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Minimizing adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making

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ABSTRACT

In some real-world decision processes, decision makers may prefer to provide their opinions using linguistic expressions instead of a single linguistic term. Particularly, they may hesitate between several linguistic terms. In this paper, we deal with the consensus issue in the hesitant linguistic group decision making (GDM) problem. Firstly, a novel distance-based consensus measure is proposed. Then, using this consensus measure we develop an optimization-based consensus model in the hesitant linguistic GDM, which minimizes the number of adjusted simple terms in the consensus building. Furthermore, a two-stage model is displayed to further optimize the solutions to the proposed consensus model, through which we obtain the unique optimal adjustment suggestion to support the consensus reaching process in the hesitant linguistic GDM. Finally, several desirable properties are proposed to justify the proposal, and two examples are used to demonstrate the validity of the models.

1. Introduction

In real-world decision-making activities, decision makers often provide their opinions linguistically. However, solving a linguistic decision problem is complex, and implies a need of linguistic computational models for computing with words (CWW) [13,22,30]. There are three different linguistic computational models in decision making: (1) the model based on type-1 fuzzy sets [7,44] (or interval type-2 fuzzy sets [25,26,39,40]), (2) the symbolic model based on ordinal scales [8,41–43] and (3) the model based on the 2-tuple representation [10,11,14,37].

However, the linguistic computational models mentioned above are based on single linguistic term sets. In some decision-making situations, it is more comfortable for decision makers to use linguistic expressions to provide their opinions instead of using a single linguistic term. In recent years, several studies based on linguistic expressions have been proposed [1,21,23,34,45]. Particularly, if decision makers are not confident of their opinions, they may hesitate between several different linguistic terms [2,20,31,32,38,48]. Rodríguez et al. [31] introduced the concept of a hesitant fuzzy linguistic term set (HFLTS) by using comparative terms to provide a linguistic and computational basis to enrich linguistic elicitation based on hesitant linguistic approach. Rodríguez et al. [32] extended the use of context-free grammars to develop a group decision

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making (GDM) model based on HFLTSs. Wei et al. [38] defined operations on HFLTSs, and gave possibility degree formulas for comparing HFLTSs and also presented two new linguistic aggregation operators for HFLTSs. Liu and Rodríguez [20] proposed a new representation of HFLTSs by means of a fuzzy envelope to carry out the CWW processes. Beg and Rashid [2] proposed a new method to aggregate the opinions of decision makers on different criteria, regarding a set of alternatives, where the opinions of decision makers are represented by HFLTSs. Zhu and Xu [48] introduced the concept of hesitant fuzzy linguistic preference relation and defined several consistency measures for hesitant fuzzy linguistic preference relations. Agell et al. [1] and Roselló et al. [34] introduced a complete description of the order-of-magnitude qualitative space, which is related to the HFLTS proposed in Rodríguez et al. [31]. Agell et al. [1] and Roselló et al. [34] considered a set of consecutive linguistic labels based on order-of-magnitude qualitative space to represent the uncertainty.

Generally, at the beginning of GDM problems, decision makers' opinions may differ substantially. As a result, consensus processes are proposed to help decision makers reach a consensus. In consensus processes, full and unanimous agreement for every decision maker is often not necessary, so "soft" consensus has been presented [5,17,18]. Afterward, a number of studies for modeling the consensus process based on "soft" consensus have been presented (e.g., [6,15,16,24,28,29,34]). Feedback mechanism is one of the key elements in the consensus process, and the most important issue in feedback mechanism is to provide the adjustment suggestions to help decision makers reach a higher consensus level. It is natural that decision makers often hope to minimize adjustments between the original and adjusted individual opinions. Dong et al. [9] proposed a consensus operator, which provided an alternative consensus model for GDM problems to minimize the deviation between original and adjusted individual opinions. Ben-Arieh and Easton [3] and Ben-Arieh et al. [4] considered that the cost of moving each decision maker's opinion 1 unit distance is different. Based on the concept of consensus cost, they proposed the minimum cost consensus model and the maximum expert consensus model. Subsequently, Zhang et al. [46,47] extended the minimum cost consensus models and proposed a novel framework to achieve minimum cost consensus under aggregation operators.

As mentioned above, consensus models with minimum adjustments have been proposed in numeric environments [3,4,46,47] (or linguistic environments with single linguistic term [9]). But none of these consensus studies relates to the hesitant linguistic assessments. Actually in some real-world decision processes, decision makers may hesitate about their opinions. Therefore, we focus on the theories of HFLTSs that allow for handling of imprecise and vague assessments, and hope to solve the open problem: reaching a consensus with minimum adjustments in the hesitant linguistic GDM context. In order to do this, we must tackle the following two challenges:

(1) How to measure the consensus level among decision makers in hesitant linguistic GDM problems.
(2) How to design a procedure to provide adjustment suggestions, which helps the decision makers reach a consensus in the hesitant linguistic GDM context. Particularly, we hope to minimize the adjustments between original and adjusted hesitant linguistic opinions in the consensus building.

Motivated by these challenges, in this paper we define a distance between two HFLTSs, which reflects the number of different simple terms between two HFLTSs. For example, let $S = \{s_0 = \textit{very poor}, s_1 = \textit{poor}, s_2 = \textit{average}, s_3 = \textit{good}, s_4 = \textit{very good}\}$ be a linguistic term set, and let $Q = \{s_1, s_2, s_3\}$ and $N = \{s_3, s_4\}$ be two HFLTSs of S. The distance between Q and N is defined as the number of different simple terms between Q and N, i.e., the number of simple terms in the set $(Q \cup N) - (Q \cap N) = \{s_1, s_2, s_4\}$. Based on this idea, a novel consensus measure is proposed for measuring the consensus level in hesitant linguistic GDM problems. Furthermore, we design an optimization-based two-stage procedure to provide optimal adjustment suggestions to help the decision makers reach a consensus in the hesitant linguistic GDM context.

The purpose of this paper is to provide tools to help the decision makers manage the consensus reaching process in hesitant linguistic GDM problems. The rest of this paper is organized as follows. Section 2 provides background regarding the HFLTSs developed by Rodríguez et al. [31] and proposes the hesitant linguistic GDM problem. A distance-based approach for measuring the consensus level in the hesitant linguistic GDM is provided in Section 3. Following this, Section 4 proposes an optimization-based two-stage model with minimum adjustments to obtain the optimal adjusted individual opinions. Subsequently, several desired properties are investigated in Section 5. Finally, two illustrative examples are provided in Section 6, and concluding remarks are included in Section 7.

2. Background and the proposed problem

In this section, we review the concept and related operation laws of HFLTS, and then propose the hesitant linguistic GDM problem.

2.1. Hesitant fuzzy linguistic term sets

The basic notations and operational laws of linguistic variables were introduced in [14]. Let $S = \{s_j | j = 0, \ldots, g\}$ be a linguistic term set with odd granularity $g + 1$, where the term s_j represents a possible value for a linguistic variable. The linguistic term set is usually required to satisfy the following additional characteristics:

(1) The set is ordered: $s_i \leqslant s_j$ if and only if $i \leqslant j$;
(2) There is a negation operator: $neg(s_j) = s_{g-j}$.

Torra [36] introduced the hesitant fuzzy set. Similar to the situations that are described and managed by hesitant fuzzy sets in [36], decision makers may hesitate between several linguistic terms before assessing an alternative. Bearing this idea in mind, Rodríguez et al. [31] gave concepts regarding HFLTSs as follows:

Definition 2.1 [31]. Let $S = \{s_j | j = 0, \ldots, g\}$ be a linguistic term set, where $g + 1$ is odd. A hesitant fuzzy linguistic term set (HFLTS), M^S, is an ordered finite subset of consecutive linguistic terms of S.

In this paper, if $s_j \in M^S$, we define that s_j is a simple term in M^S. For example, s_2 is a simple term in $M^S = \{s_2, s_3, s_4\}$.

Once the concept of HFLTS has been introduced, some operation laws can be performed on HFLTSs.

Let $S = \{s_j | j = 0, \ldots, g\}$ be a linguistic term set, where $g + 1$ is odd. Let M^S, M_1^S and M_2^S be three HFLTSs of S.

Definition 2.2 [31]. The upper bound M^{S+} and lower bound M^{S-} of the HFLTS M^S are defined as:

(1) $M^{S+} = \max(s_i) = s_j$, where $s_i \in M^S$;

(2) $M^{S-} = \min(s_i) = s_j$, where $s_i \in M^S$.

Definition 2.3 [31]. The envelope of the HFLTS M^S, denoted as $env(M^S)$, is a linguistic interval whose limits are obtained by means of lower bound M^{S-} and upper bound M^{S+}, i.e.,

$$env(M^S) = [M^{S-}, M^{S+}]. \tag{1}$$

Based on the concept of the envelope of the HFLTS, $env(M^S)$, the definition of the comparison between two HFLTSs is defined as Definition 2.4.

Definition 2.4 [31]. The comparison between M_1^S and M_2^S is defined as follows:

(1) $M_1^S > M_2^S$ if $env\left(M_1^S\right) > env\left(M_2^S\right)$;
(2) $M_1^S = M_2^S$ if $env\left(M_1^S\right) = env\left(M_2^S\right)$.

The latest proposals regarding the HFLTSs include the formulas for comparing HFLTSs [38], the aggregation operators of HFLTSs [2,38] and the consistency measures for hesitant fuzzy linguistic preference relations [48], etc. Rodríguez et al. [33] provided an overview in order to present a clear view about the different concepts, tools and trends related to the use of hesitant fuzzy sets in decision making.

2.2. Proposed problem in the hesitant linguistic GDM

The GDM is defined as a decision situation where two or more decision makers take part and provide their opinions in order to reach a collective decision. Let $D = \{d_i | i = 1, \ldots, n\} (n \geqslant 2)$ denote a set of n decision makers, and let $S = \{s_j | j = 0, \ldots, g\}$ be a linguistic term set. As pointed out in Section 1, in some situations, it is more appropriate for decision makers to provide their preferences using HFLTSs instead of single linguistic term sets. So, in this paper, we consider that the decision makers provide their original preferences by HFLTSs of S, denoted by $A_i (i = 1, \ldots, n)$.

Then, the decision problem is how to obtain a collective opinion with an acceptable consensus level in the hesitant linguistic GDM. In order to deal with this open decision problem, two key challenges were pointed out in Section 1, and they will be tackled in the following sections.

In order to improve readability, the main notations used in this paper are listed as follows:

$S = \{s_j | j = 0, \ldots, g\}$: Linguistic term set.

$D = \{d_i | i = 1, \ldots, n\}$: The set of decision makers.

$A = (A_1, \ldots, A_n)$: Original individual opinions of n decision makers, where A_i is the original individual opinion of d_i.

$\overline{A} = (\overline{A_1}, \ldots, \overline{A_n})$: Adjusted individual opinions of n decision makers, where $\overline{A_i}$ is the adjusted individual opinion of d_i, obtained by model P_1 (that will be presented in model (11) in Section 4.1).

$\overline{A^c}$: Adjusted collective opinion obtained by model P_1.

$\overline{OA} = (\overline{OA_1}, \ldots, \overline{OA_n})$: Adjusted individual opinions of n decision makers, where $\overline{OA_i}$ is the adjusted individual opinion of d_i, obtained by model P_2 (that will be presented in model (26) in Section 4.3).

$\overline{OA^c}$: Adjusted collective opinion obtained by model P_2.

M^S: A HFLTS of S.

H^S: Set of the HFLTSs of S, $H^S = \{M^S | M^S \text{ is a HFLTS of } S\}$.

CL_i: The consensus level of d_i.
$\overline{CL}$: The established consensus level for all decision makers.

3. Measuring the consensus level in the hesitant linguistic GDM

Consensus measure is used to measure the similar degree of preferences among a group of decision makers, which is the basis for constructing a consensus model. Usually, using different consensus measures obtains different consensus models in GDM problems. In this section, we develop a novel distance-based approach for measuring the consensus level in the hesitant linguistic GDM.

Let M^S be a HFLTS of S, and let $\#(M^S)$ denote the number of simple linguistic terms in M^S. For example, if $M^S = \{s_2, s_3, s_4, s_5\}$, then $\#(M^S) = 4$. For two arbitrary HFLTSs, Q and N, here we propose the following equation to measure the difference between Q and N, as Eq. (2):

$$d(Q, N) = \#(Q \cup N) - \#(Q \cap N). \tag{2}$$

The value of $d(Q, N)$ has a definite implication, and measures the number of different simple terms between Q and N. For example, $Q = \{s_1, s_2, s_3\}$ and $N = \{s_3, s_4\}$, the number of different simple terms between Q and N is $\#(Q \cup N) - \#(Q \cap N) = 3$. Then, we use $d(Q, N) = 3$ to measure the difference between Q and N.

Furthermore, we prove this difference measure between two HFLTSs is a distance metric as Theorem 3.1.

Theorem 3.1. *Let Q, N and P be three HFLTSs. Then, the following conditions will be satisfied:*

(1) $d(Q, N) = 0$ *if and only if* $Q = N$;
(2) $d(Q, N) = d(N, Q)$;
(3) $d(Q, P) + d(P, N) \geqslant d(Q, N)$;
(4) $0 \leqslant d(Q, N) \leqslant \#(Q \cup N)$.

Proof of Theorem 3.1 is provided in Appendix.

Note 1. To our knowledge, Falcó et al. [12] proposed a distance measure for sets of consecutive labels, which is based on the computing with position indexes (see Definition 4.1). The distance measure proposed in [12] can guarantee the accuracy when the linguistic term set distributes uniform and symmetrical. The new distance measure for HFLTSs proposed in this paper has a definite implication, which measures the number of different simple terms between two HFLTSs. The new proposed distance measure for HFLTSs can not only be used in the hesitant linguistic GDM with the uniformly and symmetrically distributed term set, but also in the hesitant linguistic GDM with the linguistic term set that not uniformly and symmetrically distributed.

Usually, distance based approaches are used to measure the consensus level among decision makers' opinions. Based on Eq. (2), an approach for measuring consensus level in the hesitant linguistic GDM can be proposed as Definition 3.1.

Definition 3.1. Let A_i represent the individual opinion of d_i and let A^c represent the collective opinion. Here, we define the consensus level of d_i as Eq. (3):

$$CL_i = 1 - \frac{d(A_i, A^c)}{\#(A_i \cup A^c)}, \tag{3}$$

i.e.,

$$CL_i = \frac{\#(A_i \cap A^c)}{\#(A_i \cup A^c)}. \tag{4}$$

Clearly, $CL_i \in [0, 1]$. The value of CL_i has a definite implication, and measures the proportion of the same simple terms between A_i and A^c. The larger CL_i value indicates the higher consensus level associated with d_i. When $CL_i = 1$, the decision maker d_i achieves full agreement with the collective opinion.

4. Minimizing the number of adjusted simple terms in the consensus building

When decision makers' opinions differ substantially, the consensus process assists the decision makers to adjust their opinions to improve the consensus level. In the consensus process, the feedback mechanism plays an important role and the core issue in the feedback mechanism is to provide the adjustment suggestions.

In this section, using the novel distance-based consensus measure, we develop an optimization-based two-stage procedure to obtain the optimal adjusted suggestions in the hesitant linguistic GDM context, which minimizes the number of adjusted simple terms in the consensus building.

4.1. Basic ideas and model

Usually, distance-based approaches are used to measure the adjustments between original and adjusted individual opinions. Here, we use the distance defined by Eq. (2) to measure the adjustments between A_i and $\overline{A_i}(i = 1, \ldots, n)$, i.e.,

$$d(A_i, \overline{A_i}) = \#(A_i \cup \overline{A_i}) - \#(A_i \cap \overline{A_i}). \tag{5}$$

In order to preserve the original preference information as much as possible, we hope to minimize the adjustments between the original and adjusted individual opinions of all decision makers. Namely,

$$\min_{\overline{A_i}} \sum_{i=1}^{n} d(A_i, \overline{A_i}) = \min_{\overline{A_i}} \sum_{i=1}^{n} \left(\#(A_i \cup \overline{A_i}) - \#(A_i \cap \overline{A_i}) \right). \tag{6}$$

Here, in the sense of the distance defined by Eq. (2), the value of $\min_{\overline{A_i}} \sum_{i=1}^{n} d(A_i, \overline{A_i})$ has a definite implication, and measures the minimum number of adjusted simple terms of all decision makers in consensus building.

Meanwhile, the main work in consensus building is to find adjusted individual opinions with an established consensus level. In this paper, $\overline{CL} \in [0, 1]$ is established as the threshold of CL_i. Based on Definition 3.1, we have $CL_i \geqslant \overline{CL}(i = 1, \ldots, n)$, i.e.,

$$\frac{\#\left(\overline{A_i} \cap \overline{A^c}\right)}{\#\left(\overline{A_i} \cup \overline{A^c}\right)} \geqslant \overline{CL} \quad (i = 1, \ldots, n). \tag{7}$$

Furthermore, $\overline{A_i}(i = 1, \ldots, n)$ and $\overline{A^c}$ are HFLTSs, i.e.,

$$\overline{A_i} \in H^s (i = 1, \ldots, n), \tag{8}$$

and

$$\overline{A^c} \in H^s. \tag{9}$$

Additionally, in order to get relatively precise collective opinion $\overline{A^c}$, the upper bound of $\#(\overline{A^c})$ is limited as β, i.e.,

$$\#(\overline{A^c}) \leqslant \beta. \tag{10}$$

Generally, we suggest $\beta \leqslant 3$.

According to Eqs. (6)–(10), an optimization-based consensus model in the hesitant linguistic GDM is constructed as:

$$\begin{cases} \min_{\overline{A_i}} \sum_{i=1}^{n} \left(\#(A_i \cup \overline{A_i}) - \#(A_i \cap \overline{A_i}) \right) \\ s.t. \begin{cases} \frac{\#(\overline{A_i} \cap \overline{A^c})}{\#(\overline{A_i} \cup \overline{A^c})} \geqslant \overline{CL} & i = 1, \ldots, n \\ \overline{A_i} \in H^s & i = 1, \ldots, n \\ \overline{A^c} \in H^s \\ \#(\overline{A^c}) \leqslant \beta \end{cases} \end{cases} \tag{11}$$

In model (11), the constraint condition $\frac{\#(\overline{A_i} \cap \overline{A^c})}{\#(\overline{A_i} \cup \overline{A^c})} \geqslant \overline{CL}(i = 1, \ldots, n)$ guarantees all the decision makers can reach the established consensus threshold $\overline{CL}$. Meanwhile, the constraint conditions $\overline{A_i} \in H^s (i = 1, \ldots, n)$ and $\overline{A^c} \in H^s$ guarantee the adjusted opinions are HFLTSs. In this paper, denote model (11) as P_1, which we call minimum adjusted simple terms model (MASTM).

4.2. Procedure to solve minimum adjusted simple terms model with a mixed 0–1 linear programming model

In this subsection, we present a mixed 0–1 linear programming to obtain the optimal solutions to minimum adjusted simple terms model (i.e., P_1).

In order to transform P_1 into a mixed 0–1 linear programming model, three binary variables x_i^j, y_i^j and z^j are introduced, i.e.,

$$x_i^j = \begin{cases} 1 & s_j \in A_i \\ 0 & s_j \notin A_i \end{cases} \quad i = 1, \ldots, n;\ j = 0, \ldots, g; \tag{12}$$

$$y_i^j = \begin{cases} 1 & s_j \in \overline{A_i} \\ 0 & s_j \notin \overline{A_i} \end{cases} \quad i = 1, \ldots, n;\ j = 0, \ldots, g; \tag{13}$$

and let

$$z^j = \begin{cases} 1 & s_j \in \overline{A^c} \\ 0 & s_j \notin \overline{A^c} \end{cases} \quad j = 0, \ldots, g. \tag{14}$$

Then, Lemmas 4.1, 4.2, 4.3 are proposed.

Lemma 4.1

$$\#(A_i \cup \overline{A}_i) - \#(A_i \cap \overline{A}_i) = \sum_{j=0}^{g} \left| x_i^j - y_i^j \right| \ (i = 1, \ldots, n). \tag{15}$$

Proof of Lemma 4.1 is provided in Appendix.

Lemma 4.2

$$\frac{\#\left(\overline{A}_i \cap \overline{A^c}\right)}{\#\left(\overline{A}_i \cup \overline{A^c}\right)} = \frac{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j - \sum_{j=0}^{g} \left| y_i^j - z^j \right|}{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j + \sum_{j=0}^{g} \left| y_i^j - z^j \right|} \ (i = 1, \ldots, n). \tag{16}$$

Proof of Lemma 4.2 is provided in Appendix.

Lemma 4.3

(1) $\overline{A}_i (i = 1, \ldots, n)$ *is a HFLTS if and only if the following conditions are satisfied:*

1) $\sum_{j=0}^{g-1} \left| y_i^{j+1} - y_i^j \right| \leqslant 2 \quad i = 1, \ldots, n;$

2) $y_i^0 + y_i^g \leqslant 1 \quad i = 1, \ldots, n.$

(2) $\overline{A^c}$ *is a HFLTS if and only if the following conditions are satisfied:*

1) $\sum_{j=0}^{g-1} \left| z^{j+1} - z^j \right| \leqslant 2;$

2) $z^0 + z^g \leqslant 1.$

Proof of Lemma 4.3 is provided in Appendix.

Based on Lemmas 4.1, 4.2, 4.3, we can obtain Theorem 4.1.

Theorem 4.1. *P_1 can be transformed into model (17):*

$$\begin{cases} \min \sum_{i=1}^{n} \sum_{j=0}^{g} \left| x_i^j - y_i^j \right| \\ s.t. \begin{cases} \frac{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j - \sum_{j=0}^{g} |y_i^j - z^j|}{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j + \sum_{j=0}^{g} |y_i^j - z^j|} \geqslant \overline{CL} & i = 1, \ldots, n; \\ \sum_{j=0}^{g-1} \left| y_i^{j+1} - y_i^j \right| \leqslant 2 & i = 1, \ldots, n; \\ y_i^0 + y_i^g \leqslant 1 & i = 1, \ldots, n; \\ \sum_{j=0}^{g-1} \left| z^{j+1} - z^j \right| \leqslant 2 \\ z^0 + z^g \leqslant 1 \\ \sum_{j=0}^{g} z^j \leqslant \beta \\ y_i^j = 0 \text{ or } 1 & i = 1, \ldots, n; \ j = 0, \ldots, g; \\ z^j = 0 \text{ or } 1 & j = 0, \ldots, g. \end{cases} \end{cases} \tag{17}$$

Theorem 4.1 shows that the optimal solution to P_1 can be obtained by solving model (17). In order to solve model (17), Theorem 4.2 is provided to transform model (17) into a mixed 0–1 linear programming model.

Theorem 4.2. *By introducing eight transformed decision variables: $b_i^j = x_i^j - y_i^j$, $c_i^j = \left| b_i^j \right|$, $e_i^j = y_i^j - z^j$, $f_i^j = \left| e_i^j \right|$, $h_i^j = y_i^{j+1} - y_i^j$, $o_i^j = \left| h_i^j \right|$, $u^j = z^{j+1} - z^j$ and $v^j = |u^j|$. Model (17) can be transformed into a mixed 0–1 linear programming model (18):*

$$\begin{cases} \min \sum_{i=1}^{n}\sum_{j=0}^{g} c_i^j \\ s.t. \begin{cases} b_i^j = x_i^j - y_i^j & i=1,\ldots,n;\ j=0,\ldots,g; \\ b_i^j \leqslant c_i^j & i=1,\ldots,n;\ j=0,\ldots,g; \\ -b_i^j \leqslant c_i^j & i=1,\ldots,n;\ j=0,\ldots,g; \\ (1-\overline{CL})\sum_{j=0}^{g} y_i^j + (1-\overline{CL})\sum_{j=0}^{g} z^j - (1+\overline{CL})\sum_{j=0}^{g} f_i^j \geqslant 0 & i=1,\ldots,n; \\ e_i^j = y_i^j - z^j & i=1,\ldots,n;\ j=0,\ldots,g; \\ e_i^j \leqslant f_i^j & i=1,\ldots,n;\ j=0,\ldots,g; \\ -e_i^j \leqslant f_i^j & i=1,\ldots,n;\ j=0,\ldots,g; \\ \sum_{j=0}^{g-1} o_i^j \leqslant 2 & i=1,\ldots,n; \\ h_i^j = y_i^{j+1} - y_i^j & i=1,\ldots,n;\ j=0,\ldots,g-1; \\ h_i^j \leqslant o_i^j & i=1,\ldots,n;\ j=0,\ldots,g-1; \\ -h_i^j \leqslant o_i^j & i=1,\ldots,n;\ j=0,\ldots,g-1; \\ y_i^0 + y_i^g \leqslant 1 & i=1,\ldots,n; \\ \sum_{j=0}^{g-1} v^j \leqslant 2 & \\ u^j = z^{j+1} - z^j & j=0,\ldots,g-1; \\ u^j \leqslant v^j & j=0,\ldots,g-1; \\ -u^j \leqslant v^j & j=0,\ldots,g-1; \\ z^0 + z^g \leqslant 1 & \\ \sum_{j=0}^{g} z^j \leqslant \beta & \\ y_i^j = 0 \text{ or } 1 & i=1,\ldots,n;\ j=0,\ldots,g; \\ z^j = 0 \text{ or } 1 & j=0,\ldots,g. \end{cases} \end{cases} \quad (18)$$

Proof of Theorem 4.2 *is provided in* Appendix.

In this paper, denote model (18) as P_1'. Theorem 4.2 guarantees the equivalence between P_1 and P_1'. So, to simplify the notation, models P_1 and P_1' are called P_1 in this paper.

4.3. Optimizing the optimal solutions to minimum adjusted simple terms model by Manhattan distance

In Section 4.2, we obtain the optimal solution(s) to P_1. However, in some situations, the optimal solution(s) to P_1 is not unique. Particularly, some of the optimal solutions are not reasonable enough (in some sense). For illustrating this idea, an example is shown as follows:

Example 4.1. Let $S^1 = \{s_0^1, \ldots, s_6^1\}$ be the established linguistic term set, which is defined in Section 6.1, and five decision makers $D = \{d_1, d_2, d_3, d_4, d_5\}$ provide their preferences by HFLTSs of S^1 to assess an alternative. We assume that the original individual opinions of these five decision makers are $A = (A_1, A_2, A_3, A_4, A_5) = (\{s_3^1, s_4^1\}, \{s_4^1, s_5^1, s_6^1\}, \{s_0^1, s_1^1\}, \{s_6^1\}, \{s_2^1, s_3^1, s_4^1\})$. Then we use P_1 to obtain the optimal solution (for details of the solving process see Section 6.1), i.e., $\overline{A} = (\overline{A_1}, \overline{A_2}, \overline{A_3}, \overline{A_4}, \overline{A_5}) = (\{s_3^1, s_4^1\}, \{s_2^1, s_3^1, s_4^1, s_5^1, s_6^1\}, \{s_0^1, s_1^1, s_2^1, s_3^1, s_4^1\}, \{s_2^1, s_3^1\}, \{s_2^1, s_3^1, s_4^1\})$, and $\overline{A^c} = \{s_2^1, s_3^1, s_4^1\}$. Here, we can construct another set of HFLTSs, $\overline{\overline{A}} = \left(\overline{\overline{A_1}}, \overline{\overline{A_2}}, \overline{\overline{A_3}}, \overline{\overline{A_4}}, \overline{\overline{A_5}}\right)$ and $\overline{\overline{A^c}}$, where $\overline{\overline{A_i}} = \begin{cases} \overline{A_i} & i = 1,2,3,5 \\ \{s_3^1, s_4^1\} & i = 4 \end{cases}$ and $\overline{\overline{A^c}} = \{s_2^1, s_3^1, s_4^1\}$. $\overline{\overline{A_i}}(i = 1,2,3,4,5)$ satisfies $\frac{\#\left(\overline{\overline{A_i}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_i}} \cup \overline{\overline{A^c}}\right)} \geqslant \overline{CL}$ and $\sum_{i=1}^{5} d\left(A_i, \overline{\overline{A_i}}\right) = \sum_{i=1}^{5} d(A_i, \overline{A_i})$. As a result, $\overline{\overline{A}}$ is also an optimal solution to P_1.

Through Example 4.1, we present two issues with P_1:

(1) In some situations, the optimal solutions to P_1 are not unique.
(2) S is an ordered linguistic term set, intuitively, the deviation between A_4(i.e., $\{s_6^1\}$) and $\overline{A_4}$ (i.e., $\{s_2^1, s_3^1\}$) is larger than the deviation between A_4 and $\overline{\overline{A_4}}$ (i.e, $\{s_3^1, s_4^1\}$), which implies some of the optimal solutions to P_1 are not reasonable enough.

So, it is necessary to further optimize the optimal solutions to P_1. In order to do this, we introduce another approach for measuring the distance between two HFLTSs. Let $s \in S$, we denote $ind(s)$ to be the position index (or lower index) of s in S. For example, if $s = s_j$, then $ind(s_j) = j$. According to the Manhattan distance, a natural distance between two HFLTSs is defined as Definition 4.1.

Definition 4.1. For two HFLTSs E and G, the distance between E and G can be defined as:

$$l(E,G) = \left|ind(E^+) - ind(G^+)\right| + |ind(E^-) - ind(G^-)|, \tag{19}$$

where E^+ and E^- respectively denote the upper bound and the lower bound of E, and G^+ and G^- respectively denote the upper bound and the lower bound of G.

Then, we optimize the optimal solutions to P_1 by minimizing the Manhattan distance between A_i and $\overline{OA_i}$, defined by Definition 4.1, i.e.,

$$\min_{\overline{OA_i}} \sum_{i=1}^{n} l(A_i, \overline{OA_i}) = \min_{\overline{OA_i}} \sum_{i=1}^{n} \left(\left| ind\left(\overline{OA_i}^+\right) - ind(A_i^+) \right| + \left| ind\left(\overline{OA_i}^-\right) - ind(A_i^-) \right| \right), \tag{20}$$

where $(\overline{OA}_1, \ldots, \overline{OA}_n, \overline{OA^c})$ is optimal solution to P_1, and thus must satisfy the conditions (21)–(25):

$$\sum_{i=1}^{n} d(A_i, \overline{OA_i}) = \sum_{i=1}^{n} \left(\#(A_i \cup \overline{OA_i}) - \#(A_i \cap \overline{OA_i}) \right) = M, \tag{21}$$

where M is the optimal objective function value of P_1,

$$\frac{\#(\overline{OA_i} \cap \overline{OA^c})}{\#(\overline{OA_i} \cup \overline{OA^c})} \geqslant \overline{CL}(i = 1, \ldots, n), \tag{22}$$

$$\overline{OA_i} \in H^s(i = 1, \ldots, n), \tag{23}$$

$$\overline{OA^c} \in H^s, \tag{24}$$

and

$$\#(\overline{OA^c}) \leqslant \beta. \tag{25}$$

Then, based on Eqs. (20)–(25), a two-stage model can be constructed as:

$$\begin{cases} \min_{\overline{OA_i}} \sum_{i=1}^{n} \left(\left| ind\left(\overline{OA_i}^+\right) - ind(A_i^+) \right| + \left| ind\left(\overline{OA_i}^-\right) - ind(A_i^-) \right| \right) \\ s.t. \begin{cases} \sum_{i=1}^{n} (\#(A_i \cup \overline{OA_i}) - \#(A_i \cap \overline{OA_i})) = M \\ \frac{\#(\overline{OA_i} \cap \overline{OA^c})}{\#(\overline{OA_i} \cup \overline{OA^c})} \geqslant \overline{CL} & i = 1, \ldots, n; \\ \overline{OA_i} \in H^s & i = 1, \ldots, n; \\ \overline{OA^c} \in H^s \\ \#(\overline{OA^c}) \leqslant \beta \end{cases} \end{cases} \tag{26}$$

In model (26), the constraint condition $\frac{\#\left(\overline{OA_i} \cap \overline{OA^c}\right)}{\#\left(\overline{OA_i} \cup \overline{OA^c}\right)} \geqslant \overline{CL}(i = 1, \ldots, n)$ guarantees all the decision makers can reach the established consensus threshold $\overline{CL}$. Meanwhile, the constraint condition $\sum_{i=1}^{n}(\#(A_i \cup \overline{OA_i}) - \#(A_i \cap \overline{OA_i})) = M$ guarantees the adjusted simple terms between original and adjusted individual opinions is minimum. The constraint conditions $\overline{OA_i} \in H^s(i = 1, \ldots, n)$ and $\overline{OA^c} \in H^s$ guarantee the adjusted opinions are HFLTSs. In this paper, denote model (26) as P_2.

Subsequently, we show that P_2 can also be transformed into a mixed 0–1 linear programming model. In order to do so, Lemma 4.4 is proposed.

Lemma 4.4. *For $i = 1, \ldots, n$, we have*

(1) $ind\left(\overline{OA_i}^-\right) = \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k\right)$;
(2) $ind\left(\overline{OA_i}^+\right) = \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} y_i^j - 1$;
(3) $ind(A_i^-) = \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(x_i^j - x_i^k\right)$;
(4) $ind(A_i^+) = \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(x_i^j - x_i^k\right) + \sum_{j=0}^{g} x_i^j - 1$.

Proof of Lemma 4.4 is provided in Appendix.

Let

$$\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(x_i^j - x_i^k\right) + \sum_{j=0}^{g} x_i^j = X_i \quad i = 1,\ldots,n; \tag{27}$$

$$\left|\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} y_i^j - X_i\right| \leqslant r_i \quad i = 1,\ldots,n; \tag{28}$$

and let

$$\left|\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} x_i^j - X_i\right| \leqslant t_i \quad i = 1,\ldots,n. \tag{29}$$

Based on Lemmas 4.1, 4.2, 4.3, 4.4, we can obtain Theorem 4.3.

Theorem 4.3. *P_2 can be transformed into model (30):*

$$\begin{cases} \min \sum_{i=1}^{n}(r_i + t_i) \\ s.t. \begin{cases} \max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(x_i^j - x_i^k\right) + \sum_{j=0}^{g} x_i^j = X_i & i = 1,\ldots,n; \\ \sum_{k=0}^{j}\left(y_i^j - y_i^k\right) - X_i + \sum_{j=0}^{g} y_i^j \leqslant r_i & i = 1,\ldots,n;\ j = 0,\ldots,g; \\ X_i - \sum_{k=0}^{j}\left(y_i^j - y_i^k\right) - \sum_{j=0}^{g} y_i^j \leqslant r_i + H(1 - w_{ij}) & i = 1,\ldots,n;\ j = 0,\ldots,g; \\ \sum_{j=0}^{g} w_{ij} \geqslant 1 & i = 1,\ldots,n; \\ \sum_{k=0}^{j}\left(y_i^j - y_i^k\right) - X_i + \sum_{j=0}^{g} x_i^j \leqslant t_i & i = 1,\ldots,n;\ j = 0,\ldots,g; \\ X_i - \sum_{k=0}^{j}\left(y_i^j - y_i^k\right) - \sum_{j=0}^{g} x_i^j \leqslant t_i + H(1 - h_{ij}) & i = 1,\ldots,n;\ j = 0,\ldots,g; \\ \sum_{j=0}^{g} h_{ij} \geqslant 1 & i = 1,\ldots,n; \\ \sum_{i=1}^{n}\sum_{j=0}^{g}\left|x_i^j - y_i^j\right| = M & \\ \frac{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j - \sum_{j=0}^{g}\left|y_i^j - z^j\right|}{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j + \sum_{j=0}^{g}\left|y_i^j - z^j\right|} \geqslant \overline{CL} & i = 1,\ldots,n; \\ \sum_{j=0}^{g-1}\left|y_i^{j+1} - y_i^j\right| \leqslant 2 & i = 1,\ldots,n; \\ y_i^0 + y_i^g \leqslant 1 & i = 1,\ldots,n; \\ \sum_{j=0}^{g-1}\left|z^{j+1} - z^j\right| \leqslant 2 & i = 1,\ldots,n; \\ z^0 + z^g \leqslant 1 & \\ \sum_{j=0}^{g} z^j \leqslant \beta & \\ y_i^j,\ w_{ij},\ h_{ij} = 0 \text{ or } 1 & i = 1,\ldots,n;\ j = 0,\ldots,g; \\ z^j = 0 \text{ or } 1 & j = 0,\ldots,g. \end{cases} \end{cases} \tag{30}$$

Proof of Theorem 4.3 *is provided in* Appendix.

Similarly to Theorem 4.2, model (30) can be equivalently transformed into a mixed 0–1 linear programming model.

Note 2. Solving models P_1 and P_2 obtains the optimal adjusted individual opinions, which should only be considered as decision aids that decision makers use as reference when modifying their individual opinions.

Note 3. In classical GDM studies, the number of decision makers in the most effective GDM context is less than 7 (see Thomas and Fink [35]). Meanwhile, the granularity of a linguistic term set used by decision makers without confusion is less than 9 (see Miller [27]). As a result, the proposed mixed 0–1 linear programming models are usually small-scale optimization problems, except in the case of the large-scale GDM that was recently proposed in [28]. Generally, mixed 0–1 linear programming models with a few hundred binary variables can be effectively solved by several software packages (e.g., Lingo and CEPLEX).

5. Further discussion regarding models P_1 and P_2

This section provides further discussion to justify models P_1 and P_2. Specifically, Section 5.1 presents several desirable properties of P_1, and Section 5.2 discusses the problem of uniqueness of solution to model P_2.

5.1. Desirable properties of model P_1

Let S, A_i, $\overline{A_i}$ and $\overline{A^c}$ be as before, then the following properties of P_1 (i.e., Properties 5.1–5.6) are satisfied:

Property 5.1. *Idempotency. If $A_i = A_{i+1}$ for $i = 1, \ldots, n-1$, then $\overline{A^c} = A_i$.*

Proof of Property 5.1 is provided in Appendix.

Property 5.2. $\min_{i=1,\ldots,n}(A_i) \leqslant \overline{A^c} \leqslant \max_{i=1,\ldots,n}(A_i)$.

Proof of Property 5.2 is provided in Appendix.

Property 5.2 guarantees that the adjusted collective opinion obtained by P_1 ranges from the minimum original individual opinion to the maximum original individual opinion.

Note 4. In [29], $M_1^S > M_2^S$ if $env\left(M_1^S\right) > env\left(M_2^S\right)$. Furthermore, in this paper we define $env\left(M_1^S\right) > env\left(M_2^S\right)$ if $\frac{ind\left(M_1^{S-}\right)+ind\left(M_1^{S+}\right)}{2} > \frac{ind\left(M_2^{S-}\right)+ind\left(M_2^{S+}\right)}{2}$.

Property 5.3. *Commutativity. Let $(Q_1, Q_2, \ldots, Q_n)$ be a permutation of $(A_1, A_2, \ldots, A_n)$, and let $\overline{Q^c}$ be the adjusted collective opinion that was obtained by P_1, and associated with $(Q_1, Q_2, \ldots, Q_n)$. Then, $\overline{A^c} = \overline{Q^c}$.*

Proof of Property 5.3 is provided in Appendix.

Property 5.4. *Monotonicity. Let $A' = (A'_1, A'_2, \ldots, A'_n)$ be another set of HFLTSs of S, where $A_i \leqslant A'_i$ for $i = 1, \ldots, n$. Let $\overline{A'^c}$ be the adjusted collective opinion that was obtained by P_1, and associated with $(A'_1, A'_2, \ldots, A'_n)$. Then, when $n \leqslant 2$, $\overline{A^c} \leqslant \overline{A'^c}$.*

Proof of Property 5.4 is provided in Appendix.

Note 5. We assume that the adjusted collective opinion obtained by P_1 also satisfies monotonicity when $n > 2$. However, it would be an open problem if the property is completely validated.

Property 5.5.

$$\#\left[\min\left\{\min_{i=1,\ldots,n}(\overline{A_i})\right\}, \max\left\{\max_{i=1,\ldots,n}(\overline{A_i})\right\}\right] \leqslant \#\left[\min\left\{\min_{i=1,\ldots,n}(A_i)\right\}, \max\left\{\max_{i=1,\ldots,n}(A_i)\right\}\right].$$

Proof of Property 5.5 is provided in Appendix.

Property 5.5 shows that the adjusted individual opinions obtained by P_1 concentrate into a smaller domain than the original individual opinions.

Property 5.6. $\overline{A_1} \cap \cdots \cap \overline{A_n} \subseteq \overline{A^c}$.

Proof of Property 5.6 is provided in Appendix.

Property 5.6 guarantees that the intersection of the adjusted individual opinions is a subset of the adjusted collective opinion. In other words, if $s_j \in S$ is used by all decision makers to assess an alternative, then the collective will be in favor of using s_j to assess the alternative.

5.2. Uniqueness of solution to model P_2

Before proposing the uniqueness of the solution to model P_2, we introduce Lemmas 5.1 and 5.2.

Lemma 5.1. *If $(\overline{A_1},\ldots,\overline{A_n},\overline{A^c})$ is an optimal solution to P_1 and $\exists p\in\{1,\ldots,n\}$, $\overline{A_p}\subset\overline{A^c}$ and $A_p\cap\overline{A^c}=\varnothing$, then $\overline{\overline{A}}=\left(\overline{\overline{A_1}},\ldots,\overline{\overline{A_n}},\overline{\overline{A^c}}\right)$, where $\overline{\overline{A_i}}=\overline{A_i}$ $(i\neq p)$ and $\overline{\overline{A^c}}=\overline{A^c}$, is an optimal solution to P_1 if and only if $\overline{\overline{A_p}}$ satisfies the following conditions:*

(1) $\overline{\overline{A_p}}\subset\overline{A^c}$;

(2) $\#\left(\overline{\overline{A_p}}\right)=\#(\overline{A_p})$.

Proof of Lemma 5.1 is provided in Appendix.

Lemma 5.2. *Let [a, b] and [c, d] be two real number intervals, and $[a,b]\cap[c,d]=\varnothing$. Let k be a constant and $0\leqslant k<d-c$. Then, the mathematical programming*

$$\begin{cases}\min_{x,y}(|x-a|+|y-b|)\\ s.t.\begin{cases}[x,y]\subset[c,d]\\ y-x=k\end{cases}\end{cases}, \tag{31}$$

where x and y are decision variables, has a unique solution.

Clearly, Lemma 5.2 can be directly obtained.

Based on Lemmas 5.1 and 5.2, the uniqueness of the solution to model P_2 is presented as Theorem 5.1.

Theorem 5.1. *The optimal solution to P_2 is unique.*

Proof of Theorem 5.1 is provided in Appendix.

Theorem 5.1 guarantees the uniqueness of solution to model P_2, through which we obtain the optimal adjustment suggestions to support the consensus reaching process in the hesitant linguistic GDM.

6. Illustrative examples

In order to show how these theoretical results work in practice, let us consider the following two examples.

6.1. Example 1

We suppose that five decision makers, $D=\{d_1,d_2,d_3,d_4,d_5\}$, want to assess an alternative using HFLTSs of the linguistic term set S^1, where

$$S^1=\{s_0^1=neither,s_1^1=very\ low,s_2^1=low,s_3^1=medium,s_4^1=high,s_5^1=very\ high,s_6^1=absolute\}.$$

Decision makers provide their original individual preferences using HFLTSs of S^1, i.e.,

$$A=(A_1,A_2,A_3,A_4,A_5)=\left(\{s_3^1,s_4^1\},\{s_4^1,s_5^1,s_6^1\},\{s_0^1,s_1^1\},\{s_6^1\},\{s_2^1,s_3^1,s_4^1\}\right).$$

According to these original individual opinions, using Eq. (12) obtains the values of $x_i^j(i=1,\ldots,5;\ j=0,\ldots,6)$, which are listed in Table 1.

We set $\overline{CL}=0.6$ and $\beta=3$. After determining the values of $x_i^j(i=1,\ldots,5;\ j=0,\ldots,6)$, $\overline{CL}$ and β, we use the mixed 0–1 linear programming model P_1' obtains model (32):

Table 1
The values of x_i^j.

x_i^j	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$i=1$	0	0	0	1	1	0	0
$i=2$	0	0	0	0	1	1	1
$i=3$	1	1	0	0	0	0	0
$i=4$	0	0	0	0	0	0	1
$i=5$	0	0	1	1	1	0	0

$$\begin{cases} \min \sum_{i=1}^{5}\sum_{j=0}^{6} c_i^j \\ s.t. \begin{cases} x_i^j - y_i^j \leqslant c_i^j & i=1,2,3,4,5; \quad j=0,1,2,3,4,5,6 \\ y_i^j - x_i^j \leqslant c_i^j & i=1,2,3,4,5; \quad j=0,1,2,3,4,5,6; \\ 0.4\sum_{j=0}^{6} y_i^j + 0.4\sum_{j=0}^{6} z^j - 1.6\sum_{j=0}^{6} f_i^j \geqslant 0 & i=1,2,3,4,5; \\ y_i^j - z^j \leqslant f_i^j & i=1,2,3,4,5; \quad j=0,1,2,3,4,5,6 \\ z^j - y_i^j \leqslant f_i^j & i=1,2,3,4,5; \quad j=0,1,2,3,4,5,6 \\ \sum_{j=0}^{5} o_i^j \leqslant 2 & i=1,2,3,4,5; \\ y_i^{j+1} - y_i^j \leqslant o_i^j & i=1,2,3,4,5; \quad j=0,1,2,3,4,5; \\ y_i^j - y_i^{j+1} \leqslant o_i^j & i=1,2,3,4,5; \quad j=0,1,2,3,4,5; \\ y_i^0 + y_i^6 \leqslant 1 & i=1,2,3,4,5; \\ \sum_{j=0}^{5} v^j \leqslant 2 & \\ z^{j+1} - z^j \leqslant v^j & j=0,1,2,3,4,5; \\ z^j - z^{j+1} \leqslant v^j & j=0,1,2,3,4,5; \\ z^0 + z^6 \leqslant 1 & \\ \sum_{j=0}^{6} z^j \leqslant 3 & \\ y_i^j = 0 \text{ or } 1 & i=1,2,3,4,5; \quad j=0,1,2,3,4,5,6; \\ z^j = 0 \text{ or } 1 & j=0,1,2,3,4,5,6. \end{cases} \end{cases} \quad (32)$$

Solving model (32) by the software package LINGO, we obtain the values of $y_i^j(i=1,\ldots,5;\ j=0,\ldots,6)$ and $z^j(j = 0, \ldots, 6)$, which are listed in Tables 2 and 3, respectively.

Subsequently, based on Eqs. (13) and (14), we can obtain the adjusted individual opinions, i.e.,

$$\overline{A} = (\overline{A_1}, \overline{A_2}, \overline{A_3}, \overline{A_4}, \overline{A_5}) = (\{s_3^1, s_4^1\}, \{s_2^1, s_3^1, s_4^1, s_5^1, s_6^1\}, \{s_0^1, s_1^1, s_2^1, s_3^1, s_4^1\}, \{s_2^1, s_3^1\}, \{s_2^1, s_3^1, s_4^1\}),$$

and the adjusted collective opinion $\overline{A^c} = \{s_2^1, s_3^1, s_4^1\}$.

Let $M = \sum_{i=1}^{5} d(A_i, \overline{A_i}) = 8$ be the optimal objective function value of P_1. After determining the value of M, we use P_2 to further optimize the optimal solutions to model (32). Finally, we can obtain the optimal adjusted individual opinions, i.e.,

$$\overline{OA} = (\overline{OA_1}, \overline{OA_2}, \overline{OA_3}, \overline{OA_4}, \overline{OA_5}) = (\{s_3^1, s_4^1\}, \{s_2^1, s_3^1, s_4^1, s_5^1, s_6^1\}, \{s_0^1, s_1^1, s_2^1, s_3^1, s_4^1\}, \{s_3^1, s_4^1\}, \{s_2^1, s_3^1, s_4^1\}),$$

and the optimal adjusted collective opinion $\overline{OA^c} = \{s_2^1, s_3^1, s_4^1\}$.

Clearly, $\sum_{i=1}^{5} d(A_i, \overline{A_i}) = 8, \sum_{i=1}^{5} l(A_i, \overline{A_i}) = 12, \sum_{i=1}^{5} d(A_i, \overline{OA_i}) = 8$ and $\sum_{i=1}^{5} l(A_i, \overline{OA_i}) = 10$. In other words, $\overline{OA}$ is also the optimal solution to P_1, however, the Manhattan distance between A_i and $\overline{OA_i}$ ($i = 1, \ldots, n$) is smaller than the Manhattan distance between A_i and $\overline{A_i}$ ($i = 1, \ldots, n$).

Table 2
The values of y_i^j.

y_i^j	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
$i = 1$	0	0	0	1	1	0	0
$i = 2$	0	0	1	1	1	1	1
$i = 3$	1	1	1	1	1	0	0
$i = 4$	0	0	1	1	0	0	0
$i = 5$	0	0	1	1	1	0	0

Table 3
The values of z^j.

	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
z^j	0	0	1	1	1	0	0

6.2. Example 2

We suppose that a committee is composed of seven decision makers, $D = \{d_1, d_2, \ldots, d_7\}$, which compare five alternatives $X = \{x_1, x_2, \ldots, x_5\}$ by using HFLTSs of the linguistic term set S^2, where

$$S^2 = \{s_0^2 = \textit{neither}, s_1^2 = \textit{very low}, s_2^2 = \textit{low}, s_3^2 = \textit{slightly low}, s_4^2 = \textit{medium}, s_5^2 = \textit{slightly high}, s_6^2 = \textit{high}, s_7^2 = \textit{very high}, s_8^2 = \textit{absolute}\}.$$

and construct, respectively, the linguistic preference relations $P^{(k)} = \left(p_{ij}^{(k)}\right)_{5\times5} (k = 1, 2, \ldots, 7; \quad i, j = 1, 2, \ldots, 5)$, where $p_{ij}^{(k)}$ denotes the linguistic preference degree of the alternative x_i over x_j for the decision maker d_k. They are listed as follows:

$$P^{(1)} = \begin{pmatrix} - & \{s_1^2, s_2^2\} & \{s_7^2\} & \{s_2^2, s_3^2\} & \{s_5^2, s_6^2\} \\ \{s_5^2, s_6^2\} & - & \{s_4^2, s_5^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_2^2\} \\ \{s_2^2\} & \{s_3^2, s_4^2\} & - & \{s_6^2, s_7^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_6^2, s_7^2\} & \{s_5^2, s_6^2, s_7^2\} & \{s_1^2, s_2^2\} & - & \{s_4^2, s_5^2, s_6^2\} \\ \{s_1^2, s_2^2\} & \{s_5^2\} & \{s_1^2, s_2^2\} & \{s_1^2, s_2^2, s_3^2\} & - \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} - & \{s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_4^2, s_5^2, s_6^2\} & - & \{s_6^2\} & \{s_3^2\} & \{s_0^2, s_1^2\} \\ \{s_0^2, s_1^2\} & \{s_4^2\} & - & \{s_7^2, s_8^2\} & \{s_4^2, s_5^2\} \\ \{s_6^2, s_7^2\} & \{s_7^2\} & \{s_0^2, s_1^2, s_2^2\} & - & \{s_5^2\} \\ \{s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_3^2\} & - \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} - & \{s_3^2, s_4^2\} & \{s_4^2, s_5^2\} & \{s_4^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_0^2, s_1^2, s_2^2\} & - & \{s_5^2, s_6^2\} & \{s_6^2, s_7^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_1^2, s_2^2\} & \{s_2^2\} & - & \{s_5^2\} & \{s_4^2\} \\ \{s_3^2\} & \{s_1^2, s_2^2\} & \{s_2^2\} & - & \{s_3^2, s_4^2, s_5^2\} \\ \{s_3^2, s_4^2\} & \{s_4^2, s_5^2\} & \{s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & - \end{pmatrix}$$

$$P^{(4)} = \begin{pmatrix} - & \{s_3^2, s_4^2\} & \{s_5^2, s_6^2\} & \{s_1^2, s_2^2\} & \{s_0^2, s_1^2\} \\ \{s_4^2, s_5^2\} & - & \{s_6^2, s_7^2\} & \{s_2^2, s_3^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_1^2, s_2^2\} & \{s_2^2, s_3^2\} & - & \{s_5^2, s_6^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_7^2, s_8^2\} & \{s_6^2, s_7^2\} & \{s_4^2, s_5^2\} & - & \{s_4^2, s_5^2, s_6^2\} \\ \{s_3^2, s_4^2\} & \{s_4^2, s_5^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_3^2, s_4^2\} & - \end{pmatrix}$$

$$P^{(5)} = \begin{pmatrix} - & \{s_1^2, s_2^2\} & \{s_6^2, s_7^2\} & \{s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_5^2, s_6^2\} & - & \{s_4^2, s_5^2, s_6^2\} & \{s_1^2, s_2^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_0^2, s_1^2\} & \{s_3^2\} & - & \{s_5^2, s_6^2, s_7^2\} & \{s_3^2, s_4^2\} \\ \{s_5^2, s_6^2, s_7^2\} & \{s_5^2, s_6^2\} & \{s_0^2, s_1^2, s_2^2\} & - & \{s_3^2, s_4^2\} \\ \{s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_3^2, s_4^2\} & \{s_5^2\} & - \end{pmatrix}$$

$$P^{(6)} = \begin{pmatrix} - & \{s_2^2, s_3^2, s_4^2\} & \{s_7^2, s_8^2\} & \{s_3^2, s_4^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_5^2, s_6^2\} & - & \{s_5^2, s_6^2\} & \{s_0^2, s_1^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_0^2, s_1^2\} & \{s_3^2, s_4^2\} & - & \{s_7^2, s_8^2\} & \{s_3^2, s_4^2, s_5^2\} \\ \{s_4^2, s_5^2\} & \{s_7^2\} & \{s_0^2, s_1^2\} & - & \{s_3^2, s_4^2\} \\ \{s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_5^2\} & - \end{pmatrix}$$

$$P^{(7)} = \begin{pmatrix} - & \{s_2^2, s_3^2\} & \{s_6^2, s_7^2, s_8^2\} & \{s_2^2, s_3^2\} & \{s_5^2\} \\ \{s_6^2, s_7^2\} & - & \{s_6^2, s_7^2\} & \{s_1^2, s_2^2\} & \{s_4^2, s_5^2\} \\ \{s_0^2, s_1^2, s_2^2\} & \{s_3^2, s_4^2\} & - & \{s_5^2, s_6^2\} & \{s_3^2, s_4^2\} \\ \{s_5^2, s_6^2\} & \{s_7^2, s_8^2\} & \{s_4^2, s_5^2\} & - & \{s_5^2, s_6^2\} \\ \{s_2^2\} & \{s_0^2, s_1^2, s_2^2\} & \{s_4^2, s_5^2\} & \{s_3^2, s_4^2\} & - \end{pmatrix}$$

Here, we set $\beta = 3$ and $\overline{CL} = 0.6$. Then, we use models P_1 and P_2 to obtain the optimal adjusted individual preferences $\overline{p_{ij}^{(1)}}, \overline{p_{ij}^{(2)}}, \ldots, \overline{p_{ij}^{(7)}}$ and the adjusted collective preference $\overline{p_{ij}^{(c)}}$, associated with $p_{ij}^{(1)}, p_{ij}^{(2)}, \ldots, p_{ij}^{(7)}$, respectively. Let $\overline{P^{(k)}} = \left(\overline{p_{ij}^{(k)}}\right)_{5\times5}$ (k = 1, 2, …, 7) and $\overline{P^{(c)}} = \left(\overline{p_{ij}^{(c)}}\right)_{5\times5}$, which are listed as follows:

$$\overline{P^{(1)}} = \begin{pmatrix} - & \{s_1^2, s_2^2, s_3^2\} & \{s_6^2, s_7^2\} & \{s_2^2, s_3^2\} & \{s_5^2, s_6^2\} \\ \{s_5^2, s_6^2\} & - & \{s_5^2, s_6^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_1^2, s_2^2\} \\ \{s_1^2, s_2^2\} & \{s_3^2, s_4^2\} & - & \{s_6^2, s_7^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_6^2, s_7^2\} & \{s_5^2, s_6^2, s_7^2\} & \{s_1^2, s_2^2\} & - & \{s_4^2, s_5^2, s_6^2\} \\ \{s_1^2, s_2^2\} & \{s_5^2, s_6^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_2^2, s_3^2, s_4^2, s_5^2\} & - \end{pmatrix}$$

$$\overline{P^{(2)}} = \begin{pmatrix} - & \{s_1^2, s_2^2, s_3^2\} & \{s_4^2, s_5^2, s_6^2, s_7^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_4^2, s_5^2, s_6^2\} & - & \{s_6^2, s_7^2\} & \{s_2^2, s_3^2\} & \{s_0^2, s_1^2\} \\ \{s_0^2, s_1^2\} & \{s_3^2, s_4^2\} & - & \{s_5^2, s_6^2, s_7^2, s_8^2\} & \{s_4^2, s_5^2\} \\ \{s_6^2, s_7^2\} & \{s_6^2, s_7^2\} & \{s_0^2, s_1^2, s_2^2\} & - & \{s_4^2, s_5^2\} \\ \{s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_3^2, s_4^2\} & - \end{pmatrix}$$

$$\overline{P^{(3)}} = \begin{pmatrix} - & \{s_2^2, s_3^2, s_4^2\} & \{s_4^2, s_5^2, s_6^2, s_7^2\} & \{s_3^2, s_4^2\} & \{s_4^2, s_5^2\} \\ \{s_4^2, s_5^2\} & - & \{s_5^2, s_6^2\} & \{s_2^2, s_3^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_1^2, s_2^2\} & \{s_2^2, s_3^2\} & - & \{s_5^2, s_6^2\} & \{s_4^2, s_5^2\} \\ \{s_5^2, s_6^2\} & \{s_5^2, s_6^2\} & \{s_1^2, s_2^2\} & - & \{s_3^2, s_4^2, s_5^2\} \\ \{s_2^2, s_3^2\} & \{s_4^2, s_5^2\} & \{s_2^2, s_3^2\} & \{s_3^2, s_4^2, s_5^2, s_6^2\} & - \end{pmatrix}$$

$$\overline{P^{(4)}} = \begin{pmatrix} - & \{s_2^2, s_3^2, s_4^2\} & \{s_5^2, s_6^2\} & \{s_2^2, s_3^2\} & \{s_4^2, s_5^2\} \\ \{s_4^2, s_5^2\} & - & \{s_6^2, s_7^2\} & \{s_2^2, s_3^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_1^2, s_2^2\} & \{s_2^2, s_3^2\} & - & \{s_5^2, s_6^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_5^2, s_6^2, s_7^2, s_8^2\} & \{s_6^2, s_7^2\} & \{s_1^2, s_2^2\} & - & \{s_4^2, s_5^2, s_6^2\} \\ \{s_2^2, s_3^2\} & \{s_4^2, s_5^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_3^2, s_4^2\} & - \end{pmatrix}$$

$$\overline{P^{(5)}} = \begin{pmatrix} - & \{s_1^2, s_2^2, s_3^2\} & \{s_6^2, s_7^2\} & \{s_2^2, s_3^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_5^2, s_6^2\} & - & \{s_5^2, s_6^2\} & \{s_1^2, s_2^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_0^2, s_1^2\} & \{s_3^2, s_4^2\} & - & \{s_5^2, s_6^2\} & \{s_3^2, s_4^2, s_5^2\} \\ \{s_5^2, s_6^2, s_7^2\} & \{s_5^2, s_6^2\} & \{s_0^2, s_1^2, s_2^2\} & - & \{s_3^2, s_4^2, s_5^2\} \\ \{s_2^2, s_3^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_4^2, s_5^2\} & - \end{pmatrix}$$

$$\overline{P^{(6)}} = \begin{pmatrix} - & \{s_2^2, s_3^2, s_4^2\} & \{s_6^2, s_7^2\} & \{s_3^2, s_4^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_5^2, s_6^2\} & - & \{s_5^2, s_6^2\} & \{s_1^2, s_2^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_0^2, s_1^2\} & \{s_3^2, s_4^2\} & - & \{s_5^2, s_6^2, s_7^2, s_8^2\} & \{s_3^2, s_4^2, s_5^2\} \\ \{s_5^2, s_6^2\} & \{s_6^2, s_7^2\} & \{s_0^2, s_1^2\} & - & \{s_3^2, s_4^2, s_5^2\} \\ \{s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_4^2, s_5^2\} & - \end{pmatrix}$$

$$\overline{P^{(7)}} = \begin{pmatrix} - & \{s_2^2, s_3^2\} & \{s_6^2, s_7^2\} & \{s_2^2, s_3^2\} & \{s_5^2, s_6^2\} \\ \{s_4^2, s_5^2, s_6^2, s_7^2\} & - & \{s_6^2, s_7^2\} & \{s_1^2, s_2^2\} & \{s_1^2, s_2^2\} \\ \{s_0^2, s_1^2, s_2^2\} & \{s_3^2, s_4^2\} & - & \{s_5^2, s_6^2\} & \{s_3^2, s_4^2, s_5^2\} \\ \{s_5^2, s_6^2\} & \{s_5^2, s_6^2, s_7^2, s_8^2\} & \{s_1^2, s_2^2\} & - & \{s_4^2, s_5^2, s_6^2\} \\ \{s_2^2, s_3^2\} & \{s_5^2, s_6^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_3^2, s_4^2\} & - \end{pmatrix}$$

$$\overline{P^{(c)}} = \begin{pmatrix} - & \{s_2^2, s_3^2\} & \{s_5^2, s_6^2, s_7^2\} & \{s_2^2, s_3^2, s_4^2\} & \{s_4^2, s_5^2, s_6^2\} \\ \{s_4^2, s_5^2, s_6^2\} & - & \{s_5^2, s_6^2, s_7^2\} & \{s_1^2, s_2^2, s_3^2\} & \{s_0^2, s_1^2, s_2^2\} \\ \{s_0^2, s_1^2, s_2^2\} & \{s_2^2, s_3^2, s_4^2\} & - & \{s_5^2, s_6^2, s_7^2\} & \{s_4^2, s_5^2\} \\ \{s_5^2, s_6^2, s_7^2\} & \{s_5^2, s_6^2, s_7^2\} & \{s_0^2, s_1^2, s_2^2\} & - & \{s_4^2, s_5^2\} \\ \{s_0^2, s_1^2, s_2^2\} & \{s_4^2, s_5^2, s_6^2\} & \{s_2^2, s_3^2\} & \{s_3^2, s_4^2, s_5^2\} & - \end{pmatrix}$$

Using this approach, the decision makers $D = \{d_1, d_2, \ldots, d_7\}$ can adjust their individual preference relations with the established consensus level.

7. Conclusions

This paper focuses on the consensus issue in the hesitant linguistic GDM problem. The main contributions presented are as follows:

(1) We provide a new method of measuring the difference between two HFLTSs, which reflects the number of different simple terms between two HFLTSs. Following this method, a novel distance-based approach is developed to measure the consensus level.
(2) We propose an optimization-based two-stage model to obtain the optimal adjusted individual opinions in the hesitant linguistic GDM, which minimizes the number of adjusted simple terms in the consensus building. Mixed 0–1 linear programming models are proposed to solve this two-stage model.
(3) Several desirable properties are proposed to justify the proposal, and the uniqueness of the solution to the proposed consensus model is proven.

Moreover, we argue that the following directions should be considered for further research:

(1) Models P_1 and P_2 are transformed into mixed 0–1 linear programming models, which can be effectively solved in small-scale GDM problems. However, nowadays societal and technological trends demand the management of larger-scale GDM problems, such as e-democracy and social networks [28]. In order to provide a decision aid for the large-scale GDM, we argue that it would be useful in any future research to see if better algorithms for obtaining the optimal solution to models P_1 and P_2 are proposed.
(2) Consensus building not only relates to mathematical models, but also to philosophical issues [19]. Therefore, it would be interesting in any future research to see hesitant behaviors based on psychology being investigated in the consensus process.

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Appendix Proofs

Proof of Theorem 3.1. Obviously, $d(Q, N)$ satisfies conditions (1), (2) and (4). Here we only have to prove that $d(Q, N)$ satisfies condition (3).

Because

$$\begin{aligned} d(Q,P) + d(P,N) &= \#(Q \cup P) - \#(Q \cap P) + \#(P \cup N) - \#(P \cap N) \\ &= \#(Q) + \#(P) - 2\#(Q \cap P) + \#(P) + \#(N) - 2\#(P \cap N) \end{aligned} \tag{33}$$

and

$$d(Q,N) = \#(Q) + \#(N) - 2\#(Q \cap N). \tag{34}$$

Based on Eqs. (33) and (34), then

$$d(Q,P) + d(P,N) - d(Q,N) = 2(\#(P) + \#(Q \cap N) - \#(Q \cap P) - \#(P \cap N)). \tag{35}$$

To discuss Eq. (35), we consider three cases:
Case 1: if $\#(Q \cap P) + \#(P \cap N) < \#(P)$, then $d(Q, P) + d(P, N) - d(Q, N) > 0$.
Case 2: if $\#(Q \cap P) + \#(P \cap N) = \#(P)$, then $d(Q, P) + d(P, N) - d(Q, N) \geqslant 0$.
Case 3: if $\#(Q \cap P) + \#(P \cap N) > \#(P)$, we have

$$\#(Q \cap P) + \#(P \cap N) = \#(P) + \#(Q \cap N \cap P),$$

then $d(Q, P) + d(P, N) - d(Q, N) \geqslant 0$.
This completes Proof of Theorem 3.1. □

Proof of Lemma 4.1. Based on Eqs. (12) and (13), we introduce $\Phi_{A_i}(x)$ to denote $\#(A_i)$, i.e., $\Phi_{A_i}(x)=\sum_{j=0}^{g}x_i^j$ and introduce $\Phi_{\overline{A_i}}(y)$ to denote $\#(\overline{A_i})$, i.e., $\Phi_{\overline{A_i}}(y)=\sum_{j=0}^{g}y_i^j$. Hence,

$$\#(A_i\cup\overline{A_i})-\#(A_i\cap\overline{A_i})=\Phi_{A_i}(x)+\Phi_{\overline{A_i}}(y)-2\sum_{j=0}^{g}x_i^j\cdot y_i^j=\sum_{j=0}^{g}x_i^j+\sum_{j=0}^{g}y_i^j-2\sum_{j=0}^{g}x_i^j\cdot y_i^j$$
$$=\sum_{j=0}^{g}\left(x_i^j\right)^2+\sum_{j=0}^{g}\left(y_i^j\right)^2-2\sum_{j=0}^{g}x_i^j\cdot y_i^j=\sum_{j=0}^{g}\left(x_i^j-y_i^j\right)^2.$$

Due to x_i^j, y_i^j being binary variables, Thus $\#(A_i\cup\overline{A_i})-\#(A_i\cap\overline{A_i})=\sum_{j=0}^{g}\left|x_i^j-y_i^j\right|$.
This completes Proof of Lemma 4.1. □

Proof of Lemma 4.2. According to Lemma 4.1, we can easily obtain:

$$\#(\overline{A_i}\cup\overline{A^c})-\#(\overline{A_i}\cap\overline{A^c})=\sum_{j=0}^{g}\left|y_i^j-z^j\right|, \tag{36}$$

while

$$\#(\overline{A_i}\cup\overline{A^c})=\sum_{j=0}^{g}y_i^j+\sum_{j=0}^{g}z^j-\#(\overline{A_i}\cap\overline{A^c}). \tag{37}$$

Based on Eqs. (36) and (37), then

$$\#(\overline{A_i}\cap\overline{A^c})=\frac{\sum_{j=0}^{g}y_i^j+\sum_{j=0}^{g}z^j-\sum_{j=0}^{g}\left|y_i^j-z^j\right|}{2}, \tag{38}$$

and

$$\#(\overline{A_i}\cup\overline{A^c})=\frac{\sum_{j=0}^{g}y_i^j+\sum_{j=0}^{g}z^j+\sum_{j=0}^{g}\left|y_i^j-z^j\right|}{2}. \tag{39}$$

According to Eqs. (38) and (39),

$$\frac{\#(\overline{A_i}\cap\overline{A^c})}{\#(\overline{A_i}\cup\overline{A^c})}=\frac{\sum_{j=0}^{g}y_i^j+\sum_{j=0}^{g}z^j-\sum_{j=0}^{g}\left|y_i^j-z^j\right|}{\sum_{j=0}^{g}y_i^j+\sum_{j=0}^{g}z^j+\sum_{j=0}^{g}\left|y_i^j-z^j\right|}.$$

This completes Proof of Lemma 4.2. □

Proof of Lemma 4.3.

Part 1: Proving that $\overline{A_i}(i=1,\ldots,n)$ is a HFLTS if and only if the following conditions are satisfied:

1) $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|\leqslant 2\ (i=1,\ldots,n)$;
2) $y_i^0+y_i^g\leqslant 1(i=1,\ldots,n)$.

Necessity: If $\overline{A_i}(i=1,\ldots,n)$ is a HFLTS, then there are three distribution cases in $\overline{A_i}$:
Case 1: $\overline{A_i}=\{s_0,\ldots,s_k\}(0\leqslant k<g)$, then $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|=1$ and $y_i^0+y_i^g=1$;
Case 2: $\overline{A_i}=\{s_k,\ldots,s_g\}(0<k\leqslant g)$, then $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|=1$ and $y_i^0+y_i^g=1$;
Case 3: $\overline{A_i}=\{s_k,\ldots,s_{k+q}\}(k>0,q\geqslant 0$ and $k+q<g)$, then $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|=2$ and $y_i^0+y_i^g=0$.

Therefore, if $\overline{A_i}(i=1,\ldots,n)$ is a HFLTS, it satisfies $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|\leqslant 2(i=1,\ldots,n)$ and $y_i^0+y_i^g\leqslant 1(i=1,\ldots,n)$.
Sufficiency: If $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|\leqslant 2(i=1,\ldots,n)$, then there are four distribution cases in $\overline{A_i}(i=1,\ldots,n)$:
Case 1: $\overline{A_i}=\{s_0,\ldots s_k\}(0\leqslant k<g)$, then $\sum_{j=0}^{g-1}\left|y_i^{j+1}-y_i^j\right|=1$.

Case 2: $\overline{A_i} = \{s_k, \dots s_g\}(0 < k \leqslant g)$, then $\sum_{j=0}^{g-1} \left| y_i^{j+1} - y_i^j \right| = 1$.

Case 3: $\overline{A_i} = \{s_k, \dots s_{k+q}\}(k > 0, q \geqslant 0$ and $k + q < g)$, then $\sum_{j=0}^{g-1} \left| y_i^{j+1} - y_i^j \right| = 2$.

Case 4: $\overline{A_i} = \{s_0, \dots s_k, s_{k+p}, \dots s_g\}(k \geqslant 0, p \geqslant 2$ and $k + p \leqslant g)$, then $\sum_{j=0}^{g-1} \left| y_i^{j+1} - y_i^j \right| = 2$.

Requiring $y_i^0 + y_i^g \leqslant 1$ for $\overline{A_i}$, Case 4 is excluded. In cases 1–3, simple terms in $\overline{A_i}$ are consecutive, i.e., $\overline{A_i}(i = 1, \dots, n)$ is a HFLTS.

Part 2: Proving that $\overline{A^c}$ is a HFLTS if and only if the following conditions are satisfied:

1) $\sum_{j=0}^{g-1} \left| z^{j+1} - z_i^j \right| \leqslant 2$.
2) $z^0 + z^g \leqslant 1$.

The proof of Part 2 is the same as the proof of Part 1.
This completes Proof of Lemma 4.3. □

Proof of Theorem 4.2. Eight transformed decision variables are introduced as: $b_i^j = x_i^j - y_i^j$, $c_i^j = \left| b_i^j \right|$, $e_i^j = y_i^j - z^j$, $f_i^j = \left| e_i^j \right|$, $h_i^j = y_i^{j+1} - y_i^j$, $o_i^j = \left| h_i^j \right|$, $u^j = z^{j+1} - z^j$ and $v^j = |u^j|$. Then

(1) $b_i^j = x_i^j - y_i^j$, $b_i^j \leqslant c_i^j$ and $-b_i^j \leqslant c_i^j$ guarantee $c_i^j \geqslant \left| b_i^j \right| = \left| x_i^j - y_i^j \right|$.
(2) $e_i^j = y_i^j - z^j$, $e_i^j \leqslant f_i^j$ and $-e_i^j \leqslant f_i^j$ guarantee $f_i^j \geqslant \left| e_i^j \right| = \left| y_i^j - z^j \right|$.
(3) $h_i^j = y_i^{j+1} - y_i^j$, $h_i^j \leqslant o_i^j$ and $-h_i^j \leqslant o_i^j$ guarantee $o_i^j \geqslant \left| h_i^j \right| = \left| y_i^{j+1} - y_i^j \right|$.
(4) $u^j = z^{j+1} - z^j$, $u^j \leqslant v^j$ and $-u^j \leqslant v^j$ guarantee $v^j \geqslant |u^j| = |z^{j+1} - z^j|$.

Therefore, P_1 can be equivalently transformed into the mixed 0–1 linear programming model P_1'.
This completes Proof of Theorem 4.2. □

Proof of Lemma 4.4. For $i = 1, \dots, n$, we assume $\overline{OA_i} = \{s_p, \dots s_{p+q}\}\left(p \geqslant 0,\ 0 \leqslant q = \sum_{j=0}^{g} y_i^j - 1 \leqslant g\right)$. Based on Eq. (13), the values $y_i^j (j = 0, \dots, g)$ of $\overline{OA_i}$ can be denoted as $\{y_i^0, y_i^1, \dots, y_i^g\} = \left\{ \underbrace{0, \dots 0}_{p-1}, \underbrace{1, \dots 1}_{\sum_{j=0}^{g} y_i^j}, 0, \dots 0 \right\}$.

For all $j \in [0, g]$, we have

$$\sum_{k=0}^{j} \left(y_i^j - y_i^k \right) = \begin{cases} 0, & j \in [0, p - 1], \\ p, & j \in \left[p, p + \sum_{j=0}^{g} y_i^j - 1\right], \\ -\sum_{j=0}^{g} y_i^j, & j \in \left[p + \sum_{j=0}^{g} y_i^j, g\right]. \end{cases} \tag{40}$$

According to (40), thus

$$ind(\overline{OA_i}^-) = p = \max_{j=0,\dots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k \right),$$

and

$$ind(\overline{OA_i}^+) = \max_{j=0,\dots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k \right) + \sum_{j=0}^{g} y_i^j - 1.$$

In the same way, we can obtain

$$ind(A_i^-) = \max_{j=0,\dots,g} \sum_{k=0}^{j} \left(x_i^j - x_i^k \right),$$

and

$$ind(A_i^+) = \max_{j=0,\dots,g} \sum_{k=0}^{j} \left(x_i^j - x_i^k \right) + \sum_{j=0}^{g} x_i^j - 1.$$

This completes Proof of Lemma 4.4. □

Proof of Theorem 4.3. The process of obtaining Theorem 4.3 can be divided into four steps:

Step 1: The constrain conditions in P_2, i.e.,

$$\begin{aligned}
&\sum_{i=1}^{n}(\#(A_i \cup \overline{OA_i}) - \#(A_i \cap \overline{OA_i})) = M \\
&\frac{\#(\overline{OA_i} \cap \overline{OA^c})}{\#(\overline{OA_i} \cup \overline{OA^c})} \geqslant \overline{CL} \qquad i = 1, \ldots, n; \\
&\overline{OA_i} \in H^s \qquad\qquad i = 1, \ldots, n; \\
&\overline{OA^c} \in H^s \\
&\#(\overline{OA^c}) \leqslant \beta
\end{aligned} \tag{41}$$

can be transformed into the following mixed 0–1 constrain conditions in model (30):

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=0}^{g}\left|x_i^j - y_i^j\right| = M \\
&\frac{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j - \sum_{j=0}^{g} |y_i^j - z^j|}{\sum_{j=0}^{g} y_i^j + \sum_{j=0}^{g} z^j + \sum_{j=0}^{g} |y_i^j - z^j|} \geqslant \overline{CL} \quad i = 1, \ldots, n; \\
&\sum_{j=0}^{g-1}\left|y_i^{j+1} - y_i^j\right| \leqslant 2 \qquad i = 1, \ldots, n; \\
&y_i^0 + y_i^g \leqslant 1 \qquad i = 1, \ldots, n; \\
&\sum_{j=0}^{g-1}\left|z^{j+1} - z^j\right| \leqslant 2 \qquad i = 1, \ldots, n; \\
&z^0 + z^g \leqslant 1 \\
&\sum_{j=0}^{g} z^j \leqslant \beta
\end{aligned} \tag{42}$$

Step 2: Let $\max_{j=0,\ldots,g} \sum_{k=0}^{j}\left(x_i^j - x_i^k\right) + \sum_{j=0}^{g} x_i^j = X_i (i = 1, \ldots, n)$, and based on Lemma 4.4, the objective function $\min_{\overline{OA_i}} \sum_{i=1}^{n}(\left|ind\left(\overline{OA_i}^+\right) - ind(A_i^+)\right| + \left|ind\left(\overline{OA_i}^-\right) - ind(A_i^-)\right|)$ in P_2 can be transformed as

$$\min \sum_{i=1}^{n}\left(\left|\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} y_i^j - X_i\right| + \left|\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} x_i^j - X_i\right|\right). \tag{43}$$

Step 3: Introducing two transformed decision variables: r_i and $t_i (i = 1, \ldots, n)$.

Similar to Proof of Theorem 4.2, let $\left|\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} y_i^j - X_i\right| \leqslant r_i$ and $\left|\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} x_i^j - X_i\right| \leqslant t_i$, then Eq. (43) can be further transformed as

$$\min \sum_{i=1}^{n}(r_i + t_i). \tag{44}$$

Meanwhile, several new constrain conditions are produced as:

$$\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} y_i^j - X_i \leqslant r_i; \tag{45}$$

$$X_i - \sum_{j=0}^{g} y_i^j - \max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) \leqslant r_i; \tag{46}$$

$$\max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} x_i^j - X_i \leqslant t_i; \tag{47}$$

$$X_i - \sum_{j=0}^{g} x_i^j - \max_{j=0,\ldots,g}\sum_{k=0}^{j}\left(y_i^j - y_i^k\right) \leqslant t_i. \tag{48}$$

Step 4: Transforming Eqs. (45)–(48) into linear constrain conditions in model (30).

(1) For $i = 1, \ldots, n,\ \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} y_i^j - X_i \leqslant r_i$ if and only if the following constraints are satisfied:

$$\sum_{k=0}^{j} \left(y_i^j - y_i^k\right) \leqslant r_i + X_i - \sum_{j=0}^{g} y_i^j (j = 0, \ldots, g). \tag{49}$$

(2) For $i = 1, \ldots, n,\ \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k\right) + \sum_{j=0}^{g} x_i^j - X_i \leqslant t_i$ if and only if the following constraints are satisfied:

$$\sum_{k=0}^{j} \left(y_i^j - y_i^k\right) - X_i + \sum_{j=o}^{g} x_i^j \leqslant t_i (j = 0, \ldots, g). \tag{50}$$

(3) For $i = 1, \ldots, n,\ X_i - \sum_{j=0}^{g} y_i^j - r_i \leqslant \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k\right)$ if and only if the following constraints are satisfied:

$$X_i - \sum_{j=0}^{g} y_i^j - r_i \leqslant \sum_{k=0}^{j} \left(y_i^j - y_i^k\right) + H(1 - w_{ij})(j = 0, \ldots, g), \tag{51}$$

and

$$\sum_{j=0}^{g} w_{ij} \geqslant 1, \tag{52}$$

where H is a very large number, $w_{ij} \in \{0, 1\}$.

(4) For $i = 1, \ldots, n$, $X_i - \sum_{j=0}^{g} x_i^j - t_i \leqslant \max_{j=0,\ldots,g} \sum_{k=0}^{j} \left(y_i^j - y_i^k\right)$ if and only if the following constraints are satisfied:

$$X_i - \sum_{j=0}^{g} x_i^j - t_i \leqslant \sum_{k=1}^{j} \left(y_i^j - y_i^k\right) + H(1 - h_{ij})(j = 0, \ldots, g), \tag{53}$$

and

$$\sum_{j=0}^{g} h_{ij} \geqslant 1, \tag{54}$$

where H is a very large number, $h_{ij} \in \{0, 1\}$.

Therefore, based on Eqs. (42), (44), (49)–(54), all the constraint conditions in P_2 can be equivalently transformed into the constraint conditions in model (30).

This completes Proof of Theorem 4.3. □

Proof of Property 5.1. The original individual opinions $A_i = A_{i+1}$ for $i = 1, \ldots, n-1$, i.e., $A_1 = A_2 = \cdots = A_n$. We use model P_1 to obtain the adjusted opinions $\overline{A_i} = A_i (i = 1, 2, \ldots, n)$ and $\overline{A^c} = A_i$, which guarantee the consensus level $CL_i = \frac{\#(\overline{A_i} \cap \overline{A^c})}{\#(\overline{A_i} \cup \overline{A^c})} = 1 (i = 1, \ldots, n)$ (i.e., there is a full and unanimous consensus among decision makers) and the number of adjusted simple terms $\sum_{i=1}^{n} d(A_i, \overline{A_i}) = 0$.

This completes Proof of Property 5.1. □

Proof of Property 5.2. From the implementation of the adjusted collective opinion, we have $\min_{i=1,\ldots,n}(\overline{A_i}) \leqslant \overline{A^c} \leqslant \max_{i=1,\ldots,n}(\overline{A_i})$. We first prove that $\min_{i=1,\ldots,n}(\overline{A_i}) \geqslant \min_{i=1,\ldots,n}(A_i)$. Using reduction to absurdity, we assume that $\min_{i=1,\ldots,n}(\overline{A_i}) < \min_{i=1,\ldots,n}(A_i)$. Afterward, we assume that $P \subset \{1, 2, \ldots n\}$ and $\overline{A_p} < \min_{i=1,\ldots,n}(A_i)$ for $p \in P$. Let $\bar{\bar{A}} = (\overline{\overline{A_1}}, \ldots, \overline{\overline{A_n}})$, where

$$\overline{\overline{A_i}} = \begin{cases} \min_{i=1,\ldots,n}(A_i), & \text{for i } \in \text{P} \\ \overline{A_i}, & \text{for i } \notin \text{P} \end{cases}.$$

We find that

$$\frac{\#\left(\overline{\overline{A^c}} \cap \max\left(\overline{\overline{A_i}}\right)\right)}{\#\left(\overline{\overline{A^c}} \cup \max\left(\overline{\overline{A_i}}\right)\right)} \geqslant \overline{CL}$$

and

$$\frac{\#\left(\overline{\overline{A^c}} \cap \min_{i=1,\ldots,n}\left(\overline{\overline{A_i}}\right)\right)}{\#\left(\overline{\overline{A^c}} \cup \min_{i=1,\ldots,n}\left(\overline{\overline{A_i}}\right)\right)} \geqslant \overline{CL}.$$

Thus, $\bar{\bar{A}}$ is a feasible solution to P_1. Since

$$\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) - \sum_{i=1}^{n} d\left(A_i, \overline{A_i}\right) = \sum_{i \in P}\left(\left(\#\left(A_i \cup \overline{\overline{A_i}}\right) - \#\left(A_i \cap \overline{\overline{A_i}}\right)\right) - \left(\#\left(A_i \cup \overline{A_i}\right) - \#\left(A_i \cap \overline{A_i}\right)\right)\right) < 0,$$

we have that $\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) < \sum_{i=1}^{n} d(A_i, \overline{A_i})$, which contradicts the fact that $(\overline{A_1}, \ldots, \overline{A_n})$ is the optimal solution of P_1. Similarly, we can prove that $\max_{i=1,\ldots,n}\{\overline{A_i}\} \leqslant \max_{i=1,\ldots,n}\{A_i\}$.

This completes Proof of Property 5.2. □

Proof of Property 5.3. Let $\{\sigma(1), \ldots, \sigma(n)\}$ be a permutation of $\{1, \ldots, n\}$ such that $A_{\sigma(i-1)} \geqslant A_{\sigma(i)}$ for $i = 2, \ldots, n$ and $\{\rho(1), \ldots, \rho(n)\}$ be a permutation of $\{1, \ldots, n\}$ such that $Q_{\sigma(i-1)} \geqslant Q_{\sigma(i)}$ for $i = 2, \ldots, n$.

Since $(Q_1, Q_2, \ldots, Q_n)$ is a permutation of $(A_1, A_2, \ldots, A_n)$, we have $\sigma(i) = \rho(i)$, $i = 1, \ldots, n$. Applying model P_1, we can obtain $\overline{A^c} = \overline{Q^c}$.

This completes Proof of Property 5.3. □

Proof of Property 5.4. Without loss of generality, we first suppose that for $i < j$, $A_i < A_j$ and $A'_i < A'_j$.

When $n = 1$, it is obvious that $\overline{A^c} = \overline{A_1} = A_1 \leqslant A'_1 = \overline{A'_1} = \overline{A'^c}$.

When $n = 2$, using reduction to absurdity, we assume that $\overline{A'_1} < \overline{A_1}$ or $\overline{A'_2} < \overline{A_2}$. Here, we consider three cases:

Case 1: $\overline{A'_1} < \overline{A_1}$ and $\overline{A'_2} \geqslant \overline{A_2}$.

In this case, let $\bar{\bar{A}} = \left(\overline{\overline{A_1}}, \overline{\overline{A_2}}\right)$, where $\overline{\overline{A_1}} = \overline{A'_1}$, $\overline{\overline{A_2}} = \min(\overline{A'_2}, A_2)$, we have that

$$\frac{\#\left(\overline{\overline{A_i}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_i}} \cup \overline{\overline{A^c}}\right)} \geqslant \overline{CL} \text{ for } i = 1, 2$$

and

$$\sum_{i=1}^{2} d\left(A_i, \overline{\overline{A_i}}\right) \leqslant \sum_{i=1}^{2} d(A_i, \overline{A_i}).$$

This contradicts the fact that $(\overline{A_1}, \ldots, \overline{A_n})$ is the optimal adjusted opinion of $(A_1, A_2, \ldots, A_n)$.

Case 2: $\overline{A'_1} \geqslant \overline{A_1}$ and $\overline{A'_2} < \overline{A_2}$.

In this case, let $\bar{\bar{A'}} = \left(\overline{\overline{A'_1}}, \overline{\overline{A'_2}}\right)$, where $\overline{\overline{A'_1}} = \max\left(A'_1, \overline{A_1}\right)$, $\overline{\overline{A'_2}} = \overline{A_2}$, we have that

$$\frac{\#\left(\overline{\overline{A'_i}} \cap \overline{\overline{A'^c}}\right)}{\#\left(\overline{\overline{A'_i}} \cup \overline{\overline{A'^c}}\right)} \geqslant \overline{CL} \text{ for } i = 1, 2$$

and

$$\sum_{i=1}^{2} d\left(A'_i, \overline{\overline{A'_i}}\right) \leqslant \sum_{i=1}^{2} d\left(A'_i, \overline{A'_i}\right).$$

This contradicts the fact that $\left(\overline{A'_1}, \ldots, \overline{A'_n}\right)$ is the optimal adjusted opinion of $(A'_1, \ldots, A'_n)$.

Case 3: $\overline{A'_1} < \overline{A_1}$ and $\overline{A'_2} < \overline{A_2}$.

In this case, $A_1 < A'_1 \leqslant \overline{A'_1} < \overline{A_1}$ and $\overline{A'_2} < \overline{A_2} \leqslant A_2 < A'_2$. Thus, it cannot absolutely guarantee the value of $\sum_{i=1}^{2} d\left(A'_i, \overline{A'_i}\right)$ is smaller than $\sum_{i=1}^{2} d\left(A'_i, \overline{A_i}\right)$, which contradicts the fact that $\left(\overline{A'_1}, \ldots, \overline{A'_n}\right)$ is the optimal adjusted opinion of $(A'_1, A'_2, \ldots, A'_n)$.

Based on the three cases, we have $\overline{A_1} \leqslant \overline{A'_1}$ and $\overline{A_2} \leqslant \overline{A'_2}$. Consequently, $\overline{A^c} \leqslant \overline{A'^c}$.

This completes Proof of Property 5.4. □

Proof of Property 5.5. Because of $\min_{i=1,\ldots,n}(A_i) \leqslant \min_{i=1,\ldots,n}(\overline{A_i})$ and $\max_{i=1,\ldots,n}(\overline{A_i}) \leqslant \max_{i=1,\ldots,n}(A_i)$, then $\min\{\min_{i=1,\ldots,n}(A_i)\} \leqslant \min\left\{\min_{i=1,\ldots,n}(\overline{A_i})\right\}$ and $\max\left\{\max_{i=1,\ldots,n}(\overline{A_i})\right\} \leqslant \max\{\max_{i=1,\ldots,n}(A_i)\}$.

Therefore, $\#\left[\min\left\{\min_{i=1,\ldots,n}(\overline{A_i})\right\}, \max\left\{\max_{i=1,\ldots,n}(\overline{A_i})\right\}\right] \leqslant \#[\min\{\min_{i=1,\ldots,n}(A_i)\}, \max\{\max_{i=1,\ldots,n}(A_i)\}]$.

This completes Proof of Property 5.5. □

Proof of Property 5.6. There are two cases:

(1) If $\overline{A_1} \cap \cdots \cap \overline{A_n} = \varnothing$, then $\overline{A_1} \cap \cdots \cap \overline{A_n} \subseteq \overline{A^c}$.

(2) If $\overline{A_1} \cap \cdots \cap \overline{A_n} \neq \varnothing$, then using reduction to absurdity, we assume that $\overline{A_1} \cap \cdots \cap \overline{A_n} \neg \subset \overline{A^c}$, thus we can construct another HFLTSs of S: $\bar{\bar{A}} = (\overline{\overline{A_1}}, \ldots, \overline{\overline{A_n}})$ and $\overline{\overline{A^c}}$, where $\overline{\overline{A_i}} = \overline{A_i}(i = 1, \ldots, n)$ and $\overline{\overline{A^c}} = (\overline{A_1} \cap \cdots \cap \overline{A_n}) \cup \overline{A^c}$.

Then, we can find that

$$\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) = \sum_{i=1}^{n} d(A_i, \overline{A_i}),$$

and for $i = 1, \ldots, n$,

$$\frac{\#\left(\overline{\overline{A_i}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_i}} \cup \overline{\overline{A^c}}\right)} = \frac{\#\left(\overline{A_i} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{A_i} \cup \overline{\overline{A^c}}\right)} > \frac{\#\left(\overline{A_i} \cap \overline{A^c}\right)}{\#\left(\overline{A_i} \cup \overline{A^c}\right)},$$

which contradicts the fact that $\overline{A^c}$ is the optimal solution to P_1.

Therefore, $\overline{A_1} \cap \cdots \cap \overline{A_n} \subseteq \overline{A^c}$.

This completes Proof of Property 5.6. □

Proof of Lemma 5.1. (1) Sufficiency. For $i \neq p$, it satisfies

$$\frac{\#\left(\overline{\overline{A_i}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_i}} \cup \overline{\overline{A^c}}\right)} = \frac{\#\left(\overline{A_i} \cap \overline{A^c}\right)}{\#\left(\overline{A_i} \cup \overline{A^c}\right)} \geqslant \overline{CL}.$$

For $i = p$, since $\overline{\overline{A^c}} = \overline{A^c}$, $\overline{\overline{A_p}} \subset \overline{A^c}$, $\#\left(\overline{\overline{A_p}}\right) = \#(\overline{A_p})$ and $\overline{A_p} \subset \overline{A^c}$, it satisfies

$$\frac{\#\left(\overline{\overline{A_p}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_p}} \cup \overline{\overline{A^c}}\right)} = \frac{\#\left(\overline{\overline{A_p}}\right)}{\#\left(\overline{A^c}\right)} = \frac{\#\left(\overline{A_p} \cap \overline{A^c}\right)}{\#\left(\overline{A_p} \cup \overline{A^c}\right)} \geqslant \overline{CL}.$$

Furthermore, $\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) = \sum_{i=1, i \neq p}^{n} d(A_i, \overline{A_i}) + d\left(A_p, \overline{\overline{A_p}}\right)$,

since $\#\left(\overline{\overline{A_p}}\right) = \#(\overline{A_p})$ and $\overline{\overline{A_p}} \cap A_p = \varnothing$, then

$$d\left(A_p, \overline{\overline{A_p}}\right) = \#\left(A_p \cup \overline{\overline{A_p}}\right) - \#\left(A_p \cap \overline{\overline{A_p}}\right) = \#(A_p) + \#(\overline{A_p}) = d(A_p, \overline{A_p}),$$

Thus it satisfies $\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) = \sum_{i=1}^{n} d(A_i, \overline{A_i})$.

Therefore, $\bar{\bar{A}} = \left(\overline{\overline{A_1}}, \ldots, \overline{\overline{A_n}}, \overline{\overline{A^c}}\right)$ is the optimal solution to P_1.

(2) Necessity. For $i = p$, Since $\overline{\overline{A^c}} = \overline{A^c}$,

$$\frac{\#\left(\overline{\overline{A_p}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_p}} \cup \overline{\overline{A^c}}\right)} = \frac{\#\left(\overline{\overline{A_p}} \cap \overline{A^c}\right)}{\#\left(\overline{\overline{A_p}} \cup \overline{A^c}\right)}.$$

There are three cases between $\overline{\overline{A_p}}$ and $\overline{A^c}$:

1) If $\overline{\overline{A_p}} \cap \overline{A^c} = \varnothing$, then $\frac{\#\left(\overline{\overline{A_p}} \cap \overline{A^c}\right)}{\#\left(\overline{\overline{A_p}} \cup \overline{A^c}\right)} = 0$. Therefore, $\bar{\bar{A}} = \left(\overline{\overline{A_1}}, \ldots, \overline{\overline{A_n}}, \overline{\overline{A^c}}\right)$ is not an optimal solution to P_1.

2) If $\overline{\overline{A_p}} \cap \overline{A^c} \neq \varnothing$ and $\overline{\overline{A_p}} \neg \subset \overline{A^c}$, since $\overline{A_p} \subset \overline{A^c}$ and $A_p \cap \overline{A^c} = \varnothing$, then

$$d\left(A_p, \overline{\overline{A_p}}\right) \neq d(A_p, \overline{A_p}).$$

As a result, $\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) \neq \sum_{i=1}^{n} d(A_i, \overline{A_i})$.

Therefore, $\bar{\bar{A}} = \left(\overline{\overline{A_1}}, \ldots, \overline{\overline{A_n}}, \overline{\overline{A^c}}\right)$ is not an optimal solution to P_1.

3) If $\overline{\overline{A_p}} \subset \overline{A^c}$, then

$$\frac{\#\left(\overline{\overline{A_p}} \cap \overline{A^c}\right)}{\#\left(\overline{\overline{A_p}} \cup \overline{A^c}\right)} = \frac{\#\left(\overline{\overline{A_p}}\right)}{\#(\overline{A^c})}.$$

When $\#\left(\overline{\overline{A_p}}\right) = \#(\overline{A_p}),\ d\left(A_p, \overline{\overline{A_p}}\right) = d(A_p, \overline{A_p})$, then it guarantees

$$\sum_{i=1}^{n} d\left(A_i, \overline{\overline{A_i}}\right) = \sum_{i=1}^{n} d(A_i, \overline{A_i})$$

and

$$\frac{\#\left(\overline{\overline{A_p}} \cap \overline{\overline{A^c}}\right)}{\#\left(\overline{\overline{A_p}} \cup \overline{\overline{A^c}}\right)} = \frac{\#\left(\overline{A_p} \cap \overline{A^c}\right)}{\#\left(\overline{A_p} \cup \overline{A^c}\right)} \geqslant \overline{CL}.$$

Therefore, if $\overline{\overline{A}} = \left(\overline{\overline{A_1}}, \cdots, \overline{\overline{A_n}}, \overline{\overline{A^c}}\right)$, where $\overline{\overline{A_i}} = \overline{A_i}\ (i \neq p)$ and $\overline{\overline{A^c}} = \overline{A^c}$, is an optimal solution to P_1, then $\overline{\overline{A_p}}$ satisfies $\overline{\overline{A_p}} \subset \overline{A^c}$ and $\#\left(\overline{\overline{A_p}}\right) = \#(\overline{A_p})$.

This completes Proof of Lemma 5.1. □

Proof of Theorem 5.1. Based on Lemma 5.1, we prove the uniqueness of solution to model P_2 as follows:

(1) For $i \neq p,\ \overline{OA_i} = \overline{A_i}$ is the unique optimal solution to P_2.
(2) For $i = p$, $\overline{A_p}$ can be further optimized to $\overline{OA_p}$ by applying model P_2 to minimize the Manhattan distance between A_p and $\overline{OA_p}$.

$$\begin{cases} \min_{\overline{OA_p}} \left(\left| ind(\overline{OA_p}^-) - ind\left(A_p^-\right) \right| + \left| ind(\overline{OA_p}^+) - ind\left(A_p^+\right) \right| \right) \\ s.t. \begin{cases} \left[ind(\overline{OA_p}^-), ind(\overline{OA_p}^+) \right] \subset \left[ind(\overline{A^c}^-), ind(\overline{A^c}^+) \right] \\ ind(\overline{OA_p}^+) - ind(\overline{OA_p}^-) = \#(\overline{A_p}) \end{cases} \end{cases} . \tag{55}$$

According to Lemma 5.2, the above mathematical programming (55) has a unique solution.
Therefore, $\overline{OA} = (\overline{OA_1}, \ldots, \overline{OA_n})$ is the unique optimal solution to P_2.

This completes Proof of Theorem 5.1. □

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