Consistency of Reciprocal Preference Relations

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***Abstract*— The consistency of reciprocal preference relations is studied. Consistency is related with rationality, which is associated with the *transitivity property*. For fuzzy preference relations many properties have been suggested to model transitivity and, consequently, consistency may be measured according to which of these different properties is required to be satisfied. However, we will show that many of them are not appropriate for reciprocal preference relations. We put forward a functional equation to model consistency of reciprocal preference relations, and show that self-dual uninorms operators are the solutions to it. In particular, Tanino's multiplicative transitivity property being an example of such type of uninorms seems to be an appropriate consistency property for fuzzy reciprocal preferences.**

I. Introduction

Preference relations are usually assumed to model experts' preferences in decision making problems [6]. In the classical preference modelling, given two alternatives, an expert judges them in one of the following ways:

(i) one alternative is preferred to another;
(ii) the two alternatives are indifferent to him/her;
(iii) he/she is unable to compare them.

According to these cases, three binary relations can be defined:

(i) the strict preference relation P: $(x, y) \in P$ if and only if the expert prefers x to y ($x \succ y$);
(ii) the indifference relation I: $(x, y) \in I$ if and only if the expert is indifferent between x and y ($x \sim y$);
(iii) the incomparability relation J: $(x, y) \in J$ if and only if the expert unable to compare x and y.

Fishburn in [6] defines indifference as the absence of strict preference. He also points out that indifference might arise in three different ways:

(a) when an expert truly feels that there is no real difference, in a preference sense, between the alternatives;
(b) when the expert is uncertain as to his/her preference between the alternatives because 'he might find their comparison difficult and may decline to commit himself[/herself] to a strict preference judgement while not being sure that he[/she] regards [them] equally desirable (or undesirable)';
(c) or when both alternatives are considered incomparable on a preference basis by the expert.

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It is obvious from the third case that Fishburn treats the incomparability relation as an indifference relation, i.e., J is empty (there is no incomparability). Asymmetry is considered by Fishburn [6] as an "obvious" condition for preferences: if an expert prefers x to y, then he[/she] should not simultaneously prefers y to x.

Using a numerical representation of preferences on a set of alternatives X, we have [5]:

$$r_{ij} = 1 \quad \Leftrightarrow \quad x_i \succ x_j$$
$$r_{ij} = 0 \quad \Leftrightarrow \quad x_j \succ x_i$$

Clearly, this can be extended by adding the indifference case:

$$r_{ij} = 0.5 \quad \Leftrightarrow \quad x_i \sim x_j$$

However, given three alternatives x_i, x_j, x_k such that x_i is preferred to x_j and x_j to x_k, the question whether the "degree or strength of preference" of x_i over x_j exceeds, equals, or is less than the "degree or strength of preference" of x_j over x_k cannot be answered by the classical preference modelling. The implementation of the degree of preference between alternatives may be essential in many situations, and this can be modelled using fuzzy preference relations [2].

A fuzzy preference relation R on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function

$$\mu_R : X \times X \longrightarrow [0, 1].$$

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $R = (r_{ij})$ being $r_{ij} = \mu_R(x_i, x_j) \ \ \forall i, j \in \{1, \ldots, n\}$.

In this approach, given two alternatives an experts provides

(i) a value in the range $(0.5, 1]$ to quantify the "degree or strength of preference" of an alternative when preferred to another;
(ii) the value 0.5 when the two alternatives are indifferent to him/her;
(iii) no value when he/she is uncertain as to his/her preference between the alternatives or he/she is unable to compare them [9].

The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time which facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception of the alternatives and, as a consequence, the provided preferences could not be rational. Usually, rationality is related with consistency, which is associated with the *transitivity property* [10]. Transitivity seems like a reasonable criterion of coherence for an individual's preferences: if x is preferred to y and y is preferred to z, common sense suggests that x should be preferred to z.

Many properties have been suggested to model transitivity of a fuzzy preference relation and, consequently, consistency may be measured according to which of these different properties is required to be satisfied. One of these properties is the additive transitivity property, which is equivalent to Saaty's consistency property for multiplicative preference relations [10]. However, as we will show this consistency property is in conflict with the corresponding scale used for providing the preference values. In order to overcome this conflict, a set of conditions will be put forward for a fuzzy preference relation to be considered 'fully consistent.' Under this set of conditions we show that consistency of preferences should be modelled using uninorm operators. In particular, Tanino's multiplicative transitivity property [22], being an example of such type of uninorms, seems to be an appropriate consistency property for fuzzy reciprocal preferences.

The rest of the paper is set out as follows. Preliminaries on consistency of preferences are provided in Section II. In Section III, a set of conditions for a fuzzy preference relation to be considered 'fully consistent' will be established. Self-dual uninorms operators are shown to be the solutions to this set of conditions in Section IV. Finally, conclusions are drawn in Section V.

II. Consistency of Preferences

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [12]:

- The first level of rationality requires indifference between any alternative x_i and itself.
- The second one requires that if an expert prefers x_i to x_j, that expert should not simultaneously prefer x_j to x_i. This asymmetry condition is viewed as an "obvious" condition/criterion of consistency for preferences [6]. This rationality condition is modelled by the property of reciprocity in the pairwise comparison between any two alternatives, which is seen by Saaty as basic in making paired comparisons [19].
- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

This hierarchical structure also requires for a particular level of rationality to be compatible with the upper ones: the third level of rationality should imply or be compatible with the second level of rationality, and this with the first one. This necessary compatibility between the rationality assumptions could be used as a criterion for considering a particular condition modelling any one of the rationality levels as adequate or inadequate.

A preference relation verifying the third level of rationality is usually called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [20].

In a crisp context, where an expert provides his/her opinion on the set of alternatives X by means of a binary preference relation, R, the concept of consistency has traditionally been defined in terms of acyclicity [21], i.e. the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ with $x_j R x_{j+1} \forall j = 1, \ldots, k$. Clearly, this condition as said before is closely related to the transitivity of the binary relation and its corresponding binary indifference relation.

In a fuzzy context, the traditional requirement to characterise consistency has followed the way of extending the classical requirements of binary preference relations. Thus, consistency is also based on the notion of transitivity, in the sense that if alternative x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k, which is normally referred in this context as *weak transitivity*. However, the main difference in this case with respect to the classical one is that consistency has been modelled in many different ways due to the role the intensity of preference has [7], [10], [13]–[15], [19], [22], [24]. Indeed, many properties or conditions have been suggested as rational ones for a fuzzy preference relation to be considered a consistent one. Among these properties we can cite :

- Max-min transitivity [4], [24]:
  $r_{ik} \geq \min\{r_{ij}, r_{jk}\}$
- Restricted max-min transitivity [22]:
  $\min\{r_{ij}, r_{jk}\} \geq 0.5 \Rightarrow r_{ik} \geq \min\{r_{ij}, r_{jk}\}$
- Max-max transitivity [4], [24]:
  $r_{ik} \geq \max\{r_{ij}, r_{jk}\}$
- Restricted max-max transitivity [22]:
  $\min\{r_{ij}, r_{jk}\} \geq 0.5 \Rightarrow r_{ik} \geq \max\{r_{ij}, r_{jk}\}$
- Multiplicative transitivity [22]:
  $\frac{r_{ji}}{r_{ij}} \cdot \frac{r_{kj}}{r_{jk}} = \frac{r_{ki}}{r_{ik}}$
- Additive transitivity [22]:
  $(r_{ij} - 0.5) + (r_{jk} - 0.5) = r_{ik} - 0.5$

We note that these conditions are stronger than weak transitivity, and therefore a fuzzy preference relation might be transitive but not consistent (see [10], [17], [22]).

As aforementioned, the value 0.5 is usually used to model the first level of rationality in the case fuzzy preference relations, and therefore we have

$$r_{ii} = 0.5 \ \forall i. \qquad (1)$$

The second level of rationality of fuzzy preferences is modelled using the following reciprocity property

$$r_{ij} + r_{ji} = 1 \ \forall i, j. \qquad (2)$$

Clearly, reciprocity property implies indifference, and therefore both properties are compatible.

Max-max transitivity cannot be verified under reciprocity. Indeed, if $R = (r_{ij})$ is reciprocal and verifies max-max transitivity, then:

$$\begin{aligned} \forall i, j, k: \quad r_{ki} &= 1 - r_{ik} \\ &\leq 1 - \max\{r_{ij}, r_{jk}\} \\ &= \min\{1 - r_{ij}, 1 - r_{jk}\} \\ &= \min\{r_{ji}, r_{kj}\} \end{aligned}$$

From max-max transitivity we have that:

$$\forall i, j, k : r_{ki} \geq \max\{r_{kj}, r_{ji}\}$$

and therefore we have that max-max transitivity and reciprocity can be verified only when

$$\forall i, j, k : r_{ik} = r_{ij} = r_{jk} = 0.5.$$

Max-min transitivity and reciprocity imply:

$$\begin{array}{rcl} \forall i, j, k : \quad r_{ik} & = & 1 - r_{ki} \\ & \leq & 1 - \min\{r_{kj}, r_{ji}\} \\ & = & \max\{1 - r_{kj}, 1 - r_{ji}\} \\ & = & \max\{r_{jk}, r_{ij}\} \end{array}$$

Therefore, max-min transitivity under reciprocity can be rewritten as

$$\forall i, j, k : \min\{r_{ij}, r_{jk}\} \leq r_{ik} \leq \max\{r_{ij}, r_{jk}\}.$$

The restricted versions of max-max and max-min do not imply reciprocity. For example, the following fuzzy preference relation

$$R = \begin{pmatrix} 0.5 & 0.6 & 0.8 \\ 0.4 & 0.5 & 0.7 \\ 0.1 & 0.3 & 0.5 \end{pmatrix}$$

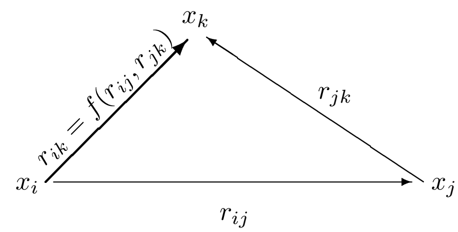
verifies both restricted transitivity properties but it is not reciprocal. This does not imply that they are incompatible with the reciprocity property. In fact, a fuzzy preference relation can be reciprocal and still verify both restricted transitivity properties, as the one we would have obtained by changing the values r_{13} for 0.9 or the value r_{31} for 0.2. The same applies to max-min transitivity and multiplicative transitivity.

Additive transitivity implies both reciprocity and indifference and we might conclude that it is the most adequate property among the above list to model consistency of fuzzy preferences. However, additive transitivity is in conflict with scale used for providing the preference values. Indeed, if we have a set of three alternatives $\{x_1, x_2, x_3\}$ with $r_{12} = 0.75$ and $r_{23} = 0.9$, additive transitivity leads to the meaningless negative value $r_{13} = 1.5 - r_{12} - r_{23} = -0.15$. This could be avoided by using the set of real numbers as the range of possible preference values, which assumes that the human judgement is capable of comparing the relative dominance of any two objects [19]. But if this is to be assumed then there would not be any reason for the scale $[0, 1]$ to be changed and therefore additive transitivity might not be considered the most suitable condition to model consistency of fuzzy preference relations.

III. Consistency Function of Reciprocal Preferences

The assumption of experts being able to quantify their preferences in the domain [0,1] instead of $\{0, 1\}$ or a set with finite cardinality, as it may be a set of linguistic labels [11], [18], underlies unlimited computational abilities and resources from the experts. Taking these unlimited computational abilities and resources into account we may formulate that an expert's preferences are consistent when for any three alternatives x_i, x_j, x_k their preference values are related in the 'exact' form

$$r_{ik} = f(r_{ij}, r_{jk}) \quad \forall i, j, k \tag{3}$$



being f a function $f\colon [0, 1] \times [0, 1] \rightarrow [0, 1]$.

In practical cases expression (3) might obviously not be verified even when the preference values of a preference relation are transitive, i.e., they comply with the weak transitivity property. However, the assumption of modelling consistency using the expression (3) can be used to introduce levels of consistency, which in group decision making situations could be exploited by assigning a relative importance weight to each one of the experts in arriving to a collective preference opinion. Also, expression (3) can be used as a principle for deriving missing values. Indeed, using just those preference values provided by an expert, expression (3) could be used to estimate those preference values which were not given by that expert because he/she was uncertain as to his/her preference between the alternatives or he/she is unable to compare them. By doing this, we assure that the estimated values are 'compatible' with the rest of the information provided by that expert.

In what follows we will set out a set of conditions or properties to be verified by such a function f.

The first condition to impose to function f is that it must be monotonic (increasing), i.e. if any of the preference values r_{ij}, r_{jk} increases while the other remains fixed then the preference value r_{ik} will not decrease.

Property 1 (Monotonicity):

$$f(x, y) \geq f(x', y') \text{ if } x \geq x' \text{and } y \geq y' \tag{4}$$

Equation (3) implies that

$$f(r_{ij}, r_{jk}) = f(r_{il}, r_{lk}) \quad \forall i, j, k, l.$$

On the other hand we have $r_{ij} = f(r_{il}, r_{lj})$ and $r_{lk} = f(r_{lj}, r_{jk})$. Putting these expressions together we have

$$f(f(r_{il}, r_{lj}), r_{jk}) = f(r_{il}, f(r_{lj}, r_{jk})) \quad \forall i, j, k, l.$$

Thus function f must be associative.

Property 2 (Associativity):

$$f(f(x, y), z) = f(x, f(y, z)) \quad \forall x, y, z \in [0, 1] \tag{5}$$

The application of equation (3) and the assumed reciprocity property of preferences give

$$\begin{array}{rclcl} \forall i, j, k : \quad r_{ki} & = & f(r_{kj}, r_{ji}) & = & f(1 - r_{jk}, 1 - r_{ij}) \\ r_{ki} & = & 1 - r_{ik} & = & 1 - f(r_{ij}, r_{jk}) \end{array}$$

and hence

$$f(1 - r_{jk}, 1 - r_{ij}) = 1 - f(r_{ij}, r_{jk}) \ \ \forall i, j, k.$$

Property 3 (Reciprocity):

$$f(x, y) + f(1 - y, 1 - x) = 1 \ \ \forall x, y \in [0, 1] \qquad (6)$$

Making $y = 1 - x$ and $x = y = 0.5$ in property 3 we have respectively:

Property 4 (Indifference):

$$f(x, 1 - x) = 0.5 \ \ \forall x \in [0, 1] \qquad (7)$$

Property 5 (Transitivity of Indifference):

$$f(0.5, 0.5) = 0.5 \qquad (8)$$

From properties 2 and 4 we derive the following:

$$\begin{aligned} \forall i, k: \quad f(0.5, r_{ik}) &= f(f(r_{ik}, 1 - r_{ik}), r_{ij}) \\ &= f(r_{ik}, f(1 - r_{ik}, r_{ik})) \\ &= f(r_{ik}, 0.5) \end{aligned}$$

From equation (3) and property 2 we have that

$$\begin{aligned} \forall i, k: \quad r_{ik} &= f(r_{ij}, r_{jk}) \\ &= f(r_{ij}, f(r_{ji}, r_{ik})) \\ &= f(f(r_{ij}, r_{ji}), r_{ik}). \end{aligned}$$

By property 4, $f(r_{ij}, r_{ji}) = 0.5$ which reduces the previous expression to

$$r_{ik} = f(0.5, r_{ik}) \ \forall i, k.$$

Therefore, we have that 0.5 must be the identity element of function f.

Property 6 (Identity element):

$$f(0.5, x) = f(x, 0.5) = x \ \ \forall x \in [0, 1] \qquad (9)$$

The following result can be easily proved from properties 1 and 6:

Proposition 1:

- $\min\{r_{ij},\ r_{jk}\} \geq 0.5 \Rightarrow f(r_{ij}, r_{jk}) \geq \max\{r_{ij},\ r_{jk}\}$
- $\max\{r_{ij},\ r_{jk}\} \leq 0.5 \Rightarrow f(r_{ij}, r_{jk}) \leq \min\{r_{ij},\ r_{jk}\}$
- $r_{ij} \leq 0.5 \leq r_{jk} \Rightarrow r_{ij} \leq f(r_{ij}, r_{jk}) \leq r_{jk}$

This result means that a reciprocal preference relation that verifies expression (3) also verifies restricted max-min and restricted max-max transitivity properties. Clearly, this result rules out the property max-min transitivity as a candidate for modelling the consistency of reciprocal preference relations.

From proposition 1 we derive the following two results:

Corollary 1:

$$\begin{aligned} x \geq 0.5 \Rightarrow & f(x, 1) = f(1, x) = 1 \\ x \leq 0.5 \Rightarrow & f(x, 0) = f(0, x) = 0 \end{aligned}$$

Corollary 2: $f(0, 0) = 0 \ \wedge \ f(1, 1) = 1$

A problem arises when $(x, y) \in \{(0, 1), (1, 0)\}$. Indeed, on the one hand, by property 3 we would have that $f(0, 1) = f(1, 0) = 0.5$. On the other hand, properties 2, 6 and corollary 1 imply

$$\begin{aligned} x \geq 0.5 \Rightarrow x =& f(0.5, x) = f(f(0, 1), x) \\ =& f(0, f(1, x)) = f(0, 1) = 0.5 \\ x \leq 0.5 \Rightarrow x =& f(x, 0.5) = f(x, f(0, 1)) \\ =& f(f(x, 0), 1) = f(0, 1) = 0.5 \end{aligned}$$

Thus, the value $f(0, 1) = f(1, 0) = 0.5$ implies that $0.5 = x \ \forall x \in [0, 1]$. Therefore, properties 3 and 4 must be true for $(x, y) \in [0, 1]^2 \backslash \{(0, 1), (1, 0)\}$.

If $f(0, 1)$ ($f(1, 0)$) exists then

$$f(0, 1) = f(0, f(1, x)) = f(f(0, 1), x) \ \forall x \geq 0.5$$

$$f(0, 1) = f(f(x, 0), 1) = f(x, f(0, 1)) \ \forall x \geq 0.5$$

There are two alternative cases to the value 0.5. If $f(0, 1) > 0.5$ then $f(0, 1) = f(f(0, 1), 1) = 1$, while if $f(0, 1) < 0.5$ then $f(0, 1) = f(0, f(0, 1)) = 0$. Therefore, we have that in all cases:

Proposition 2: $f(0, 1), f(1, 0) \in \{0, 1\}$

Another desirable property to be verified by function f should be that of continuity as it is expected that a slight change of the values in (r_{ij}, r_{jk}) should produce a slight change in the value r_{ik}. Continuity is not possible to be achieved in $(0, 1)$ nor in $(1, 0)$. Indeed, the following is true

$$\lim_{x \to 0} f(x, 1 - x) \neq f(0, 1) \ \wedge \ \lim_{x \to 0} f(1 - x, x) \neq f(1, 0).$$

To conclude this section of properties of function f, we note that if there exist alternatives x_j, x_k and x_l such that

$$f(r_{ij}, r_{jk}) = f(r_{ij}, r_{jl}) \ \ \forall i$$

then applying properties 6, 4 and 2 we have that

$$\begin{aligned} r_{jk} =& f(0.5, r_{jk}) = f(f(r_{ji}, r_{ij}), r_{jk}) \\ =& f(r_{ji}, f(r_{ij}, r_{jk})) = f(r_{ji}, f(r_{ij}, r_{jl})) \\ =& f(f(r_{ji}, r_{ij}), r_{jl}) = f(0.5, r_{jl}) \\ =& r_{jl} \end{aligned}$$

Obviously, when $f(r_{kj}, r_{ji}) = f(r_{lj}, r_{ji}) \ \ \forall i$ then we also obtain $r_{kj} = r_{lj}$.

This property is usually known with the name of "cancellative." Due to the problems with the definition of function f when $(x, y) \in \{(0, 1), (1, 0)\}$, we have that:

Property 7 (Cancellative):

$$f(x, y) = f(x, z) \ \forall x \in]0, 1[\Rightarrow y = z$$

$$f(y, x) = f(z, x) \ \forall x \in]0, 1[\Rightarrow y = z$$

Summarising, a solution to the functional equation (3) for reciprocal preference values is any function $f\colon [0, 1] \times [0, 1] \longrightarrow [0, 1]$ with the following properties:

- f is continuous, monotonic increasing, associative, cancellative and reciprocal in $[0, 1]^2 \backslash \{(0, 1), (1, 0)\}$.
- $f(0, 1), f(1, 0) \in \{0, 1\}$.
- f has 0.5 as its identity element.

IV. Uninorms and Consistency of Reciprocal Preferences

Uniorms were introduced by Yager and Rybalov in 1996 [23] as a generalisation of the t-norm and t-conorm. Uninorms share the properties commutativity, associativity and monotonicity with t-norms and t-conorms. It is the boundary condition or identity element the one that is used to generalise t-norms and t-conorms. The identity element of t-norms is the number 1, while for t-conorms the identity element is 0. Uninorms can have an identity element lying anywhere in the unit interval $[0, 1]$.

Clearly, function f in the previous section share all properties of a uninorm except perhaps commutativity, which cannot be directly derived from the above set of properties. However, commutativity of f can be derived indirectly from associativity, cancellativity and continuity of f. Indeed, the following result was proved by Aczél in [1]:

Theorem 1: Let I be a (closed, open, half-open, finite or infinite) proper interval of real numbers. Then $F\colon I^2 \longrightarrow I$ is a continuous operation on I^2 which satisfies the associativity equation

$$F(F(x, y), z) = F(x, F(y, z)) \forall x, y, z \in I$$

and is cancellative, that is,

$$F(x_1, y) = F(x_2, y) \text{ or } F(y, x_1) = F(y, x_2) \quad \text{implies } x_1 = x_2 \text{ for any } z \in I$$

if, and only if, there exists a continuous and strictly monotonic function $\phi\colon J \longrightarrow I$ such that

$$F(x, y) = \phi\left[\phi^{-1}(x) + \phi^{-1}(y)\right] \;\; \forall x, y \in I \tag{10}$$

Here J is one of the real intervals

$$]-\infty, \gamma], \;\;]-\infty, \gamma[, \;\; [\delta, \infty[, \;\;]\delta, \infty[, \text{ or }]-\infty, \infty[\tag{11}$$

for some $\gamma \leq 0 \leq \delta$. Accordingly I has to be open at least from one side.

The function in (10) is unique up to a linear transformation of the variable ($\phi(x)$ may be replaced by $\phi(Cx)$, $C \neq 0$ but by no other function.)

We note that although function F in theorem 1 was not assumed to be commutative, the result (10) shows that it is. Also, function F is strictly monotonic as a result of Aczél theorem. Therefore, the assumption of modelling consistency of reciprocal preferences in $[0, 1]$ using the functional expression (3) has as solution f a uninorm with identity element 0.5 which is strictly increasing.

Fodor, Yager and Rybalov in [8] provide a representation theorem for almost continuous uninorms U, i.e. uninorms with identity element in $]0, 1[$ continuous on $[0, 1]^2 \backslash \{(0, 1), (1, 0)\}$. This representation theorem coincides with (10), with generator function $\phi^{-1}\colon [0, 1] \longrightarrow [-\infty, \infty]$ such that $h(0) = -\infty$, $h(1) = \infty$. Furthermore, such a uninorm must be self-dual with respect a strong negation N with fixed point e, i.e.

$$U(N(x), N(y)) = N(U(x, y))$$
$$N(e) = e$$

Indifference and reciprocity of preferences in $[0, 1]$ is based on the use of the strong negation $N(x) = 1 - x$. Thus, the solutions to the functional equation (3) for reciprocal preference values are self-dual uninorms with respect to $N(x) = 1 - x$.

Interestingly, multiplicative transitivity property

$$\frac{r_{ji}}{r_{ij}} \cdot \frac{r_{kj}}{r_{jk}} = \frac{r_{ki}}{r_{ik}} \quad \forall i, j, k.$$

introduced by Tanino when $r_{ij} > 0 \;\; \forall i, j$, can be expressed, under the assumption of reciprocity, as

$$\begin{aligned}
\forall i,j,k:\quad & r_{ij} \cdot r_{jk} \cdot (1 - r_{ik}) = r_{ik} \cdot r_{kj} \cdot r_{ji} && \Leftrightarrow \\
& r_{ij} \cdot r_{jk} - r_{ij} \cdot r_{jk} \cdot r_{ik} = r_{ik} \cdot r_{kj} \cdot r_{ji} && \Leftrightarrow \\
& r_{ik} \cdot r_{kj} \cdot r_{ji} + r_{ij} \cdot r_{jk} \cdot r_{ik} = r_{ij} \cdot r_{jk} && \Leftrightarrow \\
& r_{ik} \cdot (r_{kj} \cdot r_{ji} + r_{ij} \cdot r_{jk}) = r_{ij} \cdot r_{jk} && \Leftrightarrow \\
& r_{ik} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + r_{ji} \cdot r_{jk}} && \Leftrightarrow \\
& r_{ik} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + (1 - r_{ij}) \cdot (1 - r_{jk})}
\end{aligned}$$

This type of transitivity has been studied by De Baets et al. in [3] under the name of 'isostochastic transitivity'. Clearly, multiplicative transitivity is the restriction to $[0, 1]^2 \backslash \{(0, 1), (1, 0)\}$ of the well known andlike uninorm

$$U(x, y) = \begin{cases} 0, & (x, y) \in \{(0, 1), (1, 0)\} \\ \dfrac{xy}{xy + (1 - x)(1 - y)}, & \text{otherwise} \end{cases} \tag{12}$$

This 'multiplicative' uninorm is self-dual with respect to the negator operator $N(x) = 1 - x$ and has the generator function $\phi^{-1}(x) = \ln \frac{x}{1-x}$ [16]. The behaviour of uninorms on the squares $[0, 0.5] \times [0, 0.5]$ and $[0.5, 1] \times [0.5, 1]$ is closely related to t-norms and as t-conorms [8]. For the above multiplicative uninorm (12), we have that

$$U(x, y) = \frac{T_U(2x, 2y)}{2} \;\; \forall x, y \in [0, 0.5]$$

with

$$T_U(x, y) = \frac{xy}{2 - (x + y - xy)} \;\; \forall x, y \in [0, 1]$$

being the well known Einstein product.

On the evidence obtained so far, we conclude that from the many properties or conditions suggested as rational ones for a fuzzy preference relation to be considered a consistent one, Tanino's multiplicative transitivity property is the most appropriate for the case of reciprocal fuzzy preference relations.

V. Conclusions

Rationality is related with consistency, which is associated with the *transitivity property*. For fuzzy preference relations many properties have been suggested to model transitivity. However, it has been shown that also many of them are not appropriate as they are in conflict with the corresponding scale used for providing the preferences or because they are incompatible with the reciprocity and indifference properties, which are seen as basic in making paired comparisons. In this paper, we have proved that under a set of conditions consistency of preferences is to be modelled using uninorm operators. In particular, for reciprocal fuzzy preference relations we have that consistency should be modelled by Tanino's multiplicative transitivity property. Using this consistency property, in the future we will address the issue of measuring the level of consistency of a reciprocal fuzzy preference relation, which could be very useful in the aggregation processes of GDM problems. Also, the estimation of the preference values of those pairs of alternatives an expert is unable to compare will be addressed by designing a multiplicative consistency based estimation procedure.

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