A Consensus Model for Group Decision Making with Incomplete Unbalanced Fuzzy Linguistic Preference Relations

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Abstract— *We present a consensus model for group decision making with unbalanced fuzzy linguistic preference relations, i.e., assuming that the preferences are assessed on linguistic term sets whose terms are not symmetrically and uniformly distributed. This consensus model can manage incomplete information situations, that is, situations where the experts do not give all the preference values that they are usually requested. In addition, both consistency and consensus measures are used and it allows to achieve consistent solutions with a great level of agreement.*

Keywords— Group decision making, consensus, unbalanced fuzzy linguistic preference relations, incomplete information, consistency.

1 Introduction

A Group Decision Making (GDM) problem is usually understood as a decision problem which consists in finding the best alternative(s) from a set of feasible alternatives according to the preferences provided by a group of experts characterized by their experience and knowledge. To do this, experts have to express their preferences by means of a set of evaluations over the set of alternatives. In this contribution, we assume that experts use preference relations [1, 2, 3] in an unbalanced fuzzy linguistic context [4, 5] (see Fig. 1).

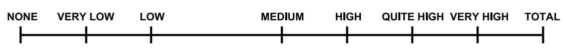

Figure 1: Example of an unbalanced fuzzy linguistic term set.

In these problems, a difficulty that has to be addressed is the lack of information. Since each expert has his/her own experience concerning the problem being studied, there may be cases where an expert would not be able to express the preference degree between two or more of the available alternatives. This may be due to an expert not possessing a precise or sufficient level of knowledge of part of the problem, or because that expert is unable to discriminate the degree to which some options are better than others.

To solve GDM problems, the experts are faced by applying two processes before obtaining a final solution [6, 7]: *the consensus process* and *the selection process*. The consensus process is defined as a dynamic and iterative group discussion process, coordinated by a moderator helping experts to bring their opinions closer. If the consensus level is lower than a specified threshold, the moderator would urge experts to discuss their opinions further in an effort to bring them closer. Otherwise, the moderator would apply the selection process which consists in obtaining the final solution to the problem from the opinions expressed by the experts. Clearly, it is preferable that the experts achieve a great agreement among their opinions before applying the selection process and, therefore, we focus on the consensus process.

The aim of this paper is to present a consensus model to deal with GDM problems in which experts use incomplete unbalanced fuzzy linguistic preference relations (FLPRs) to provide their preferences. We use two kinds of consensus measures to guide the consensus reaching process, *consensus degrees*, which evaluate the agreement of all the experts, and *proximity measures*, which evaluate the agreement between the experts' individual opinions and the group opinion. However, this consensus model will not only be based on consensus measures but also on consistency measures. To compute them, first, all missing values are estimated using an estimation procedure based on the Tanino's consistency principle [3]. Both consistency and consensus measures are used to design a feedback mechanism, and, in such a way, we substitute the actions of the moderator and give advice to the experts on how they should change and complete their opinions to obtain a solution with a high consensus degree (making experts' opinions closer).

The rest of the paper is set out as follows. Section 2 deals with the preliminaries necessary to develop our consensus model. In Section 3, the consensus model for GDM problems with incomplete unbalanced FLPRs is presented. Finally, some concluding remarks are pointed out in Section 4.

2 Preliminaries

2.1 Methodology to Manage Unbalanced Fuzzy Linguistic Information

To manage unbalanced fuzzy linguistic information, we propose a methodology similar to those proposed in [4, 5]. This methodology is based on the transformation of the unbalanced fuzzy linguistic information in a *Linguistic Hierarchy (LH)* [8], which is the linguistic representation domain that allows us to develop comparison and combination processes of unbalanced fuzzy linguistic information.

A LH is a set of levels, where each level represents a linguistic term set with different granularity from the remaining

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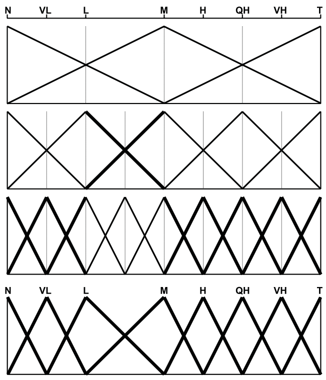

Figure 2: Representation for an unbalanced linguistic fuzzy term set.

levels of the hierarchy. Each level is denoted as $l(t, n(t))$, where t is a number indicating the level of the hierarchy, and $n(t)$ is the granularity of the linguistic term set of t. Then, a LH can be defined as the union of all levels t: $LH = \bigcup_t l(t, n(t))$. Given a LH, we denote as $\mathcal{S}^{n(t)}$ the linguistic term set of LH corresponding to the level t of LH characterized by a cardinality $n(t)$: $\mathcal{S}^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$.

The procedure to represent unbalanced fuzzy linguistic information presents the following steps:

1. Find a level t^- of LH to represent the subset of linguistic terms $\mathcal{S}_{un}^L$ on the left of the mid linguistic term of unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$.
2. Find a level t^+ of LH to represent the subset of linguistic terms $\mathcal{S}_{un}^R$ on the right of the mid linguistic term of $\mathcal{S}_{un}$.
3. Represent the mid term of $\mathcal{S}_{un}$ using the mid terms of the levels t^- and t^+.

If there does not exist a level t^- or t^+ in LH to represent $\mathcal{S}_{un}^L$ or $\mathcal{S}_{un}^R$, respectively, then the procedure applies the following recursive algorithm, which is defined, in this case, assuming that there does not exist t^-, as it happens with the unbalanced fuzzy linguistic term set given in Fig. 1:

1. Represent $\mathcal{S}_{un}^L$:
   (a) Identify the mid term of $\mathcal{S}_{un}^L$, called $\mathcal{S}_{mid}^L$.
   (b) Find a level t_2^- of the left sets of LH^L to represent the left term subset of $\mathcal{S}_{un}^L$, where LH^L represents the left part of LH.
   (c) Find a level t_2^+ of the right sets of LH^L to represent the right term subset of $\mathcal{S}_{un}^L$.
   (d) Represent the mid term $\mathcal{S}_{mid}^L$ using the levels t_2^- and t_2^+.
2. Find a level t^+ of LH to represent the subset of linguistic terms $\mathcal{S}_{un}^R$.
3. Represent the mid term of $\mathcal{S}_{un}$ using the levels t_2^+ and t^+.

For example, applying this algorithm, the representation of the unbalanced fuzzy linguistic term set $\mathcal{S}_{un} = \{N, VL, L, M, H, QH, VH, T\}$ shown in Fig. 1, using a linguistic hierarchy LH, would be as it is shown in Fig. 2.

To operate with the linguistic information in LH, the 2-tuple fuzzy linguistic model [9] is used.

Definition 2.1. Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function $\Delta\colon [0, g] \longrightarrow S \times [-0.5, 0.5)$:

$$(\beta) = (s_i, \alpha), \text{ with } \begin{cases} s_i, & i = round(\beta) \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5), \end{cases} \quad (1)$$

where $round(\cdot)$ is the usual round operation, s_i has the closest index label to "β", and "α" is the value of the symbolic translation. In addition, for all Δ, there exists Δ^{-1}, defined as $\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$.

Finally, transformation functions between labels from different levels to make processes of computing with words in multigranular linguistic information contexts without loss of information were defined in [8].

Definition 2.2. [8] Let $LH = \bigcup_t l(t, n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $\mathcal{S}^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic label in level t to a label in level t' is defined as $TF_{t'}^t : l(t, n(t)) \longrightarrow l(t', n(t'))$ such that

$$TF_{t'}^t(s_i^{n(t)}, \alpha^{n(t)}) = \Delta_{t'}\left(\frac{\Delta_t^{-1}(s_i^{n(t)}, \alpha^{n(t)}) \cdot (n(t') - 1)}{n(t) - 1}\right). \quad (2)$$

2.2 *Incomplete Unbalanced FLPRs*

In this paper, we deal with GDM problems where the experts e_h express their preferences relations $P^h = (p_{ik}^h)$ on the set of alternatives X using an unbalanced linguistic fuzzy term set, $\mathcal{S}_{un} = \{s_0, \ldots, s_{mid}, \ldots, s_g\}$, which has a minimum label, called s_0, a maximum label, called s_g, and the remaining labels are non-uniformly and non-symmetrically distributed around the central one, called s_{mid} (Fig. 1). Therefore, $p_{ik}^h \in \mathcal{S}_{un}$ represents the preference of alternative x_i over alternative x_k for the experts e_h assessed on the unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$.

Definition 2.3. An unbalanced FLPR P^h on a set of alternatives X is characterized by a membership function $\mu_{P^h} : X \times X \longrightarrow \mathcal{S}_{un}$. If it is not possible to give the preference degree for every pair of alternatives, we have an incomplete unbalanced FLPR.

When cardinality of X is small, the preference relation may be conveniently represented by a $n \times n$ matrix $P^h = (p_{ik}^h)$, being $p_{ik}^h = \mu_{P^h}(x_i, x_k), \forall i, k \in \{1, \ldots, n\}$ and $p_{ik}^h \in \mathcal{S}_{un}$.

2.3 *Consistency Measures*

For GDM problems with preference relations, some properties are usually assumed desirable to avoid contradictions within the preferences expressed by the experts, that is, to avoid inconsistent opinions. One of them is the *additive transitivity*,

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$$cp_{ik}^h = TF_t^{t'}(\Delta_{t'}\left(\frac{\sum_{j=1;i\neq k\neq j}^{n}(\Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j1})) + \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j2})) + \Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j3})))}{3(n-2)}\right)). \quad (3)$$

$$cp_{ik}^h = TF_t^{t'}(\Delta_{t'}\left(\frac{\sum_{j\in H_{ik}^{h1}}\Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j1})) + \sum_{j\in H_{ik}^{h2}}\Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j2})) + \sum_{j\in H_{ik}^{h3}}\Delta_{t'}^{-1}(TF_{t'}^t((cp_{ik}^h)^{j3}))}{(\#H_{ik}^{h1} + \#H_{ik}^{h2} + \#H_{ik}^{h3})}\right)). \quad (4)$$

which was defined for fuzzy preference relations [1, 3] as:

$$(p_{ij}^h - 0.5) + (p_{jk}^h - 0.5) = (p_{ik}^h - 0.5),\ \forall i,j,k \in \{1,\ldots,n\}. \quad (5)$$

In the case of an unbalanced fuzzy linguistic context, previously to carry out any computation task, we have to choose a level $t' \in \{t^-, t_2^-, t^+, t_2^+\}$, such that $n(t') = max\{n(t^-), n(t_2^-), n(t^+), n(t_2^+)\}$. Then, once a result is obtained, it is transformed to the correspondent level $t \in \{t^-, t_2^-, t^+, t_2^+\}$ by means of $TF_t^{t'}$ for expressing the result in the unbalanced fuzzy linguistic term set $\mathcal{S}_{un}$. In this way, the unbalanced fuzzy linguistic additive transitivity for unbalanced FLPRs is defined as:

$$\begin{array}{l} TF_t^{t'}(\Delta_{t'}[(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}))) + \\ (\Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid})))]) = \\ TF_t^{t'}(\Delta_{t'}[(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid})))]), \end{array} \quad (6)$$

being $p_{ij}^h = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_{jk}^h = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_{ik}^h = (s_z^{n(t)}, \alpha_3)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, s_{mid} is the mid term of $\mathcal{S}_{un}$ and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

Expression (6) can be rewritten as:

$$\begin{array}{l} p_{ik}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \\ \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))),\ \forall i,j,k \in \{1,\ldots,n\}. \end{array} \quad (7)$$

Equation (7) can be used to calculate an estimated value of a preference degree p_{ik}^h $(i \neq k)$ using an intermediate alternative x_j in three different ways:

1. From $p_{ik}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0))))$, we obtain:

$$\begin{array}{l} (cp_{ik}^h)^{j1} = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) + \\ \Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))). \end{array} \quad (8)$$

2. From $p_{jk}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ji}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0))))$, we obtain:

$$\begin{array}{l} (cp_{ik}^h)^{j2} = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{jk}^h)) - \\ \Delta_{t'}^{-1}(TF_{t'}^t(p_{ji}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))). \end{array} \quad (9)$$

3. From $p_{ij}^h = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(p_{kj}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0))))$, we obtain:

$$\begin{array}{l} (cp_{ik}^h)^{j3} = TF_t^{t'}(\Delta_{t'}(\Delta_{t'}^{-1}(TF_{t'}^t(p_{ij}^h)) - \\ \Delta_{t'}^{-1}(TF_{t'}^t(p_{kj}^h)) + \Delta_{t'}^{-1}(TF_{t'}^t(s_{mid}, 0)))). \end{array} \quad (10)$$

The overall estimated value cp_{ik}^h of p_{ik}^h is obtained as the average of all possible $(cp_{ik}^h)^{j1}$, $(cp_{ik}^h)^{j2}$ and $(cp_{ik}^h)^{j3}$ values as shown in (3).

When the information provided is completely consistent, then $(cp_{ik}^h)^{jl} = p_{ik}^h$, $\forall j, l$. The error between a preference value and its estimated one in $[0,1]$ is defined as follows:

$$\varepsilon p_{ik}^h = \frac{|\Delta_{t'}^{-1}(TF_{t'}^t(cp_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^t(p_{ik}^h))|}{n(t') - 1}. \quad (11)$$

We should point out that some estimated values of an incomplete unbalanced FLPR could lie outside the $\mathcal{S}_{un}$, i.e., we may have $cp_{ik}^h < s_0$ or $cp_{ik}^h > s_g$. In order to normalize the expression domains, the following function is used:

$$f(cp_{ik}^h) = \begin{cases} s_0, & \text{if } cp_{ik}^h < s_0 \\ s_g, & \text{if } cp_{ik}^h > s_g \\ cp_{ik}^h, & \text{otherwise.} \end{cases} \quad (12)$$

Thus, it can be used to define the consistency level between the preference degree p_{ik}^h and the rest of the preference values of the unbalanced FLPR as follows:

$$cl_{ik}^h = 1 - \varepsilon p_{ik}^h. \quad (13)$$

Easily, we can define the consistency measures for particular alternatives and for the whole unbalanced FLPR.

Definition 2.4. The consistency measure, $cl_i^h \in [0,1]$, associated to a particular alternative x_i of an unbalanced FLPR P^h is defined as:

$$cl_i^h = \frac{\sum_{k=1;i\neq k}^{n}(cl_{ik}^h + cl_{ki}^h)}{2(n-1)}. \quad (14)$$

Definition 2.5. The consistency level, $cl^h \in [0,1]$, of an unbalanced FLPR P^h is defined as follows:

$$cl^h = \frac{\sum_{i=1}^{n} cl_i^h}{n}. \quad (15)$$

When working with an incomplete unbalanced FLPR, (3) cannot be used to obtain the estimate of a known preference value. In this case, the following sets can be defined [10]:

$$\begin{array}{l} A = \{(i,j) \mid i,j \in \{1,\ldots,n\} \wedge i \neq j\} \\ MV^h = \{(i,j) \in A \mid p_{ij}^h \text{ is unknown}\} \\ EV^h = A \setminus MV^h \\ H_{ik}^{h1} = \{j \neq i,k \mid (i,j),(j,k) \in EV^h\} \\ H_{ik}^{h2} = \{j \neq i,k \mid (j,i),(j,k) \in EV^h\} \\ H_{ik}^{h3} = \{j \neq i,k \mid (i,j),(k,j) \in EV^h\} \\ EV_i^h = \{(a,b) \mid (a,b) \in EV^h \wedge (a = i \vee b = i)\}, \end{array} \quad (16)$$

Then, the estimated value of a particular preference degree p_{ik}^h $((i,k) \in EV^h)$ can be calculated as shown in (4) assuming $(\#H_{ik}^{h1} + \#H_{ik}^{h2} + \#H_{ik}^{h3}) \neq 0$.

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function estimate_p(h,i,k)
1) $(cp^h_{ik})^1 = (s_0, 0),\ (cp^h_{ik})^2 = (s_0, 0),\ (cp^h_{ik})^3 = (s_0, 0)$
2) if $\#H^{h1}_{ik} \neq 0$, then $(cp^h_{ik})^1 = TF^{t'}_t(\Delta_{t'}(\sum_{j \in H^{h1}_{ik}} \Delta^{-1}_{t'}(TF^t_{t'}((cp^h_{ik})^{j1}))))$
3) if $\#H^{h2}_{ik} \neq 0$, then $(cp^h_{ik})^2 = TF^{t'}_t(\Delta_{t'}(\sum_{j \in H^{h2}_{ik}} \Delta^{-1}_{t'}(TF^t_{t'}((cp^h_{ik})^{j2}))))$
4) if $\#H^{h3}_{ik} \neq 0$, then $(cp^h_{ik})^3 = TF^{t'}_t(\Delta_{t'}(\sum_{j \in H^{h3}_{ik}} \Delta^{-1}_{t'}(TF^t_{t'}((cp^h_{ik})^{j3}))))$
5) Calculate $cp^h_{ik} = TF^{t'}_t(\Delta_{t'}\left(\frac{\Delta^{-1}_{t'}(TF^t_{t'}((cp^h_{ik})^1)) + \Delta^{-1}_{t'}(TF^t_{t'}((cp^h_{ik})^2)) + \Delta^{-1}_{t'}(TF^t_{t'}((cp^h_{ik})^3))}{(\#H^{h1}_{ik} + \#H^{h2}_{ik} + \#H^{h3}_{ik})}\right))$
end function

2.4 Estimation Procedure of Missing Values

The procedure estimates missing information in an expert's incomplete unbalanced FLPR using only the preference values provided by that particular expert. It is designed using (4) and estimates missing information values by means of two different tasks:

1. **Elements to be estimated in step *t* of the procedure:**

$$EMV^h_t = \{(i,k) \in MV^h \setminus \bigcup_{l=0}^{t-1} EMV^h_l \mid i \neq k \wedge \exists j \in \{H^{h1}_{ik} \cup H^{h2}_{ik} \cup H^{h3}_{ik}\}\}, \tag{17}$$

   and $EMV^h_0 = \emptyset$ (by definition).

2. **Estimation of a particular missing value:** In order to estimate a particular value, the function $estimate\_p(h,i,k)$ at the top of the page is used.

3 Consensus model

In this section, we present a consensus model for GDM problems where experts provide their preferences using incomplete unbalanced FLPRs. To solve GDM problems with this kind of preference relations, firstly, it is necessary to deal with the missing values [10]. The previous consistency based procedure of missing values allows us to measure the consistency levels of each expert. This consistency information is used in this section to propose a consensus model based not only on consensus criteria but also on consistency criteria. We consider that both criteria are important to guide the consensus process in an incomplete decision framework. In such a way, we get that experts change their opinions toward agreement positions in a consistent way, which is desirable to achieve a consistent and consensus solution. The main characteristics of the proposed consensus model are the following:

- It is designed to guide the consensus process of incomplete unbalanced fuzzy linguistic GDM problems.
- It uses a consistency based procedure to calculate the incomplete unbalanced fuzzy linguistic information.
- It is based both consensus criteria and consistency criteria. The proposed consensus model is designed with the aim of obtaining the maximum possible consensus level while trying to achieve a high level of consistency in experts' preferences.
- A feedback mechanism is defined using the above criteria. It substitutes the moderator's actions, avoiding the possible subjectivity that he/she can introduce, and gives advice to the experts to find out the changes they need to make in their opinions to obtain a solution with certain consensus and consistency degrees simultaneously.

In particular, the consensus model develops its activity in five phases that will be described in further detail in the following subsections: 1) computing missing information, 2) computing consistency measures, 3) computing consensus measures, 4) controlling the consistency/consensus state, and 5) feedback mechanism.

3.1 Computing Missing Information

In this first step, for each incomplete unbalanced FLPR P^h, we obtain its corresponding complete unbalanced FLPR $\bar{P}^h$ using the estimation procedure described in Section 2.4.

3.2 Computing Consistency Measures

To compute consistency measures, first, for each $\bar{P}^h$, we compute its corresponding unbalanced FLPR $CP^h = (cp^h_{ik})$ according to expression (3). Second, we apply (13)-(15) to $(\bar{P}^h, CP^h)$ $(\forall h)$ to compute the consistency measures $CL^h = (cl^h_{ik}), cl^h_i, cl^h, \forall i,k \in \{1,\ldots,n\}$. Finally, we define a global consistency measure among all experts to control the global consistency situation as follows:

$$CL = \frac{\sum_{h=1}^m cl^h}{m}. \tag{18}$$

3.3 Computing Consensus Measures

As in [6, 10], we compute two different kinds of measures: consensus degrees and proximity measures. Consensus degrees are used to measure the actual level of consensus in the process, while the proximity measures give information about how close to the collective solution every expert is. These measures are given on three different levels for a preference relation: pairs of alternatives, alternatives and relation. It will allow us to find out the consensus state of the process at different levels. For example, we will be able to identify which experts are close to the consensus solution, or in which alternatives the experts are having more trouble to reach consensus.

3.3.1 Consensus Degrees

For each pair of experts (e_h, e_l) $(h = 1,\ldots,m-1,\ l = h+1,\ldots,m)$, a similarity matrix, $SM^{hl} = (sm^{hl}_{ik})$, is defined, where

$$sm^{hl}_{ik} = 1 - \frac{|\Delta^{-1}_{t'}(TF^t_{t'}(\bar{p}^h_{ik})) - \Delta^{-1}_{t'}(TF^t_{t'}(\bar{p}^l_{ik}))|}{n(t') - 1}, \tag{19}$$

being $\bar{p}^h_{ik} = (s^{n(t)}_v, \alpha_1),\ t \in \{t^-, t^-_2, t^+, t^+_2\},\ \bar{p}^l_{ik} = (s^{n(t)}_w, \alpha_2), t \in \{t^-, t^-_2, t^+, t^+_2\}$, and $t' \in \{t^-, t^-_2, t^+, t^+_2\}$.

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Then, a consensus matrix, $CM = (cm_{ik})$, is calculated by aggregating all the similarity matrices using the arithmetic mean as the aggregation function ϕ:

$$cm_{ik} = \phi(sm_{ik}^{hl},\ h = 1, \ldots, m-1,\ l = h+1, \ldots, m). \quad (20)$$

Once the consensus matrix, CM, is computed, we proceed to calculate the consensus degrees at the three different levels:

1. **Level 1.** *Consensus degree on pairs of alternatives,* cop_{ik}. It measures the consensus degree amongst all the experts on the pair of alternatives (x_i, x_k).

$$cop_{ik} = cm_{ik};\ \forall\, i, k = 1, \ldots, n \wedge i \neq k. \quad (21)$$

2. **Level 2.** *Consensus degree on alternatives,* ca_i. It measures the consensus degree amongst all the experts on the alternative x_i.

$$ca_i = \frac{\sum_{k=1;k\neq i}^{n} (cop_{ik} + cop_{ki})}{2(n-1)}. \quad (22)$$

3. **Level 3.** *Consensus degree on the relation,* cr. It measures the global consensus degree amongst all the experts and is used to control the consensus situation.

$$cr = \frac{\sum_{l=1}^{n} ca_i}{n}. \quad (23)$$

3.3.2 Proximity Measures

These measures evaluate the agreement between the individual experts' opinions and the group opinion. To compute them for each expert, we need to obtain the collective unbalanced FLPR, $P^c = (p_{ik}^c)$, which summarizes preferences given by all the experts and is calculated by means of the aggregation of the set of individual unbalanced FLPRs $\{\bar{P}^1, \ldots, \bar{P}^m\}$. In this way, to obtain P^c we use the unbalanced fuzzy linguistic version of an IOWA operator [11, 12], which uses both consensus and consistency criteria as inducing variable. Thus, we obtain each collective unbalanced fuzzy linguistic preference degree p_{ik}^c according to the most consistent and consensual individual unbalanced fuzzy linguistic preference degrees.

Thus, to obtain each p_{ik}^c according to the most consistent and consensual individual unbalanced fuzzy linguistic preference degrees, we propose to use an unbalanced fuzzy linguistic IOWA operator with the consistency/consensus values, $\{z_{ik}^1, z_{ik}^2, \ldots, z_{ik}^m\}$, as the values of the order inducing variable, i.e.,

$$\begin{array}{l} p_{ik}^c = \Phi_W(\langle z_{ik}^1, \bar{p}_{ik}^1\rangle, \ldots, \langle z_{ik}^m, \bar{p}_{ik}^m\rangle) = \\ TF_t^{t'}(\Delta_{t'}(\sum_{h=1}^{m} w_h \cdot \Delta_{t'}^{-1}(TF_{t'}^{t}(\bar{p}_{ik}^{\sigma(h)})))), \end{array} \quad (24)$$

where

- σ is a permutation of $\{1, \ldots, m\}$ such that $z_{ik}^{\sigma(h)} \geq z_{ik}^{\sigma(h+1)}$, $\forall h = 1, \ldots, m-1$, i.e., $\langle z_{ik}^{\sigma(h)}, \bar{p}_{ik}^{\sigma(h)}\rangle$ is the 2-tuple with $z_{ik}^{\sigma(h)}$ the h-th highest value in the set $\{z_{ik}^1 \ldots, z_{ik}^m\}$;
- the weighting vector is computed according to the following expression:

$$w_h = Q\left(\frac{\sum_{j=1}^{h} z_{ik}^{\sigma(j)}}{T}\right) - Q\left(\frac{\sum_{j=1}^{h-1} z_{ik}^{\sigma(j)}}{T}\right), \quad (25)$$

  with $T = \sum_{j=1}^{m} z_{ik}^j$;
- and the set of values of the inducing variable $\{z_{ik}^1, \ldots, z_{ik}^m\}$ are computed as follows:

$$z_{ik}^h = (1-\delta) \cdot cl_{ik}^h + \delta \cdot co_{ik}^h, \quad (26)$$

  being co_{ik}^h the consensus measure for the preference value $\bar{p}_{ik}^h$ and $\delta \in [0, 1]$ a parameter to control the weight of both consistency and consensus criteria in the inducing variable. Usually $\delta > 0.5$ will be used to give more importance to the consensus criterion. We should note that in our framework, each value co_{ik}^h used to calculate $\{z_{ik}^1, \ldots, z_{ik}^m\}$ is defined as follows:

$$co_{ik}^h = \frac{\sum_{l=h+1}^{n} sm_{ik}^{hl} + \sum_{l=1}^{h-1} sm_{ik}^{lh}}{n-1}. \quad (27)$$

Once we have computed P^c, we can compute the proximity measures in each level of an unbalanced FLPR.

1. **Level 1.** *Proximity measure on pairs of alternatives,* pp_{ik}^h. The proximity measure of an expert e_h on a pair of alternatives (x_i, x_k) to the group's one is calculated as:

$$pp_{ik}^h = 1 - \frac{\left|\Delta_{t'}^{-1}(TF_{t'}^{t}(\bar{p}_{ik}^h)) - \Delta_{t'}^{-1}(TF_{t'}^{t}(p_{ik}^c))\right|}{n(t')-1}. \quad (28)$$

   being $\bar{p}_{ik}^h = (s_v^{n(t)}, \alpha_1)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, $p_{ik}^c = (s_w^{n(t)}, \alpha_2)$, $t \in \{t^-, t_2^-, t^+, t_2^+\}$, and $t' \in \{t^-, t_2^-, t^+, t_2^+\}$.

2. **Level 2.** *Proximity measure on alternatives,* pa_i^h. The proximity measure of an expert e_h on an alternative x_i to the group's one is calculated as follows:

$$pa_i^h = \frac{\sum_{k=1;k\neq i}^{n} (pp_{ik}^h + pp_{ki}^h)}{2(n-1)}. \quad (29)$$

3. **Level 3.** *Proximity measure on the relation,* pr^h. The proximity measure of an expert e_h on his/her unbalanced FLPR to the group's one is calculated as follows:

$$pr^h = \frac{\sum_{l=1}^{n} pa_i^h}{n}. \quad (30)$$

3.4 Controlling Consistency/Consensus State

The consistency/consensus state control process will be used to decide when the feedback mechanism should be applied to give advice to the experts or when the consensus reaching process has to come to an end. It should take into account both the consensus and consistency measures. To do that, we use a measure or level of satisfaction, called *consistenty/consensus level* (CCL) [10], which is used as a control parameter:

$$CCL = (1-\delta) \cdot CL + \delta \cdot cr, \quad (31)$$

with δ the same value used in [7]. When CCL satisfies a minimum threshold value $\gamma \in [0, 1]$, the consensus reaching process finishes and the selection process can be applied. To avoid that the consensus process does not converge, a maximum number of consensus rounds is incorporated.

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3.5 Feedback Mechanism

The feedback mechanism generates personalized advice to the experts according to the consistency and consensus criteria. It helps experts to change their preferences and to complete their missing values. This activity is carried out in two steps:

1. **Identification of the preference values.** We must identify preference values that are contributing less to reach a high consensus/consistency state. To do that, we define set APS that contains 3-tuples (h, i, k) symbolizing preference degrees p^h_{ik} that should be changed because they affect badly to that consistency/consensus state.

   (a) *Identification of experts.* We identify the set of experts EXPCH that should receive advice on how to change some of their preference values.

   $$EXCPCH = \{h \mid (1-\delta) \cdot cl^h + \delta \cdot pr^h < \gamma\}. \quad (32)$$

   (b) *Identification of alternatives.* We identify the alternatives ALT that the above experts should consider to change.

   $$ALT = \{(h, i) \mid h \in EXPCH \wedge (1-\delta) \cdot cl^h_i + \delta \cdot pa^h_i < \gamma\}. \quad (33)$$

   (c) *Identification of pairs of alternatives.* Finally, we identify preference values for every alternative and expert $(x_i; e_h \mid (h, i) \in ALT)$ that should be changed according to their proximity and consistency measures on the pairs of alternatives, i.e.,

   $$APS = \{(h, i, k) \mid (h, i) \in ALT \wedge (1-\delta) \cdot cl^h_{ik} + \delta \cdot pp^h_{ik} < \gamma\}. \quad (34)$$

   Additionally, the feedback process must provide rules for missing preference values. To do so, it has to take into account in APS all missing values that were not provided by the experts, i.e.,

   $$APS' = APS \cup \{(h, i, k) \mid p^h_{ik} \in MV_h\}. \quad (35)$$

2. **Generation of advice.** In this step, the feedback mechanism generates personalized recommendations to help the experts to change their preferences. These recommendations are based on easy recommendation rules that will not only tell the experts which preference values they should change, but will also provide them with particular values for each preference to reach a higher consistency/consensus state.

   The new preference degree of alternatives x_i over alternative x_k to recommend to the expert e_h, rp^h_{ik}, is calculated as the following weighted average of the preference value cp^h_{ik} and the collective preference value p^c_{ik}:

   $$rp^h_{ik} = TF^{t'}_t(\Delta_{t'}((1-\delta) \cdot \Delta^{-1}_{t'}(TF^t_{t'}(cp^h_{ik})) + \delta \cdot \Delta^{-1}_{t'}(TF^t_{t'}(p^c_{ik})))). \quad (36)$$

   As previously mentioned, with $\delta > 0.5$, the consensus model leads the experts towards a consensus solution rather than towards an increase on their own consistency levels.

   Finally, we should distinguish two cases:

   (a) $\forall(h, i, k) \in APS'$, if $p^h_{ik} \in EV_h$, the recommendation generated for the expert e_h is: "You should change your preference value (i, k) to a value close to rp^h_{ik}".

   (b) $\forall(h, i, k) \in APS'$, if $p^h_{ik} \in MV_h$, the recommendation generated for the expert e_h is: "You should provide a value for (i, k) close to rp^h_{ik}".

4 Concluding Remarks

In this paper, we have proposed a model of consensus for GDM with incomplete unbalanced fuzzy linguistic information. It uses two different kinds of measures to guide the consensus reaching process, consistency and consensus measures, and generates advice to experts in a discriminate way. As a consequence, this model allows us to achieve consistent and consensus solutions. In addition, it supports the consensus process automatically, without moderator.

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