131–295.

[3] C. Dellacherie, Quelques commentaires sur les prolongements de capacités, Séminaire de Probabilités 1969/1970, Strasbourg, Lecture Notes in Mathematics, Vol. 191, 1971, pp. 77–81.

[4] J.E. Hirsch, An index to quantify an individual's scientific research output, Proc. Natl. Acad. Sci. 102(45) (2005) 16569–16572.

[5] C.H. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965) 189–212.

[6] M. Machida, A study on non-additive set functions, Master Thesis, Tokyo Institute of Technology, 1990 (in Japanese).

[7] T. Murofushi, M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems 29 (1989) 201–227.

[8] Y. Narukawa, V. Torra, Multidimensional integrals and citation analysis, in: Proc. of EUROFUSE 2007, 2007, pp. 185–189.

[9] Y. Narukawa, V. Torra, Multidimensional fuzzy integrals, in: MDAI 2007, Lecture Notes in Artificial Intelligence, Vol. 4617, 2007, pp. 145–155.

[10] D. Ralescu, G. Adams, The fuzzy integral, J. Math. Anal. Appl. 75 (1980) 562–570.

[11] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. Dissertation, Tokyo Institute of Technology, 1974.

[12] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, J. Math. Anal. Appl. 122 (1987) 197–222.

[13] V. Torra, Y. Narukawa, The h-index and the number of citations: two fuzzy integrals, IEEE Trans. Fuzzy Systems 16 (3) (2008) 795–797.