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Fuzzy Sets and Systems 78 (1996) 73–87

FUZZY sets and systems

A model of consensus in group decision making under linguistic assessments

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Received October 1994; revised February 1995

Abstract

This paper presents a consensus model in group decision making under linguistic assessments. It is based on the use of linguistic preferences to provide individuals' opinions, and on the use of fuzzy majority of consensus, represented by means of a linguistic quantifier. Several linguistic consensus degrees and linguistic distances are defined, acting on three levels. The consensus degrees indicate how far a group of individuals is from the maximum consensus, and linguistic distances indicate how far each individual is from current consensus labels over the preferences. This consensus model allows to incorporate more human consistency in decision support systems.

Keywords: Group decision making; Linguistic modelling; Consensus degree

1. Introduction

Decision making problems basically consist of finding the best option from a feasible option set. As human beings are constantly making decisions in the real world, in many situations, the use of computerized decision support systems, may be much help in solving decision making problems. Thus, the designing and building of "intelligent" decision support systems has become a research field of high growth in the last few years.

In these systems the problem is how to introduce intelligence into them, that is, how to incorporate human consistency in decision making models of decision support systems. There are several approaches to the problem by means of fuzzy-logic-based tools. Many authors have provided interesting results on multicriteria decision making, multistage decision making, solving methods of group decision making, and measures for consensus formation in group decision making. In all the cases the fuzzy logic has played an important role.

The group decision problem is established in an environment where there is a question to solve, a set of possible options, and a set of individuals (experts, judges, etc.), who present their opinions or preferences over the set of possible options. A distinguished person may exist, called a moderator, responsible for directing the session until all individuals reach an agreement on the solution to choose.

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SSDI 0165-0114(95)00107-7

In a usual framework, there are a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ with their respective *relevance degrees* defined as a real numbers, such that $\mu_R(i) \in [0, 1]$ denotes the relevance degree of the alternative x_i, and a finite set of individuals $N = \{1, \ldots, m\}$ with their respective *importance degrees* also defined as real numbers, such that, $\mu_G(k) \in [0, 1]$ denotes the importance degree of individual "k". Each individual $k \in N$ provides his or her opinions on X as a fuzzy preference relation $P^k \subset XxX$, with $p_{ij}^k \in [0, 1]$ denoting the preference degree of the alternative x_i over x_j. As it is evident, if some vagueness is assumed and coherently $\mu_R(i)$, $i = 1, \ldots, n$, and $\mu_G(k)$, $k \in N$, are defined as fuzzy sets, then the considered model becomes identical to that in [11].

Sometimes, however an individual could have a vague knowledge about the preference degree of the alternative x_i over x_j and cannot estimate his preference with an exact numerical value. Then a more realistic approach may be to use linguistic assessments instead of numerical values, that is, to suppose that the variables which participate in the problem are assessed by means of linguistic terms [3, 4, 6, 8, 12, 14]. A scale of certainty expressions (linguistically assessed) would be presented to the individuals, who could then use it to describe their degrees of certainty in a preference. In this environment we have linguistic preference relations to provide individuals' opinions.

On the other hand, assuming a set of alternatives or decisions, the basic question is how to relate to different decision schemata. According to [2] there are (at least) two possibilities: a group selection process and a consensus process. The first, a calculation of some mean value decision schema of a set of decisions D would imply the choice of an *algebraic consensus* as a mapping $l: DxD \Rightarrow D$, whereas the second, the measure of distance between schemata, it could be called *topological consensus* involving a mapping $k: DxD \Rightarrow L$, where L is a complete lattice. In [6, 8] models were proposed to the first possibility under linguistic preferences. Here, we will focus on the second possibility, to develop a consensus process under linguistic preferences.

Usually, a group of individuals initially have disagreeing opinions. The consensus reaching process is a necessity of all group decision making processes, because to achieve a general consensus about selected options is a desirable goal. Consensus is traditionally meant as a full and unanimous agreement of all individuals' opinions (it is *the maximum consensus*). Obviously, this type of consensus is an ideal consensus and very difficult to achieve. Therefore, it is quite natural to look for the highest consensus, that is, the maximum possible consensus. This process is viewed as a dynamic process where a moderator, via exchange of information and rational arguments, tries to persuade the individuals to update their opinions. In each step, the degree of existing consensus and the distance from an *ideal consensus* is measured. The moderator uses the degree or of consensus to control the process. This is repeated until the group gets closer to a maximum consensus, i.e., either until the distance to the *ideal consensus* is considered sufficiently small, or until individuals' opinions become sufficiently similar. Therefore, the type of consensus obtained in this way is not an *absolute consensus* in the ordinary sense, it is a *relative and gradual consensus*. This is represented in Fig. 1.

Although this consensus framework has been considered by several authors [2, 5, 11], in all the cases the measures used for the Consensus Reaching Process in Group Decision Making are focused on the expert set and calculated in numerical context, and however, from this point of view, in this paper we develop a new fuzzy-logic-based consensus model in group decision making focused on the alternative set and calculated in linguistic context. Therefore, the novelty here is on acting in the alternative set assessed linguistically.

The proposed model calculates two types of consensus measures in this set, which are applied in three acting levels: *level of preference*, *level of alternative*, and *level of preference relation*. These measures are:

1. *Consensus degrees.* Used to evaluate the current consensus stage, and constituted by three measures: *the preference linguistic consensus degree*, *the alternative linguistic consensus degree*, and *the relation linguistic consensus degree*.

2. *Linguistic distances.* Used to evaluate the individuals' distance to social opinions, and constituted by three measures: *the preference linguistic*

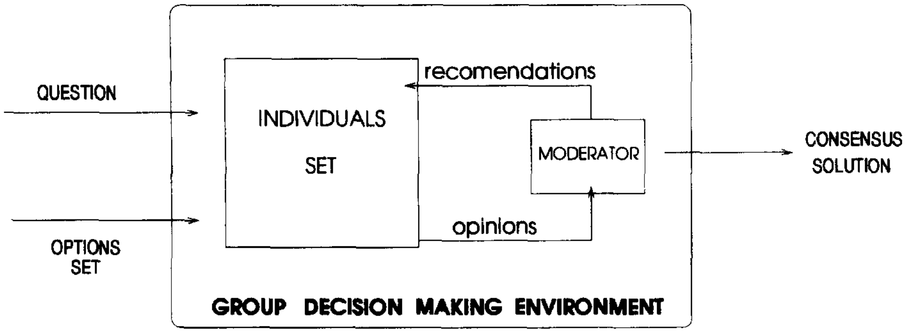

Fig. 1. Consensus reaching process in group decision making.

distance, the alternative linguistic distance, and the relation linguistic distance. These measures are calculated using three processes:

1. *Counting process.* To count the individuals' opinions over preference values.

2. *Coincidence process.* To choose the *coincidence degree*, that is, the proportion of individuals, who are in agreement in their preference values, and also to calculate the *consensus labels*, that is, the social opinion over preference values.

3. *Computing process.* To calculate each one of the above measures in its respective level.

In the following, Section 2 shows the importance of the linguistic assessments and the linguistic quantifiers in group decision making. Section 3 presents the consensus model. Then, and for the sake of illustrating the consensus reaching process described, Section 4 is devoted to develop an easy example, and finally, in Section 5 some conclusions are pointed out.

2. Linguistic assessments and linguistic quantifiers

2.1. Linguistic assessments in decision making

The linguistic approach considers the variables which participate in the problem assessed by means of linguistic terms instead of numerical values [16]. This approach is appropriate for a lot of problems, since it allows a representation of individuals' information in a more direct and adequate form whether they are unable of expressing that with precision.

A linguistic variable differs from a numerical one in that its values are not numbers, but words or sentences in a natural or artificial language. Since words, in general, are less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means of approximated characterization of phenomena, which are too complex, or too ill-defined to be amenable for description in conventional quantitative terms.

Therefore, we need a term set defining the uncertainty granularity, that is the level of distinction among different countings of uncertainty [17]. The elements of the term set will determine the granularity of the uncertainty. In [1] the use of term sets with odd cardinal was studied, representing the middle term, an assess of "approximately 0.5", being the rest terms placed symmetrically around it and the limit of granularity 11 or no more than 13.

The semantic of the elements of the term set is given by fuzzy numbers defined on the [0, 1] interval, which are described by membership functions. Because the linguistic assessments are just approximate ones given by the individuals, we can consider that linear trapezoidal membership functions are good enough to capture the vagueness of those linguistic assessments, since to obtain more accurate values may be impossible or unnecessary. This representation is achieved by the 4-tuple $(a_i, b_i, \alpha_i, \beta_i)$, the first two parameters indicate the interval in which the membership value is 1; the third and fourth parameters indicate the left and right width.

For instance, Fig. 2 shows an hierarchical structure of linguistic values or labels.

As it is clear, level 1 provides a granularity containing three labels, level 2 a granularity with nine labels, and of course, different granularity levels could be presented. In fact, in this figure level 4 presents the finest granularity in a decision process, the numerical values.

Accordingly, to establish what kind of label set to use ought to be the first priority. Then, let $S = \{s_i\}$, $i \in H = \{0, \ldots, T\}$, be a finite and totally ordered term set on $[0,1]$ in the usual sense [1, 3, 17]. Any label s_i represents a possible value for a linguistic real variable, that is, a vague property of constraint on $[0,1]$. We consider a term set with odd cardinal, where the middle label represents an uncertainty of "approximately 0.5" and the rest terms are placed symmetrically around it. Moreover, the term set must have the following characteristics:

(1) The set is ordered: $s_i \geqslant s_j$ if $i \geqslant j$.

(2) There is the negation operator: $\mathrm{Neg}(s_i) = s_j$ such that $j = T - i$.

(3) Maximization operator: $\mathrm{Max}(s_i, s_j) = s_i$ if $s_i \geqslant s_j$.

(4) Minimization operator: $\mathrm{Min}(s_i, s_j) = s_i$ if $s_i \leqslant s_j$.

For example, this is the case of the following term set of level 2:

$$S = \{s_6 = P, s_5 = VH, s_4 = H, s_3 = M,$$
$$s_2 = L, s_1 = VL, s_0 = N\},$$

where

$P = Perfect$, $VH = Very\text{-}High$, $H = High$,

$M = Medium$, $L = Low$, $VL = Very\text{-}Low$,

$N = None$.

2.2. Linguistic quantifiers in decision making

The fuzzy linguistic quantifiers were introduced by Zadeh in 1983 [18]. Linguistic quantifiers are typified by terms such as *most*, *at least half*, *all*, *as many as possible* and assumed a quantifier Q to be a fuzzy set in $[0,1]$. Zadeh distinguished between two types of quantifiers, absolute and proportional or relative. Absolute quantifiers are used to represent amounts that are absolute in nature. These quantifiers are closely related to the concepts of the count of number of elements. Zadeh suggested that these absolute quantifier values can be represented as fuzzy subsets of the non-negative reals, R^+. In particular, he suggested that an absolute quantifier can be represented by a fuzzy subset Q, where for

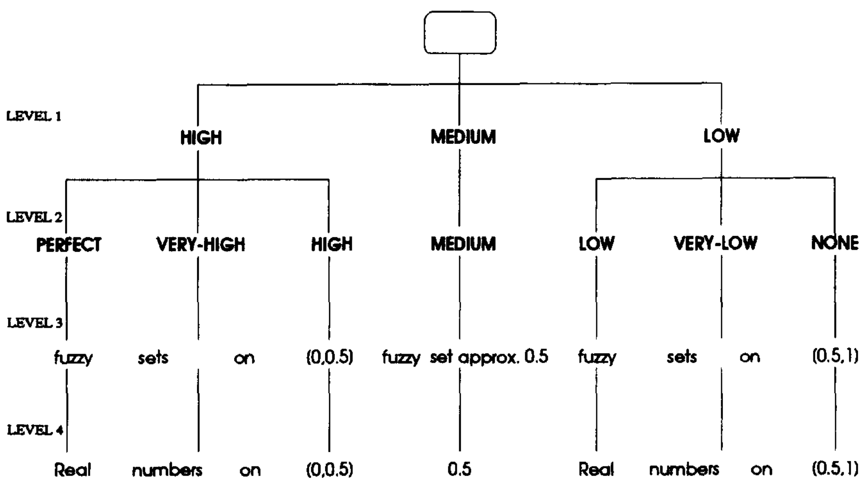

Fig. 2. Hierarchy of labels.

any $r \in R^+$, $Q(r)$ indicates the degree to which the value r satisfies the concept represented by Q, and, relative quantifiers represent proportion type statements. Thus, if Q is a relative quantifier, then Q can be represented as a fuzzy subset of $[0,1]$ such that for each $r \in [0,1]$, $Q(r)$ indicates the degree to which r portion of objects satisfies the concept devoted by Q.

An absolute quantifier $Q: R^+ \to [0,1]$ satisfies

$$Q(0) = 0 \quad \text{and } \exists k \text{ such that } Q(k) = 1.$$

A relative quantifier, $Q: [0,1] \to [0,1]$, satisfies

$$Q(0) = 0, \quad \text{and } \exists r \in [0,1] \text{ such that } Q(r) = 1.$$

A non-decreasing quantifier satisfies

$$\forall a, b \text{ if } a > b \quad \text{then } Q(a) \geqslant Q(b).$$

The membership function of a non-decreasing relative quantifier can be represented as

$$Q(r) = \begin{cases} 0 & \text{if } r < a, \\ \dfrac{r-a}{b-a} & \text{if } a \leqslant r \leqslant b, \\ 1 & \text{if } r > b, \end{cases}$$

with $a, b, r \in [0,1]$.

In order to create a more flexible framework to the moderator, we will use two types of relative quantifiers. One, numerical valued described above, and denoted Q^1, and the other linguistically valued on a label set $L = \{l_i\}$, $i \in J = \{0, \ldots, U\}$, denoted Q^2,

$$Q^2: [0,1] \to L$$

and defined as follows:

$$Q^2(r) = \begin{cases} l_0 & \text{if } r < a, \\ l_i & \text{if } a \leqslant r \leqslant b, \\ l_U & \text{if } r > b. \end{cases}$$

l_0 and l_U are the minimum and maximum labels in L, respectively, and

$$l_i = \operatorname*{Sup}_{l_q \in M} \{l_q\}$$

with

$$M = \left\{ l_q \in L \colon \mu_{l_q}(r) = \operatorname*{Sup}_{t \in J} \left\{ \mu_{l_t}\left(\frac{r-a}{b-a}\right) \right\} \right\}$$

with $a, b, r \in [0,1]$. Another definition of Q^2 can be found in [15].

Some examples of relative quantifiers are shown in Fig. 3, where the parameters (a, b) are $(0.3, 0.8)$, $(0, 0.5)$ and $(0.5, 1)$, respectively.

Remark. It is important to quote that we will use the fuzzy quantifier in order to represent the concept of *fuzzy majority*, essential in our consensus model. Q^1 will be used to aggregate linguistic labels for obtaining the *label consensus relation* and the *linguistic distances*; and Q^2 will be used for obtaining the *consensus degrees*. In the following, for simplicity and without loss of

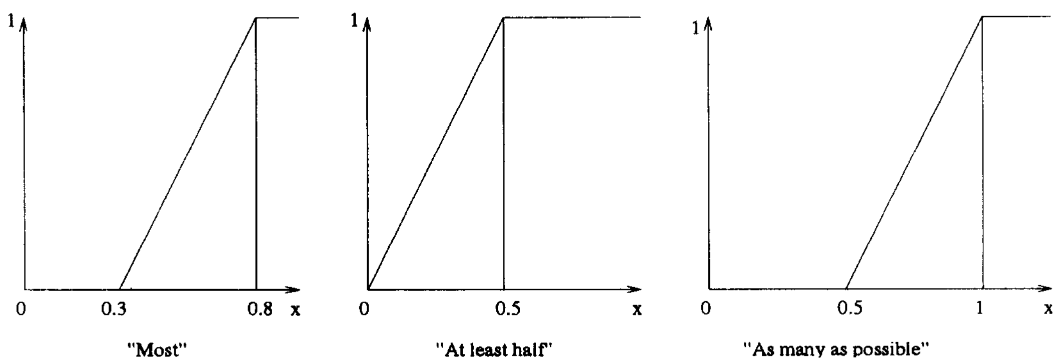

Fig. 3. Linguistic quantifiers.

generality, we still assume that the term set for the preferences among options (S) and for evaluating the consensus degree (L) are the same and denoted S.

3. A consensus model under linguistic assessments

As we said at the beginning, we are assuming a set of alternatives $X = \{x_1, \ldots, x_n\}$ and a set of individuals $N = \{1, \ldots, m\}$. For each alternative $x_i \in X$ we will suppose define a *relevance degree* $\mu_R(i) \in [0, 1]$, 0 standing for 'definitely irrelevant' and 1 standing for 'definitely relevant', through all intermediate values. Similarly, for each individual $k \in N$ we will assume known an *importance degree* $\mu_G(k) \in [0, 1]$ with a meaning clear enough. Then, the described model considers that each individual $k \in N$ provides his or her opinions on X as preference relation linguistically assessed into the term set, S,

$$\phi_{P^k}: XxX \to S,$$

where $\phi_{P^k}(x_i, x_j) = p_{ij}^k \in S$ represents the linguistically assessed preference degree of the alternative x_i over x_j. We assume that P^k is reciprocal in the sense, $p_{ij}^k = \mathrm{Neg}(p_{ji}^k)$, and by definition $p_{ii}^k = s_0$ (the minimum label in S).

As previously mentioned, we present a consensus model based on the idea of counting the number of individuals that are in agreement over the linguistic value assigned to each preference, and the aggregation of that information under fuzzy majority. Its most important feature is that it incorporates two *consensus measure* types:

(i) *Linguistic consensus degrees*: To evaluate the current consensus existing among individuals.

(ii) *Linguistic distances*: To evaluate the distance of individuals' opinions to the current *consensus labels* of preferences.

Both measures, used jointly, describe with a great exactness the current consensus situation, and help the moderator a great deal in the consensus reaching process.

Graphically, this consensus model is reflected in Fig. 4.

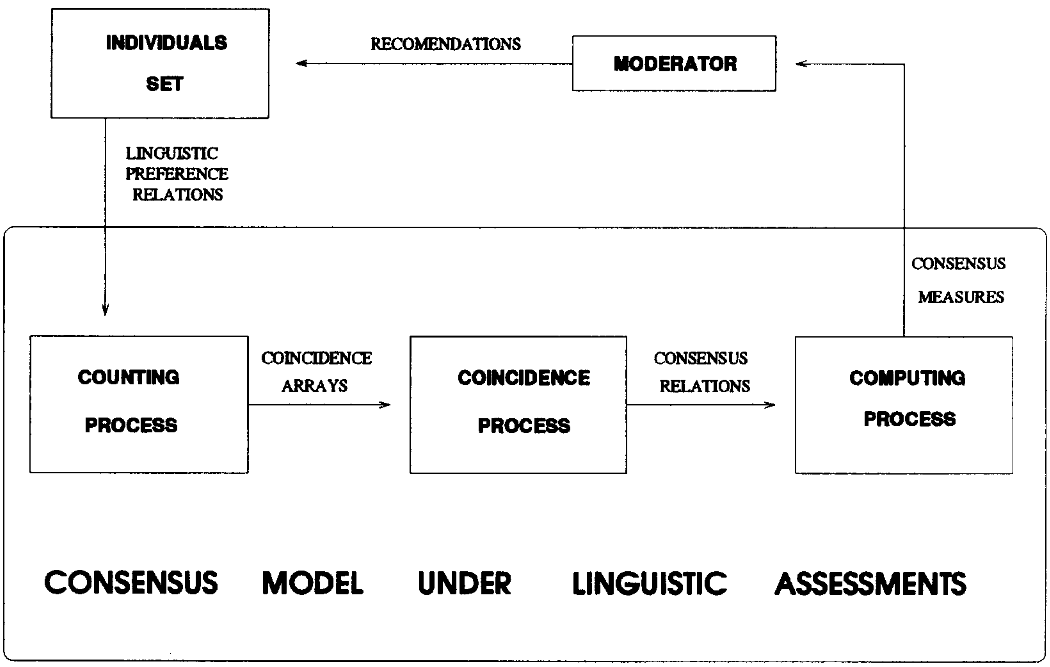

Fig. 4. Consensus model.

The consensus model is composed of three parts.

1. *Counting process.* From linguistic preference relations given by individuals, the number of individuals who are in agreement over the preference value of each alternative pair (x_i, x_j) is calculated. To do that, we will define two arrays, called *coincidence arrays*.

2. *Coincidence process.* Using the above information two relations are calculated:
   (a) *Labels consensus relation* (*LCR*), which contains the *consensus labels* over each preference, and
   (b) *Individuals consensus relation* (*ICR*), which contains the *coincidence degrees* of each preference.

3. *Computing process.* Finally, in this process several *consensus measures* over the above consensus relations are calculated. These measures enhance the moderator's knowledge upon the current consensus situation, and are used by the moderator to direct the consensus formation process.

These processes are repeated periodically, until an *acceptable consensus degree* is achieved.

3.1. Counting process

The main goal of this process is to calculate the number of individuals who have chosen each label assigned as preference value of each alternative pair. This information is stored in the *coincidence arrays*, as follows.

Previously, from the set of linguistic preference relations $P^k, k \in N$, we define an array V_{ij} for the $T + 1$ possible labels that can be assigned as preference value. Each component $V_{ij}[s_t]$, $i, j = 1, \ldots, n$, $t = 0, \ldots, T$, is a set of individuals' identification numbers, who selected the value s_t as preference value of the pair (x_i, x_j). Each V_{ij} is calculated according to the expression

$$V_{ij}[s_t] = \{k \mid p_{ij}^k = s_t, k = 1 \ldots m\} \quad \forall s_t \in S.$$

As we have mentioned, we define a pair of arrays, called *coincidence arrays*, to store information referred to the number of individuals and their respective *importance degrees*:

- The first, symbolized as V_{ij}^C, and called *individuals coincidence array*, contains in each position s_t the number of individuals, who coincide to assign the label s_t as preference value. The components of this array are obtained as

$$V_{ij}^C[s_t] = \sharp(V_{ij}[s_t]) \quad \forall s_t \in S,$$

where $\sharp$ stands for the cardinal of the term set.

- The second, symbolized as V_{ij}^G, and called *degrees coincidence array*, contains in each position s_t the arithmetic average of *individuals' importance degrees*, who coincide to assign the label s_t as preference value:

$$V_{ij}^G[s_t] = \begin{cases} (\sum_{z \in V_{ij}[s_t]} \mu_G(z)) / V_{ij}^C[s_t] & \text{if } V_{ij}^C[s_t] > 1 \\ 0 & \text{otherwise} \end{cases} \quad (\forall s_t \in S).$$

3.2. Coincidence process

This process is based on the idea of *coincidence* of individuals and preference values. We consider that *coincidence* exists over a label assigned to a preference value, when more than one individual has chosen that label.

In this process we pursue two goals:

(i) to find out the *consensus label* over the preference value of each alternative pair (x_i, x_j); thus, we will obtain the *label consensus relation* (LCR),

(ii) to calculate the *coincidence degree*, that is, the number of individuals, who selected each one of the above *consensus labels*; then we will obtain the *individual consensus relation* (ICR).

Both goals are achieved from the *coincidence arrays* (V_{ij}^C, V_{ij}^G). The first goal is necessary for calculating the *linguistic distance* of each individual, and the second goal is essential to calculate the *consensus degrees*.

The relations are calculated under the consensus policy, called *average consensus policy*, which obtains consensus values (labels or number of individuals) as the average under fuzzy majority of the *coincidence arrays* values. In what follows, we are going to show how to obtain both relations.

3.2.1. Label consensus relation (LCR)

Each element (i, j) of the *label consensus relation*, denoted by LCR_{ij}, represents the consensus label over the preference of each alternative pair (x_i, x_j). It is obtained as the aggregation of linguistic indexes s_t of the components of $V_{ij}^C[s_t]$ such that $V_{ij}^C[s_t] > 1$. That is, the aggregation of those linguistic labels s_t, which have been chosen by more than one individual to evaluate the preference of the alternative pair (x_i, x_j).

To aggregate linguistic labels, we use an aggregation operator by direct computation on labels, the LOWA operator, defined in [9], as

$$F(a_1, \ldots, a_m) = W \cdot B^{\mathrm{T}} = \mathcal{C}\{w_k, b_k,\ k = 1, \ldots, m\}$$
$$= w_1 \odot b_1 \oplus (1 - w_1) \odot \mathcal{C}\{\beta_h, b_h,\ h = 2, \ldots, m\},$$

where $W = [w_1, \ldots, w_n]$ is a weighting vector, such that $w_i \in [0,1]$ and $\sum_i w_i = 1$; $\beta_h = w_h / \sum_2^m w_k$, $h = 2, \ldots, m$, and B is associated ordered label vector. Each element $b_i \in B$ is the ith largest label in the collection $a_1, \ldots, a_m$. If $w_j = 1$ and $w_i = 0$ with $i \neq j$ $\forall i$, then the convex combination is defined as

$$\mathcal{C}\{w_i, b_i,\ i = 1, \ldots, m\} = b_j.$$

The weights w_i are computed according to [13] by means of a nondecreasing relative quantifier Q, which represents the above-mentioned concept of fuzzy majority:

$$w_i = Q(i/m) - Q((i-1)/m), \quad i = 1, \ldots, m.$$

In our case, the fuzzy quantifier is Q^1, and we write the LOWA operator as F_{Q^1} to indicate that we calculate the weights by means of Q^1. Before calculating LCR, we define the following label sets M_{ij}, for each alternative pair (x_i, x_j),

$$M_{ij} = \{s_y \mid V_{ij}^C[s_y] > 1,\ s_y \in S\}$$

which contain linguistic labels which have been chosen by more than one individual to evaluate the preference of the respective alternative pair, according to the *coincidence concept*.

From the above label sets and using the LOWA operator with a selected quantifier Q^1, we calculate each LCR_{ij}, according to the following expression:

$$LCR_{ij} = \begin{cases} F_{Q^1}(l_1, \ldots, l_q) & \text{if } \sharp(M_{ij}) > 1 \text{ and } l_k \in M_{ij},\ k = 1, \ldots, q, \\ l_q & \text{if } \sharp(M_{ij}) = 1 \text{ and } l_q \in M_{ij}, \\ \text{Undefined} & \text{otherwise,} \end{cases}$$

where $q = \sharp(M_{ij})$.

The LOWA operator F_{Q^1} based on the concept of *fuzzy majority* represented by the relative quantifier is applied, to obtain the *average of the selected labels.*

3.2.2. Individuals consensus relation (ICR)

Each element (i, j) of the *individuals consensus relation*, denoted by ICR_{ij}, represents the proportional number of individuals whose preference values have been used to calculate the consensus label LCR_{ij}. It is obtained as an arithmetic average of the components $(V_{ij}^C[s_t], V_{ij}^G[s_t])$ of the *coincidence arrays.*

Since we are interested to know the *average of individuals' importance degrees*, we define two components for each ICR_{ij}. The first ICR_{ij}^1 containing the proportional number of individuals, and the second ICR_{ij}^2, containing their respective *average of importance degrees.* Each component of ICR_{ij} is obtained as follows:

$$ICR_{ij}^1 = \begin{cases} \dfrac{\sum_{s_y \in M_{ij}} (V_{ij}^C[s_y]/m)}{\sharp(M_{ij})} & \text{if } \sharp(M_{ij}) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$ICR_{ij}^2 = \begin{cases} \dfrac{\sum_{s_y \in M_{ij}} V_{ij}^G[s_y]}{\sharp(M_{ij})} & \text{if } \sharp(M_{ij}) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

3.3. Computing process

This process constitutes the last step of our consensus model, where the *linguistic consensus measures* are calculated. These measures are moderator's reference points to control and to monitor the consensus reaching process. As we mentioned in the introduction, we define two types of *consensus measures*:

Linguistic consensus degrees. Used to evaluate current consensus existing among individuals, and therefore the distance existing to the ideal maximum consensus. This type of measure helps the moderator to decide over the necessity to continue the consensus reaching process. The linguistic consensus degrees that we define are: *preference linguistic consensus degree, alternative linguistic consensus degree, and relation linguistic consensus degree.*

Linguistic distances. Used to evaluate how far the individuals' opinions are from current consensus labels. This type of measure helps the moderator to identify which individuals are furthest from current social consensus labels, and in what preferences the distance exists. We define three linguistic distances: *preference linguistic distance, alternative linguistic distance, and relation linguistic distance.*

The three measures of both types are defined distinguishing among three levels of computation: (i) level of the preference; (ii) level of the alternative; (iii) level of the preference relation.

We define one measure of each type in its respective level. This is reflected in Fig. 5.

We calculate the *linguistic consensus degrees* using:

- *Relevance degrees of alternatives.*
- *Quantifier Q^2 to represent the concept of fuzzy majority.*
- *Individuals consensus relation, ICR.*

We calculate the *linguistic distances* using:

- *Preference relations of individuals.*
- *LOWA operator.*
- *Quantifier Q^1 to represent the concept of fuzzy majority.*
- *Labels consensus relation, LCR.*

The computing process is shown in Fig. 6.

Next, we define each linguistic consensus measure in its respective level by means of the above elements mentioned.

3.3.1. Linguistic consensus degrees

Before defining each degree, we introduce the concept of *consensus importance* over preference of pair (x_i, x_j), abbreviated $\mu_I(x_{ij})$, defined using the *alternative relevance degree* and the second component of ICR, as

$$\mu_I(x_{ij}) = \begin{cases} \dfrac{\mu_R(i) + \mu_R(j) + ICR^2_{ij}}{3} & \text{if } ICR^2_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\mu_I(x_{ij})$ represents the importance of the *consensus degree* achieved over each preference value.

Besides, we use the linguistic valued quantifier Q^2, which represents a *linguistic fuzzy majority of consensus.*

MEASURES OF CONSENSUS

LEVELS OF COMPUTATION		CONSENSUS DEGREES	LINGUISTIC DISTANCES
LEVEL 1	PREFERENCE (Xi,Xj)	PREFERENCE LINGUISTIC CONSENSUS DEGREE	PREFERENCE LINGUISTIC DISTANCE
LEVEL 2	ALTERNATIVE Xi	ALTERNATIVE LINGUISTIC CONSENSUS DEGREE	ALTERNATIVE LINGUISTIC DISTANCE
LEVEL 3	PREFERENCE RELATION P	RELATION LINGUISTIC CONSENSUS DEGREE	RELATION LINGUISTIC DISTANCE

Fig. 5. Linguistic consensus measures.

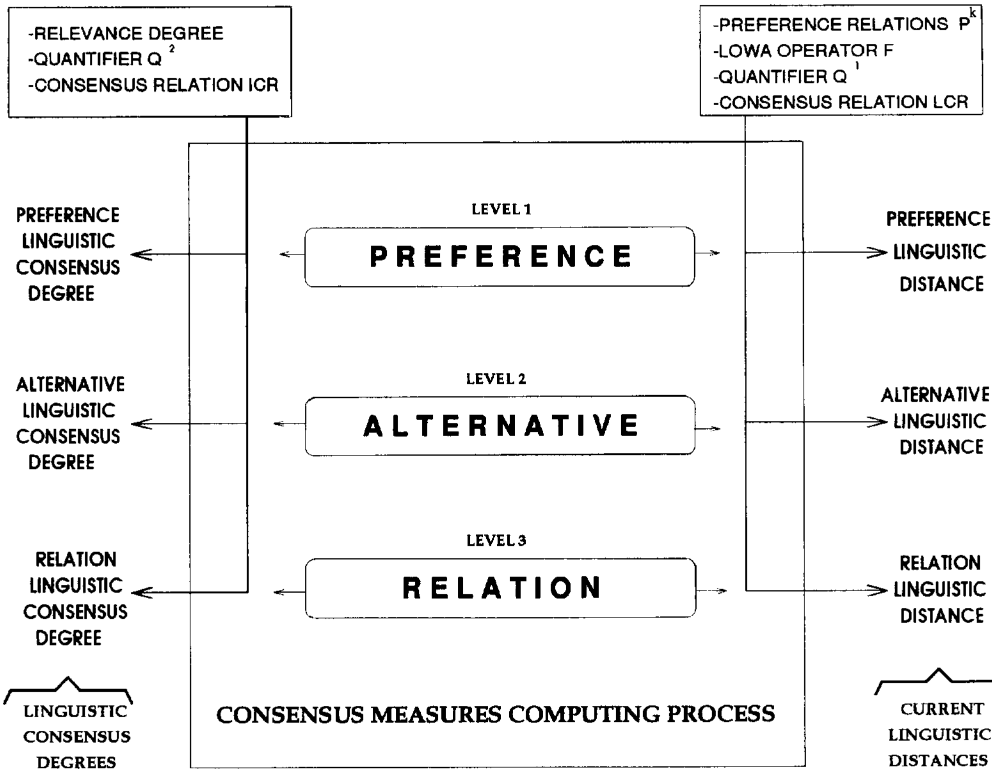

Fig. 6. Computing process.

Level 1: *Preference linguistic consensus degree.* This degree is defined over the labels assigned to the preference of each pair (x_i, x_j), and is denoted by PCR_{ij}. It indicates the consensus degree existing among all the m preference values attributed by the m individuals to the concrete preference. If we call PCR to the relation of all PCR_{ij} then PCR is calculated as follows:

$$PCR_{ij} = Q^2(ICR_{ij}^1 \wedge \mu_I(x_{ij})),$$

$$i, j = 1, \ldots, n, \text{ and } i \neq j.$$

Level 2: *Alternative linguistic consensus degree.* This degree is defined over the label set assigned to all the preferences of one alternative x_i, and is denoted by PCR_i. It allows us to measure the consensus existing over all the alternative pairs where one given is present. It is calculated as

$$PCR_i = Q^2\left[\sum_{j=1_{i \neq j}}^{n} (ICR_{ij}^1 \wedge \mu_I(x_{ij}))/(n-1)\right],$$

$$i = 1, \ldots, n.$$

Level 3: *Relation linguistic consensus degree.* This degree is defined over the preference relations of individuals' opinions, and is denoted by RC. It evaluates the social consensus, that is, the current consensus existing among all individuals over all preferences. This is calculated as follows:

$$RC = Q^2\left[\left(\sum_{i}^{n} \sum_{j=1_{i \neq j}}^{n} (ICR_{ij}^1 \wedge \mu_I(x_{ij}))\right) \Big/ (n^2 - n)\right].$$

3.3.2. Linguistic distances

In a similar way, the *linguistic distances* are defined, distinguishing the three acting levels, and using the above-cited concepts. The idea is based on the evaluation of the approximation among individuals' opinions and the *current consensus labels* of each preference.

Level 1: *Preference linguistic distance.* This distance is defined over the *consensus label* of the preference of each pair (x_i, x_j). It measures the distance between the opinions of an individual k over one preference and its respective *consensus label.* It is denoted by D_{ij}^k, and obtained as

$$D_{ij}^k = \begin{cases} p_{ij}^k - LCR_{ij} & \text{if } p_{ij}^k > LCR_{ij}, \\ LCR_{ij} - p_{ij}^k & \text{if } LCR_{ij} > = p_{ij}^k, \quad i \neq j, \\ s_T & \text{otherwise,} \end{cases}$$

with $i, j = 1, \ldots, n$ and $k = 1, \ldots, m$.

Level 2: *Alternative linguistic distance.* The distance is defined over the *consensus labels* of the preferences of one alternative x_i. It measures the distance between the preference values of an individual k over an alternative and its respective *consensus labels.* It is denoted by D_i^k, and is obtained as follows:

$$D_i^k = F_{Q^1}(D_{ij}^k, j = 1, \ldots, n, j \neq i),$$
$$k = 1, \ldots, m, \; i = 1, \ldots, n.$$

Level 3: *Relation linguistic distance.* This distance is defined over the *consensus labels* of *social preference relation LCR*. It measures the distance between the preference values of an individual k over all alternatives and their respective *consensus labels.* It is denoted by D_R^k and obtained as follows:

$$D_R^k = F_{Q^1}(D_{ij}^k, i, j = 1, \ldots, n, j \neq i),$$
$$\text{with } k = 1, \ldots, m.$$

In short, the main feature of the described model is that of being very complete, because its measures allow the moderator to have plentiful information on the current consensus stage. In a direct way: information about the *consensus degree* by means of the *linguistic consensus degrees*, information about the *consensus labels* in every preference with the *label consensus relation*, and the behavior of the individuals during the consensus process, managing the *linguistic distances*. In an indirect way: information about the individuals, who are less in agreement, and in which preference this occurs, or information about the preferences where the agreement is high.

In the following we show the use of this model in one step of the consensus formation process, with a theoretical but clear example.

4. Example

To illustrate from the practical point of view the consensus reaching process proposed, consider the following nine linguistic label set with their respective associated semantic [1]:

C	*Certain*	(1, 1, 0, 0)
EL	*Extremely_likely*	(0.98, 0.99, 0.05, 0.01)
ML	*Most_likely*	(0.78, 0.92, 0.06, 0.05)
MC	*Meaning full_chance*	(0.63, 0.80, 0.05, 0.06)
IM	*It_may*	(0.41, 0.58, 0.09, 0.07)
SC	*Small_chance*	(0.22, 0.36, 0.05, 0.06)
VLC	*Very_low_chance*	(0.1, 0.18, 0.06, 0.05)
EU	*Extermely_unlikely*	(0.01, 0.02, 0.01, 0.05)
I	*Impossible*	(0, 0, 0, 0)

represented graphically as in Fig. 7.

Let four individuals be, whose linguistic preferences using the above label set are:

$$P_1 = \begin{bmatrix} - & SC & MC & VLC \\ MC & - & IM & IM \\ SC & IM & - & VLC \\ ML & IM & ML & - \end{bmatrix},$$

$$P_2 = \begin{bmatrix} - & IM & IM & VLC \\ IM & - & MC & IM \\ IM & SC & - & VLC \\ ML & IM & ML & - \end{bmatrix},$$

$$P_3 = \begin{bmatrix} - & IM & MC & I \\ IM & - & ML & IM \\ SC & VLC & - & VLC \\ C & IM & ML & - \end{bmatrix},$$

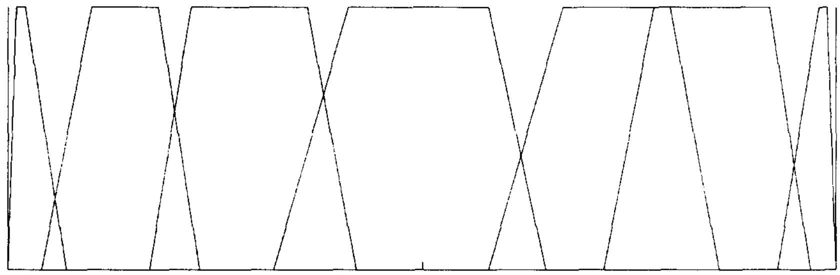

Fig. 7. Distribution of the nine linguistic labels.

$$P_4 = \begin{bmatrix} - & SC & MC & SC \\ MC & - & VLC & SC \\ SC & ML & - & VLC \\ MC & MC & ML & - \end{bmatrix},$$

respectively.

We shall use the linguistic quantifier Q = "At least half" with the pair (0.0, 0.5) for the process with its two versions, numerical and linguistically valued. Let us study the consensus model with two different *importance and relevance degree* set.

Example 1

The *importance and relevance degrees* of particular alternatives and individuals are

$$\mu_R(x_1) = 1, \quad \mu_R(x_2) = 1, \quad \mu_R(x_3) = 1, \quad \mu_R(x_4) = 1,$$

$$\mu_G(1) = 1, \quad \mu_G(2) = 1, \quad \mu_G(3) = 1, \quad \mu_G(4) = 1,$$

respectively.

Some examples of components of coincidence vectors obtained in the counting process are

$$V_{13}[MC] = \{1,3,4\}, \quad V_{23}[C] = \{\phi\},$$

$$V_{24}[SC] = \{1\}, \quad V_{34}[VLC] = \{1,2,3,4\}$$

with respective components $(V_{ij}^C[s_t], V_{ij}^G[s_t])$:

$$V_{13}^C[MC] = 3, \quad V_{23}^C[C] = 0,$$

$$V_{24}^C[SC] = 1, \quad V_{34}^C[VLC] = 4,$$

$$V_{13}^G[MC] = 1, \quad V_{23}^G[C] = 0,$$

$$V_{24}^G[SC] = 1, \quad V_{34}^G[VLC] = 1.$$

In the *coincidence process* the following relations are obtained:

Individuals consensus relations (ICR^1, ICR^2):

$$ICR^1 = \begin{bmatrix} - & 0.5 & 0.75 & 0.5 \\ 0.5 & - & 0 & 0.75 \\ 0.75 & 0 & - & 1 \\ 0.5 & 0.75 & 1 & - \end{bmatrix},$$

$$ICR^2 = \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 0 & 1 \\ 1 & 0 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix}.$$

Labels consensus relation (LCR):

$$LCR = \begin{bmatrix} - & IM & MC & VLC \\ MC & - & ? & IM \\ SC & ? & - & VLC \\ ML & IM & ML & - \end{bmatrix}.$$

Note that the symbol ? of LCR indicates undefined value in LCR_{23} and LCR_{32} because there is not a label value with coincidence value greater than 1. Note also that as the preference relations P^k are reciprocal in the sense $p_{ij}^k = Neg(p_{ji}^k)$, then the components of ICR are symmetric. Therefore, to calculate the consensus degrees we could do it using only the elements (i,j), $i = 1, \ldots, n-1$, $j = i+1, \ldots, n$. However, the reciprocity is not preserved in LCR because of the LOWA operator.

From these *consensus relations* we obtain the following *linguistic consensus degrees* and *linguistic distances*:

(A) *Consensus degrees*

(A.1) *Level of preference.* The *preference linguistic consensus degree* are

$$PCR = \begin{bmatrix} - & C & C & C \\ C & - & I & C \\ C & I & - & C \\ C & C & C & - \end{bmatrix}.$$

In this case as $\mu_I(x_{ij}) = 1, \forall i = 1, \ldots, n, j = 1, \ldots, n, i \neq j$, then each $PCR_{ij} = Q^2(ICR^1_{ij})$. As can be observed there are ten preferences where the consensus degree is total according to the fuzzy majority of consensus established by the quantifier "At least half". If we use another quantifier such as "As many as possible", then the results are

$$PCR = \begin{bmatrix} - & I & IM & I \\ I & - & I & IM \\ IM & I & - & C \\ I & IM & C & - \end{bmatrix},$$

that is, there are only two preferences where the consensus is maximum. Therefore, depending on the more or less strict nature of our idea of consensus, we must choose the adequate quantifier.

(A.2) *Level of alternative.* The alternative linguistic consensus degrees $\{PCR_i\}$ are

$PCR_1 = Q^2(0.58) = C,$

$PCR_2 = Q^2(0.4167) = ML,$

$PCR_3 = Q^2(0.58) = C, \qquad PCR_4 = Q^2(0.75) = C.$

(A.3) *Level of relation.* The *relation linguistic consensus degree RC* is

$RC = Q^2(0.5833) = C.$

Remarks. According to the concept of *linguistic fuzzy majority of consensus* introduced by the linguistically valued quantifier, Q^2, we obtain some conclusions:

1. The *social consensus degree* is total.
2. On the preferences of pairs (x_2, x_3) and (x_3, x_2) there is no consensus.
3. And, the *consensus degree* over the preferences values of alternative x_2 is the smallest one. The other alternatives present a *total consensus degree.*

(B) *The linguistic distances.* The *linguistic distances* of each individual k to social consensus labels, with $k = 1, \ldots, m$, are:

(B.1) *Level of preference.* The *preference linguistic distances* are

$$D^1 = \begin{bmatrix} - & EU & I & I \\ I & - & C & I \\ I & C & - & I \\ I & I & I & - \end{bmatrix},$$

$$D^2 = \begin{bmatrix} - & I & EU & I \\ EU & - & C & I \\ EU & C & - & I \\ I & I & I & - \end{bmatrix},$$

$$D^3 = \begin{bmatrix} - & I & I & VLC \\ EU & - & C & I \\ I & C & - & I \\ VLC & I & I & - \end{bmatrix},$$

$$D^4 = \begin{bmatrix} - & EU & I & EU \\ I & - & C & EU \\ I & C & - & I \\ EU & EU & I & - \end{bmatrix}.$$

(B.2) *Level of alternative.* The *alternative linguistic distances* with Q^1 are:

Individual 1: $D^1_1 = EU, D^1_2 = MC, D^1_3 = MC,$
$D^1_4 = I,$

Individual 2: $D^2_1 = EU, D^2_2 = ML, D^2_3 = ML,$
$D^2_4 = I,$

Individual 3: $D^3_1 = EU, D^3_2 = ML, D^3_3 = MC,$
$D^3_4 = EU,$

Individual 4: $D^4_1 = EU, D^4_2 = ML, D^4_3 = MC,$
$D^4_4 = EU.$

(B.3) *Level of relation.* The *relation linguistic distances using* Q^1 are

$D^1_R = EU, \quad D^2_R = SC, \quad D^3_R = SC, \quad D^4_R = SC.$

Remarks. We can draw some conclusions:

1. The individual 1 present less distance to the current social consensus stage.
2. All individuals are in disagreement in current preference values of the pairs (x_2, x_3) and (x_3, x_2). We must remember that $LCR_{23} = ?$ and $LCR_{32} = ?$.
3. In the preference of the pairs (x_1, x_4) and (x_4, x_1), the individual 3 is furthest away from the social opinion.
4. In the preferences of alternatives x_2 and x_3 there is more disagreement.

Example 2

The *relevance and importance degrees* of particular alternatives and individuals are:

$$\mu_R(x_1) = 0.4, \quad \mu_R(x_2) = 1,$$

$$\mu_R(x_3) = 0.3, \quad \mu_R(x_4) = 0.8,$$

$$\mu_G(1) = 0.9, \quad \mu_G(2) = 0.2,$$

$$\mu_G(3) = 0.5, \quad \mu_G(4) = 0.7.$$

Some examples of components of coincidence vectors obtained in counting process are:

$$V_{13}[MC] = \{1,3,4\}, \quad V_{23}[C] = \{\phi\},$$

$$V_{24}[SC] = \{1\}, \quad V_{34}[VLC] = \{1,2,3,4\}.$$

With their respective components $(V_{ij}^C[s_t], V_{ij}^G[s_t])$:

$$V_{13}^C[MC] = 3, \quad V_{23}^C[C] = 0,$$

$$V_{24}^C[SC] = 1, \quad V_{34}^C[VLC] = 4,$$

$$V_{13}^G[MC] = 0.7, \quad V_{23}^G[C] = 0,$$

$$V_{24}^G[SC] = 0.7, \quad V_{34}^G[VLC] = 0.575.$$

The *consensus relations* are:

Individual consensus relations

$$ICR^1 = \begin{bmatrix} - & 0.5 & 0.75 & 0.5 \\ 0.5 & - & 0 & 0.75 \\ 0.75 & 0 & - & 1 \\ 0.5 & 0.75 & 1 & - \end{bmatrix},$$

$$ICR^2 = \begin{bmatrix} - & 0.575 & 0.7 & 0.55 \\ 0.575 & - & 0 & 0.533 \\ 0.7 & 0 & - & 0.575 \\ 0.55 & 0.533 & 0.575 & - \end{bmatrix}$$

Label consensus relation

$$LCR = \begin{bmatrix} - & IM & MC & VLC \\ MC & - & ? & IM \\ SC & ? & - & VLC \\ ML & IM & ML & - \end{bmatrix}.$$

As it is observed the LCR does not change, because it does not depend on the *importance* and *relevance degrees.*

The *consensus measures* are:

(A) *Consensus degrees*

(A.1) *Level of preference.* The *importance of consensus* of the alternative pairs are

$$\begin{bmatrix} - & \mu_I(x_{12}) = 0.6583 & \mu_I(x_{13}) = 0.4667 & \mu_I(x_{14}) = 0.5833 \\ \mu_I(x_{21}) = 0.6583 & - & \mu_I(x_{23}) = 0.4333 & \mu_I(x_{24}) = 0.7767 \\ \mu_I(x_{31}) = 0.4667 & \mu_I(x_{32}) = 0.4333 & - & \mu_I(x_{34}) = 0.5583 \\ \mu_I(x_{41}) = 0.5833 & \mu_I(x_{42}) = 0.7767 & \mu_I(x_{43}) = 0.5583 & - \end{bmatrix}$$

Then PCR is

$$PCR_{Q^2} = \begin{bmatrix} - & C & ML & C \\ C & - & I & C \\ ML & I & - & C \\ C & C & C & - \end{bmatrix}.$$

(A.2) *Level of alternative.* The *alternative linguistic consensus degrees*, PCR_i, are

$$PCR_1 = Q^2(0.4889) = EL,$$

$$PCR_2 = Q^2(0.4167) = ML,$$

$$PCR_3 = Q^2(0.34167) = MC,$$

$$PCR_4 = Q^2(0.602767) = C.$$

(A.3) *Level of relation.* The *relation linguistic consensus degree* RC is

$$RC = Q^2(0.4625) = ML.$$

Remarks. As it is possible to observe in this example, because of the influence of *importance and relevance degrees*:

1. The *social consensus degree* is smaller than the above one.
2. The *total social consensus degrees* over the alternative x_1 and x_3 are lost. The alternative x_4 conserves the maximum consensus degree, and x_2 does not change.

Finally, as the *linguistic distances* do not change, because the *importance and relevance degree* are not used in their computing process, we do not compute them.

5. Conclusions

In this paper we have presented a consensus model in group decision making under linguistic assessments. All the relevant results are expressed using linguistic labels, a more natural way to communicate information. This model incorporates human consistency in decision making models. The model presents a wide spectrum of consensus measures, which allow analysing, controlling and monitoring the consensus reaching process describing the *current consensus stage*.

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