A Fuzzy Linguistic Methodology to Deal With Unbalanced Linguistic Term Sets

Francisco Herrera, Enrique Herrera-Viedma, and Luis Martínez

***Abstract*—Many real problems dealing with qualitative aspects use linguistic approaches to assess such aspects. In most of these problems, a uniform and symmetrical distribution of the linguistic term sets for linguistic modeling is assumed. However, there exist problems whose assessments need to be represented by means of unbalanced linguistic term sets, i.e., using term sets that are not uniformly and symmetrically distributed. The use of linguistic variables implies processes of *computing with words* (CW). Different computational approaches can be found in the literature to accomplish those processes. The 2-tuple fuzzy linguistic representation introduces a computational model that allows the possibility of dealing with linguistic terms in a precise way whenever the linguistic term set is uniformly and symmetrically distributed. In this paper, we present a fuzzy linguistic methodology in order to deal with unbalanced linguistic term sets. To do so, we first develop a representation model for unbalanced linguistic information that uses the concept of linguistic hierarchy as representation basis and afterwards an unbalanced linguistic computational model that uses the 2-tuple fuzzy linguistic computational model to accomplish processes of CW with unbalanced term sets in a precise way and without loss of information.**

***Index Terms*—Computing with words, linguistic aggregation, linguistic variables, unbalanced linguistic term sets.**

I. Introduction

When we face problems, depending on their aspects, we can deal with different types of information. Usually, the problems present quantitative aspects that can be assessed by means of precise numerical values. In other cases, the problems present qualitative aspects that are complex to assess by means of precise and exact values. The fuzzy linguistic approach [54]–[56] deals with qualitative aspects that are represented in qualitative terms by means of linguistic variables, providing an important tool for solving problems in different areas such as information retrieval [7], [8], [26]–[28], [35], [36], [57], services evaluation and human resources management [4], [9]–[12], [14], [40], [42], Web quality [30], [31], safety applications [37], [41], decision-making [1], [2], [13], [17], [21], [32], [34], [39], [49]–[51], aggregation operators [18], [43], [46], [48], [52], and consensus reaching [3], [6], [29].

When a problem is solved using linguistic information, it implies the need for computing with words (CW). Three linguistic computational models can be found in the specialized literature: i) the semantic model [5], [16], ii) the symbolic model [20], and iii) the model based on linguistic 2-tuples [22]. The model based on the 2-tuple has shown itself better than the other ones, due to the fact that it is able to accomplish processes of CW in a precise way besides other advantages presented in [24].

Manuscript received January 11, 2007; revised February 28, 2007. This work was supported by Research Projects TIN2006-02121 and SAINFOWEB-602.

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Digital Object Identifier 10.1109/TFUZZ.2007.896353

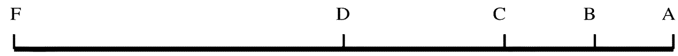

Fig. 1. Grading system evaluations.

Fig. 2. Scale with more values on the right of the midterm.

Most of the problems modeling information with linguistic assessments use linguistic variables assessed in linguistic term sets whose terms are uniformly and symmetrically distributed. However, there exist problems that need to assess their variables with linguistic term sets that are not uniformly and symmetrically distributed [16], [38], [45], [47], [48]. We shall call this type of linguistic term sets *unbalanced linguistic term sets*. In some cases, the unbalanced linguistic information appears as a consequence of the nature of the linguistic variables that participate in the problem as it happens, for example, in the grading system (Fig. 1). In others, it appears in problems dealing with scales for assessing preferences where the experts need to assess a number of terms in a side of reference domain higher than in the other one (Fig. 2).

The aim of this paper is to develop a methodology to represent, manage, and accomplish processes of CW with unbalanced linguistic term sets without loss of information. First, we define an unbalanced linguistic representation model that assigns semantics to the linguistic terms. Therefore, we outline a process to assign semantics to the linguistic terms belonging to an unbalanced linguistic term set. Then, these ideas are formalized by means of a semantic representation algorithm that represents each term by means of a parametric membership function that is assigned using a *linguistic hierarchy* structure [25]. Secondly, we present a computational model for unbalanced linguistic term sets based on the fuzzy linguistic 2-tuple [22] to accomplish the processes of CW without loss of information.

This paper is structured as follows. Section II introduces a linguistic background revising in short the fuzzy linguistic approach, the 2-tuple fuzzy linguistic representation model, and linguistic hierarchical contexts. Section III establishes the basic ideas for representing unbalanced linguistic term sets using linguistic hierarchies. Section IV presents an unbalanced linguistic representation model. Section V proposes a computational model to operate with unbalanced linguistic term sets without loss of information. Section VI shows an application as an illustrative example for dealing with unbalanced linguistic information. Lastly, some concluding remarks are pointed out.

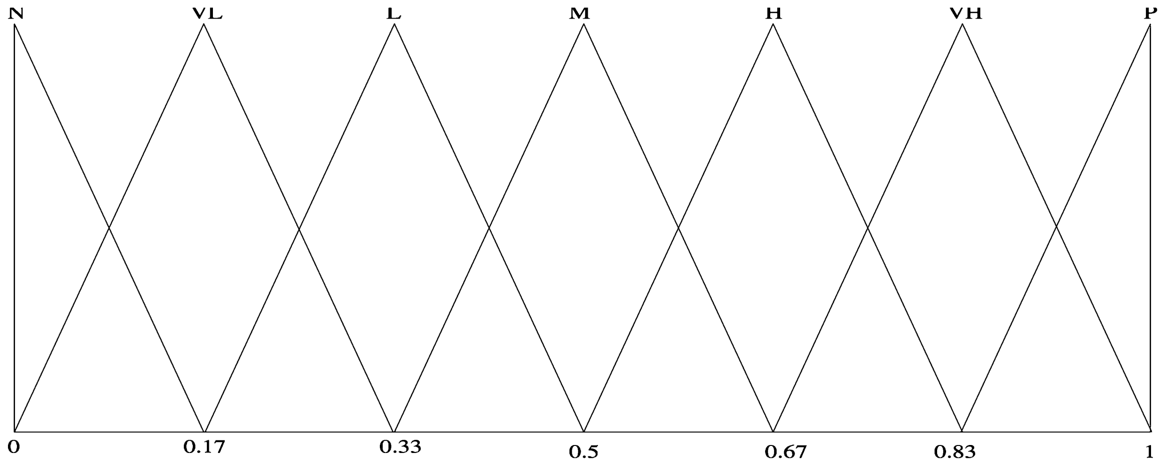

Fig. 3. A set of seven terms with its semantics.

II. PRELIMINARIES

In this section, we make a review of the fuzzy linguistic approach of the 2-tuple fuzzy linguistic representation model and its computational method. Afterwards, we review the concept of linguistic hierarchies.

A. Fuzzy Linguistic Approach

Many aspects of different activities in the real world cannot be assessed in a quantitative form but rather in a qualitative way, i.e., with vague or imprecise knowledge. In such a case, a better approach may be the use of linguistic assessments instead of numerical ones. The fuzzy linguistic approach represents qualitative aspects as linguistic values by means of linguistic variables [54]–[56].

In any fuzzy linguistic approach, we have to choose the appropriate linguistic descriptors for the term set and their semantics. Also an important parameter to be determined is the "granularity of uncertainty," i.e., the cardinality of the linguistic term set used to express the information.

One possibility of generating a linguistic term set $S = \{s_0, \ldots, s_g\}$ consists in directly supplying the term set by considering all the terms distributed on a scale where a total order is defined [53]. For example, a set of seven terms S could be $S = \{s_0 : N(\text{None}), s_1 : \text{VL}(\text{Very Low}), s_2 : L(\text{Low}), s_3 : M(\text{Medium}), s_4 : H(\text{High}), s_5 : \text{VH}(\text{Very High}), s_6 : P(\text{Perfect})\}$. Usually, in these cases, it is required that in S there exist the following.

1) A negation operator $\text{Neg}(s_i) = s_j$ such that $j = g - \text{i}$ ($\text{g} + 1$ is the cardinality of S).
2) An order $s_i \leq s_j \Longleftrightarrow i \leq j$. Therefore, there exist two linguistic comparison operators, the min and max operators.

The semantics of terms is given by fuzzy numbers defined in the [0,1] interval, which are usually described by membership functions. We consider triangular membership functions whose representation is achieved by 3-tuples (a^i, b^i, c^i), where b^i indicates the point in which the membership value is one, with a^i and c^i indicating the left and right limits of the definition domain of the membership function associated with s_i [5]. An example may be $\{\text{P} = (.83, 1, 1), \text{VH} = (.67, .83, 1), \text{H} = (.5, .67, .83), \text{M} = (.33, .5, .67), \text{L} = (.17, .33, .5), \text{VL} = (0, .17, .33), \text{N} = (0, 0, .17)\}$, which is graphically shown in Fig. 3.

Remark 1: We shall denote the upside of s_i (a^i, b^i) as $\overline{s_i}$ and the downside of s_i (b^i, c^i) as $\underline{s_i}$.

In [5], the use of term sets with an odd cardinal was studied, the midterm representing an assessment of "approximately 0,5" with the rest of the terms being placed symmetrically around it and the limit of granularity being 11 or no more than 13. This type of term sets has been widely used in decision making, evaluation processes, information retrieval, etc.

Remark 2: We must notice that, in this paper, we propose to deal with linguistic term sets in which there still exists a similar midterm but the rest of the terms are not placed symmetrically around it. This midterm will be called *central label* throughout this paper.

B. The 2-Tuple Fuzzy Linguistic Representation Model

The 2-tuple fuzzy linguistic representation model was introduced in [22] to improve several aspects of the fuzzy linguistic approach and its different computational models, as can be viewed in [24]. This model represents the linguistic information by means of a pair of values (s, α), where s is a linguistic label and α is a numerical value that represents the value of the symbolic translation.

Definition 1 [22]: Let $\beta \in [0, g]$ be a number of the interval of granularity of the linguistic term set $S = \{s_0, \ldots, s_g\}$ and let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [-.5, .5)$. Then α is called a *symbolic translation*, with *round* being the usual *rounding* operation.

This linguistic representation model defines a set of functions with the purpose of making transformations between linguistic 2-tuples and numerical values.

Definition 2 [22]: Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ a value supporting the result of a symbolic aggregation operation. Then the linguistic 2-tuple that expresses the equivalent information to β is obtained with the function $\Delta : [0, g] \longrightarrow S \times [-0.5, 0.5)$, such that

$$\Delta(\beta) = \begin{cases} s_i & i = \text{round}(\beta) \\ \alpha = \beta - i & \alpha \in [-.5, .5) \end{cases} \quad (1)$$

where s_i has the closest index label to β and α is the value of the symbolic translation.

Proposition 1 [22]: Let $S = \{s_0, \ldots, s_g\}$ be a linguistic term set and (s_i, α) be a linguistic 2-tuple. There is always a function

Δ^{-1} such that, from a linguistic 2-tuple, it returns its equivalent numerical value $\beta \in [0, g] \subset \mathcal{R}$.

Remark 3: From Definitions 1 and 2 and Proposition 1, it is obvious that the conversion of a linguistic term into a linguistic 2-tuple consists in adding a value zero as symbolic translation: $s_i \in S \Longrightarrow (s_i, 0)$.

A computational model has been developed for the 2-tuple fuzzy linguistic representation model, in which there exist the following.

1) *A 2-tuple comparison operator:* The comparison of linguistic information represented by linguistic 2-tuples is carried out according to an ordinary lexicographic order. Let (s_k, α_1) and (s_l, α_2) be two 2-tuples. Then:
   - if $k < l$ then (s_k, α_1) is smaller than (s_l, α_2);
   - if $k = l$ then
     a) if $\alpha_1 = \alpha_2$ then (s_k, α_1), (s_l, α_2) represents the same information;
     b) if $\alpha_1 < \alpha_2$ then (s_k, α_1) is smaller than (s_l, α_2);
     c) if $\alpha_1 > \alpha_2$ then (s_k, α_1) is bigger than (s_l, α_2).
2) *A 2-tuple negation operator*

$$\mathrm{Neg}((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha))). \tag{2}$$

3) *A wide range of 2-tuple aggregation operators has been developed:* Extending classical aggregation operators, such as the linguistic ordered weighted aggregation (LOWA) operator [20], the weighted average operator, the OWA operator, etc. (see [22]).

C. Linguistic Hierarchies

The concept of *linguistic hierarchies* was introduced in [15] to design *hierarchical systems of linguistic rules*. The linguistic hierarchical structure was used in [25] to improve precision in processes of CW in the multigranular linguistic information contexts [11], [13], [19], [33]. In this paper, we use them to manage unbalanced linguistic information.

A *linguistic hierarchy* is a set of levels where each level is a linguistic term set with different granularity from the remaining levels of the hierarchy. Each level belonging to a linguistic hierarchy is denoted as $\mathbf{l(t, n(t))}$, with t being a number that indicates the level of the hierarchy and $n(t)$ the granularity of the linguistic term set of t.

In the definition of linguistic hierarchies we consider linguistic terms whose membership functions are triangular shaped, uniformly and symmetrically distributed in [0,1]. In addition, the linguistic term sets have an odd value of granularity, the central label in a preference modeling framework representing the value of *indifference*.

The levels belonging to a linguistic hierarchy are ordered according to their granularity, i.e., for two consecutive levels t and $t+1$, $n(t+1) > n(t)$. This provides a linguistic refinement of the previous level.

Based on the above concepts, we define a linguistic hierarchy (LH) as the union of all levels t : $\mathrm{LH} = \bigcup_t l(t, n(t))$. To build a linguistic hierarchy, we must keep in mind that the hierarchical order is given by the increase of the granularity of the linguistic term sets in each level.

Let $S^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$ be the linguistic term set defined in the level t with $n(t)$ terms. The building of an LH must satisfy the following *linguistic hierarchy basic rules* [25].

TABLE I
LINGUISTIC HIERARCHIES

	Level 1	*Level* 2	*Level* 3	*Level* 4
$l(t, n(t))$	$l(1, 3)$	$l(2, 5)$	$l(3, 9)$	$l(4, 17)$
$l(t, n(t))$	$l(1, 7)$	$l(2, 13)$		

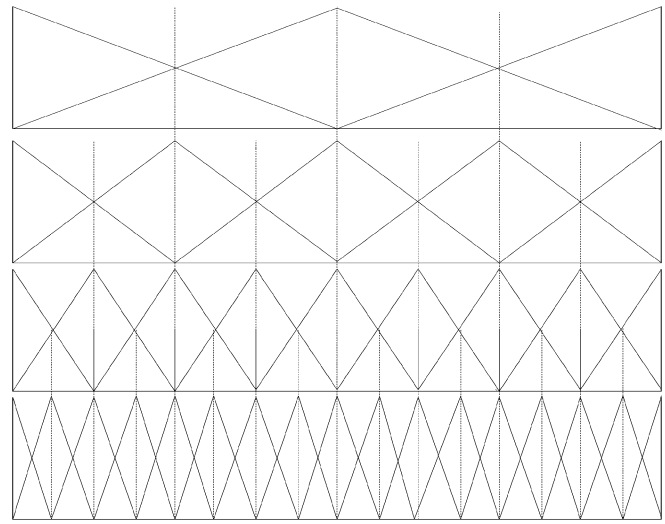

Fig. 4. Linguistic hierarchy of 3, 5, 9, and 17 labels.

1) To preserve all *former modal points* of the membership functions of each linguistic term from one level to the following one.
2) To make *smooth transitions between successive levels*. The aim is to build a new linguistic term set $S^{n(t+1)}$. A new linguistic term will be added between each pair of terms belonging to the term set of the previous level t. To carry out this insertion, we reduce the support of the linguistic labels in order to keep place for the new one located in the middle of them. A detailed description can be seen in [25].

Generally, we can say that the linguistic term set of the level $t+1$, $S^{n(t+1)}$, is obtained from its predecessor $S^{n(t)}$ as

$$l(t, n(t)) \rightarrow l(t+1, 2 \cdot n(t) - 1). \tag{3}$$

Table I shows the granularity needed in each linguistic term set of the level t depending on the value $n(t)$ defined in the first level (three and seven, respectively). A graphical example of a linguistic hierarchy is shown in Fig. 4.

In [25], we defined transformation functions between labels from different levels to make processes of CW in multigranular linguistic information contexts without loss of information.

Definition 3 [25]: Let $\mathrm{LH} = \bigcup_t l(t, n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $S^{n(t)} = \{s_0^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic label in level t to a label in consecutive level $t+c$, with $c \in \{-1, 1\}$, is defined as $\mathrm{TF}_{t+c}^t : l(t, n(t)) \longrightarrow l(t+c, n(t+c))$ such that

$$\mathrm{TF}_{t+c}^t\left(s_i^{n(t)}, \alpha^{n(t)}\right) = \Delta_{(t+c)}\left(\frac{\Delta_t^{-1}\left(s_i^{n(t)}, \alpha^{n(t)}\right) \cdot (n(t+c) - 1)}{n(t) - 1}\right). \tag{4}$$

This transformation function was recursively generalized to transform linguistic terms between any linguistic level in the linguistic hierarchy [25]. Afterwards, it has been defined in a nonrecursive way, i.e., $\mathrm{TF}_{t'}^{t} : l(t, n(t)) \longrightarrow l(t', n(t'))$, such that

$$\mathrm{TF}_{t'}^{t}\left(s_i^{n(t)}, \alpha^{n(t)}\right) = \Delta_{t'}\left(\frac{\Delta_t^{-1}\left(s_i^{n(t)}, \alpha^{n(t)}\right) \cdot (n(t') - 1)}{n(t) - 1}\right). \quad (5)$$

Proposition 2 [25]: The transformation function between linguistic terms in different levels of the linguistic hierarchy is bijective

$$\mathrm{TF}_{t}^{t'}\left(\mathrm{TF}_{t'}^{t}\left(s_i^{n(t)}, \alpha^{n(t)}\right)\right) = \left(s_i^{n(t)}, \alpha^{n(t)}\right). \quad (6)$$

This result guarantees that transformations between levels of a linguistic hierarchy are carried out without loss of information.

III. Basic Ideas for Representing Unbalanced Linguistic Information

The first step to manage unbalanced linguistic information similar to Figs. 1 and 2 using the fuzzy linguistic approach is to obtain a semantic representation, due to the fact that our aim is to manage and operate with these terms without loss of information. In this section, we introduce the basic ideas to represent, by means of a fuzzy membership function, the semantics for each term of the unbalanced term set using the linguistic hierarchy structure.

We consider an unbalanced linguistic term set $\mathcal{S}$ that has a minimum label, a maximum label, and a central label, and the remaining labels are nonuniformly and nonsymmetrically distributed around the central one (see Remark 2) on both left and right lateral sets. Consequently, to manage this type of information, we propose to divide the unbalanced linguistic term set $\mathcal{S}$ into three term subsets, i.e., $\mathcal{S} = \mathcal{S}_L \bigcup \mathcal{S}_C \bigcup \mathcal{S}_R$.

- *Left lateral set* $\mathcal{S}_L$ contains all the labels but the central label.
- *Central set* $\mathcal{S}_C$ just contains the central label.
- *Right lateral set* $\mathcal{S}_R$ contains all the labels higher than the central label.

For example, these subsets for the unbalanced linguistic term set of Fig. 1 are $\mathcal{S}_L = \{F\}$, $\mathcal{S}_C = \{D\}$, and $\mathcal{S}_R = \{C, B, A\}$.

We want to represent the labels of an unbalanced linguistic term set $\mathcal{S}$ through the levels of a linguistic hierarchy LH$= \bigcup_t l(t, n(t))$. To do so, we analyze how to represent the three term subsets $\mathcal{S}_L$, $\mathcal{S}_C$, and $\mathcal{S}_R$. We distinguish the following two possibilities.

A. Representation Using One Level of the Linguistic Hierarchy

To represent the terms of $\mathcal{S}_R(\mathcal{S}_L)$, we observe whether the following condition is satisfied:

$$\exists t \in \mathrm{LH}, \frac{n(t) - 1}{2} = \#(\mathcal{S}_R), \quad \text{or,} \quad \frac{n(t) - 1}{2} = \#(\mathcal{S}_L) \quad (7)$$

with $\#(\mathcal{S}_R)$, $\#(\mathcal{S}_L)$ being the cardinality of $\mathcal{S}_R$ and $\mathcal{S}_L$, respectively.

When the condition shown in (7) is satisfied, i.e., there exists one level t in the LH whose granularity of the subset is the same granularity as the $\mathcal{S}$ lateral subset, then the basic representation procedure of the labels of the lateral subset $\mathcal{S}_R(\mathcal{S}_L)$ is the following.

1) To assign the labels from $S_R^{n(t)}(S_L^{n(t)})$ to $\mathcal{S}_R(\mathcal{S}_L)$, i.e., $\mathcal{S}_R \leftarrow S_R^{n(t)}(\mathcal{S}_L \leftarrow S_L^{n(t)})$.
2) The central subset $\mathcal{S}_C = \{s_C\}$ is assigned depending on the lateral set represented—$\mathcal{S}_R$ or $\mathcal{S}_L$. When we are dealing with the lateral set $\mathcal{S}_R$, the semantics assigned to s_C will be the downside of the central label $s_C^{n(t)} \in S^{n(t)}$, i.e., $\underline{s_C} \leftarrow \underline{s_C^{n(t)}}$, while if we are dealing with the lateral set $\mathcal{S}_L$, the semantics assigned to s_C will be the upside, i.e., $\overline{s_C} \leftarrow \overline{s_C^{n(t)}}$.

B. Representation Using Two Levels

If the condition shown in (7) is not satisfied, the representation of $\mathcal{S}_R(\mathcal{S}_L)$ depends on the distribution of $\mathcal{S}$. In such a case, we describe the distribution of $\mathcal{S}$ by means of a set of five values

$$\{(\#(\mathcal{S}_L), \mathrm{density}_{\mathcal{S}_L}), \#(\mathcal{S}_C), (\#(\mathcal{S}_R), \mathrm{density}_{\mathcal{S}_R})\} \quad (8)$$

with $\mathrm{density}_{\mathcal{S}_L}$ and $\mathrm{density}_{\mathcal{S}_R}$ being symbolic variables assessed in the set $\{\mathrm{middle}, \mathrm{extreme}\}$, which indicates whether the higher granularity of the right(left) lateral set of $\mathcal{S}$ is concentrated near the central label or near the maximum(minimum) label. This description for the grading system term set $\mathcal{S} = \{F, D, C, B, A\}$ (see Fig. 1) is $\{(1, \mathrm{extreme}), 1, (3, \mathrm{extreme})\}$. Assuming this description of $\mathcal{S}$, the procedure to represent the lateral set $\mathcal{S}_R(\mathcal{S}_L)$ is:

a) selecting hierarchical levels in order to assign the semantics;
b) representation process of the lateral set;
c) representation of the central set.

Remark 4: To simplify the explanation, we focus just on $\mathcal{S}_R$, although the procedure is symmetrically analogous for $\mathcal{S}_L$.

1) Selecting Hierarchical Levels to Assign the Semantics: Given that (7) is not satisfied, then we look for two levels t and $t{+}1$ in LH, such that

$$\frac{n(t) - 1}{2} < \#(\mathcal{S}_R) < \frac{n(t+1) - 1}{2}. \quad (9)$$

Then the terms of $\mathcal{S}_R$ will be represented by means of the right lateral subsets of levels t and $t{+}1$ called assignable sets and noted as $\mathrm{AS}_R^{n(t)}$ and $\mathrm{AS}_R^{n(t+1)}$, respectively.

Remark 5: We propose a semantic construction model using two levels of the linguistic hierarchy, even though the number of levels to model the semantics of an unbalanced linguistic term set could be greater. That implies, however, a greater complexity in the semantic construction model, and the results would not suffer a remarkable enhancement.

The above assignable label sets contain the semantics that can be assigned to the terms of $\mathcal{S}_R$ and they will vary along the representation process, due to the fact the same label from the assignable sets cannot be assigned twice. Initially, these assignable sets are composed of $\mathrm{AS}_R^{n(t)} = S_R^{n(t)} = \{s_{((n(t)-1)/2)+1}^{n(t)}, \ldots, s_{n(t)-1}^{n(t)}\}$, and $\mathrm{AS}_R^{n(t+1)} = S_R^{n(t+1)} = \{s_{((n(t+1)-1)/2)+1}^{n(t+1)}, \ldots, s_{n(t+1)-1}^{n(t+1)}\}$. In the representation process, the cardinalities of $\mathrm{AS}_R^{n(t)}$ and

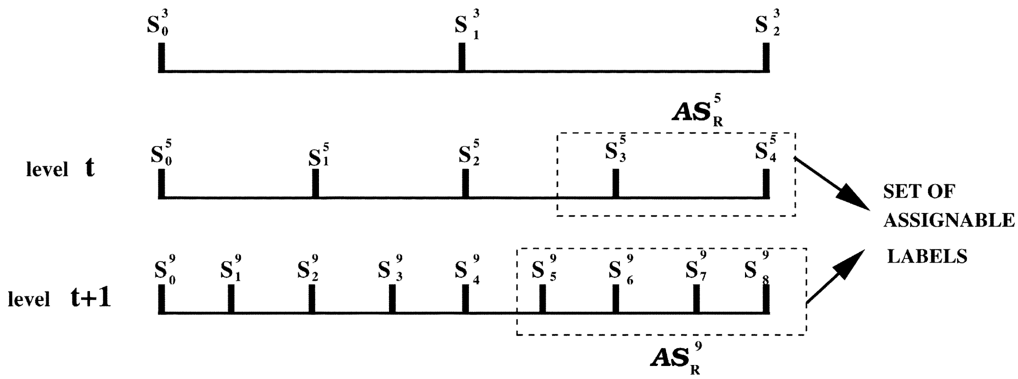

Fig. 5. Assignable term sets.

TABLE II
DECISION RULE TO REPRESENT $\mathcal{S}_R$ FROM THE ASSIGNABLE SETS

IF $density_{\mathcal{S}_R}$= "extreme"
THEN
$\mathcal{S}_{RE}$ is represented on $AS_R^{n(t+1)}$, i.e., $\mathcal{S}_{RE} \subset AS_R^{n(t+1)}$
$\mathcal{S}_{RC}$ is represented on $AS_R^{n(t)}$, i.e., $\mathcal{S}_{RC} \subset AS_R^{n(t)}$
ELSE (maximum density in the middle of $\mathcal{S}$)
$\mathcal{S}_{RE}$ is represented on $AS_R^{n(t)}$, i.e., $\mathcal{S}_{RE} \subset AS_R^{n(t)}$
$\mathcal{S}_{RC}$ is represented on $AS_R^{n(t+1)}$, i.e., $\mathcal{S}_{RC} \subset AS_R^{n(t+1)}$

$\mathrm{AS}_R^{n(t+1)}$ will decrease after each semantic assignment and we only can assure that $\mathrm{AS}_R^{n(t)} \subseteq S_R^{n(t)}$ and $\mathrm{AS}_R^{n(t+1)} \subseteq S_R^{n(t+1)}$.

Example 1: For example, if we use the LH shown in Fig. 4 to represent the labels of the term set $\mathcal{S} = \{F, D, C, B, A\}$ shown in Fig. 1, then the assignable sets for the right lateral set $\mathcal{S}_R = \{C, B, A\}$ are those dashed rectangles in Fig. 5.

Once the initial assignable sets have been selected, it is necessary to decide how to use the assignable sets $\mathrm{AS}_R^{n(t)}$ and $\mathrm{AS}_R^{n(t+1)}$ to represent the labels of $\mathcal{S}_R$. This decision depends on the distribution of $\mathcal{S}_R$, i.e., it depends on the value of the variable density$_{\mathcal{S}_R}$.

The idea consists in representing the side of $\mathcal{S}_R$ with the highest level of granularity from $\mathrm{AS}_R^{n(t+1)}$ and the side of $\mathcal{S}_R$ with the lowest level of granularity from $\mathrm{AS}_R^{n(t)}$. Hence, we distinguish in $\mathcal{S}_R$ two label subsets $\mathcal{S}_R = \mathcal{S}_{RC} \bigcup \mathcal{S}_{RE}$, with $\mathcal{S}_{RC}$ being the subset that contains the labels close to the central label of $\mathcal{S}$ and $\mathcal{S}_{RE}$ the subset that contains the labels close to the maximum label of $\mathcal{S}$. Then, the decision rule to represent the labels of $\mathcal{S}_R$ from the assignable sets $\mathrm{AS}_R^{n(t)}$ and $\mathrm{AS}_R^{n(t+1)}$ is shown in Table II.

2) Representation Process of a Lateral Set: The representation process will assign semantics to all the labels of the lateral set $\mathcal{S}_R$ by means of an iterative process using both assignable label sets $\mathrm{AS}_R^{n(t+1)}$ and $\mathrm{AS}_R^{n(t)}$. To control the updating of the assignable sets after each semantic assignment during this process, a representation rule R^{Rep} will be defined. To define this representation rule, we take into account how an LH is built [see (3)]. Every label of level $\mathbf{t}$, $s_j^{n(t)} \in \mathrm{AS}_R^{n(t)}$, $s_j^{n(t)} \neq s_C^{n(t)}$, has associated two labels of level $\mathbf{t+1}$, $s_{2\cdot j}^{n(t+1)} \in \mathrm{AS}_R^{n(t+1)}$ and $s_{2\cdot j-1}^{n(t+1)} \in \mathrm{AS}_R^{n(t+1)}$.

The representation process starts using $\mathrm{AS}_R^{n(t+1)}$ to represent the labels of $\mathcal{S}_{\mathcal{R}}$ situated in the side with highest density. Once the first label has been represented, the representation rule fixes the assignable sets for the following labels and keeps assigning semantics. This representation rule acts as follows. $\mathbf{R}^{\mathbf{Rep}}$: *when a label* $s_i^R \in \mathcal{S}_R$ *is represented by means of a label* $s_k^{n(t+1)} \in \mathrm{AS}_R^{n(t+1)}$, $k = 2 \cdot j$ *or* $k = 2 \cdot j - 1$, *then* $s_k^{n(t+1)}$ *is eliminated from* $\mathrm{AS}_R^{n(t+1)}$ *and its associated label* $s_j^{n(t)} \in \mathrm{AS}_R^{n(t)}$ *is also eliminated if it has not been already eliminated.*

Therefore, the iterative process to represent $\mathcal{S}_R$ consists in assigning semantics to its terms from the assignable set $\mathrm{AS}_R^{n(t+1)}$ and applying the representation rule $\mathrm{R}^{\mathrm{Rep}}$ until the number of unrepresented labels of $\mathcal{S}_R$ coincides with the number of assignable labels in $\mathrm{AS}_R^{n(t)}$. At that moment, the unrepresented labels are assigned directly from $\mathrm{AS}_R^{n(t)}$.

Example 2: Assuming the framework shown in Example 1 with $\mathcal{S} = \{F, D, C, B, A\}$, density$_R$ $= \{\text{extreme}\}$ and the assignable sets shown in Fig. 5

$$
\begin{aligned}
&\mathcal{S}_R = \{C, B, A\} \quad \mathcal{S}_{RE} = \{B, A\} \quad \mathcal{S}_{RC} = \{C\} \\
&\mathrm{AS}_R^5 = \{s_3^5, s_4^5\} \\
&\mathrm{AS}_R^9 = \{s_5^9, s_6^9, s_7^9, s_8^9\} \\
&S_{RE} \subset \mathrm{AS}_R^9 \quad \text{and} \quad S_{RC} \subset \mathrm{AS}_R^5.
\end{aligned}
$$

The associations between the labels of both levels are the following.

- The label $s_4^5 \in \mathrm{AS}_R^5$ is associated with the labels $s_7^9, s_8^9 \in \mathrm{AS}_R^9$.
- The label $s_3^5 \in \mathrm{AS}_R^5$ is associated with the labels $s_5^9, s_6^9 \in \mathrm{AS}_R^9$.

The representation process for this example starts using the label $s_8^9 \in \mathrm{AS}_R^9$ to represent the linguistic assessment $A \in S_{RE}$. Then the representation rule eliminates the label s_8^9 from AS_R^9 and s_4^5 from AS_R^5 (see Fig. 6).

The process goes on representing the linguistic term $B \in S_{RE}$ with the label $s_7^9 \in \mathrm{AS}_R^9$; therefore the rule does not eliminate any label from AS_R^5 because s_4^5 has already been eliminated (see Fig. 7).

So if we carry on with the iterative process, the last unrepresented label of the right lateral set is the linguistic assessment C. Then such term is represented in AS_R^5 by means of the only label s_3^5. The representation process for S_R is already finished and the unbalanced linguistic terms A, B, C are represented by

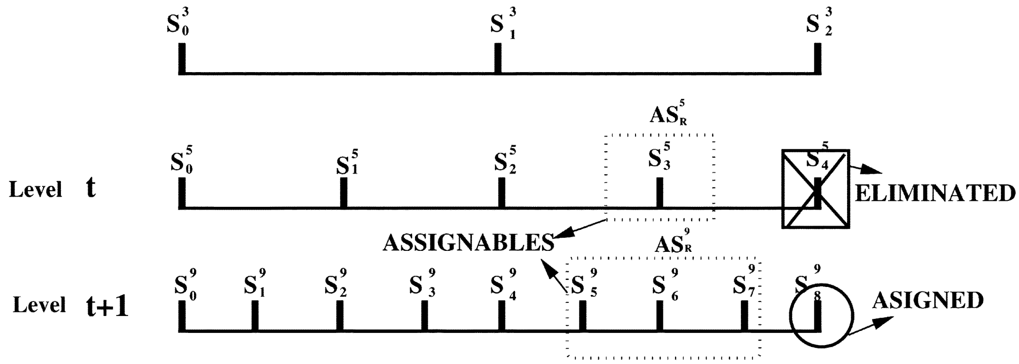

Fig. 6. $\mathrm{R}^{\mathrm{Rep}}$ if $\text{density}_{\mathcal{S}_R}$ = ``extreme" (first assignment).

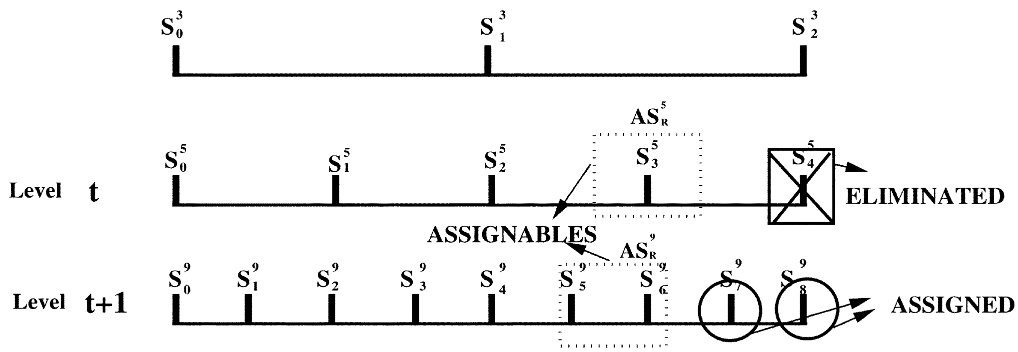

Fig. 7. $\mathrm{R}^{\mathrm{Rep}}$ if $\text{density}_{\mathcal{S}_R}$ = ``extreme" (second assignment).

means of the labels belonging to LH: s_8^9, s_7^9, and s_3^5, respectively (see Fig. 8).

Example 3: Suppose the same framework as in Example 2 but an unbalanced linguistic term set with similar five terms but $\text{density}_R = \{\text{middle}\}$. Then $\mathcal{S}_R = \{C, B, A\}, \mathcal{S}_{RE} = \{A\}, \mathcal{S}_{RC} = \{C, B\}, \mathrm{AS}_R^5 = \{s_3^5, s_4^5\}, \mathrm{AS}_R^9 = \{s_5^9, s_6^9, s_7^9, s_8^9\}, \mathcal{S}_{RE} \subset \mathrm{AS}_R^5$, and $\mathcal{S}_{RC} \subset \mathrm{AS}_R^9$. The representation rule would act as is shown in Fig. 9.

3) Improving the Representation Process: The above iterative representation process needs several rounds to represent the labels of $\mathcal{S}_{RE}$ and $\mathcal{S}_{RC}$ from the assignable sets $\mathrm{AS}_R^{n(t)}$ and $\mathrm{AS}_R^{n(t+1)}$ according to the decision rule (Table II).

It is clear that this process can be improved and simplified if we can calculate a priori the number of labels of $\mathrm{AS}_R^{n(t)}$ and $\mathrm{AS}_R^{n(t+1)}$, noted as lab_t and lab_{t+1}, respectively, which will be used to represent the labels of $\mathcal{S}_{RE}$ and $\mathcal{S}_{RC}$. In such a case, all the labels of $\mathcal{S}_R$ can be represented in just one round because we know how many labels and which ones will represent the labels of $\mathcal{S}_{RE}$ and $\mathcal{S}_{RC}$ from the initial assignable sets.

Obviously $\text{lab}_t + \text{lab}_{t+1} = \#(S_R)$. The following proposition allows us a way to compute both values.

Proposition 3: The number of labels utilized from $\mathrm{AS}_R^{n(t)}$, lab_t, to represent the labels of $\mathcal{S}_{RE}$ is computed as $\text{lab}_t = ((n(t+1) - 1)/2) - \#(\mathcal{S}_R)$.

Proof: On the one hand, we know that an LH is built in such a way that a label of a lateral set in the level t has associated two labels of the level $t+1$. Then lab_t labels of level t have associated ($2 \cdot \text{lab}_t$) labels of level $t+1$. On the other hand, following the representation rule $\mathrm{R}^{\mathrm{Rep}}$, we know that when two labels of level $t+1$ are used in the representation process, then its associated label of level t is eliminated. Therefore, we have $\text{lab}_{t+1} = ((n(t+1) - 1)/2) - (2 \cdot \text{lab}_t)$, and as $\text{lab}_{t+1} = \#(\mathcal{S}_R) - \text{lab}_t$, then it is satisfied $((n(t+1)-1)/2) - (2 \cdot \text{lab}_t) = \#(\mathcal{S}_R) - \text{lab}_t$, and consequently $\text{lab}_t = ((n(t+1) - 1)/2) - \#(\mathcal{S}_R)$. □

Example 4: Using the framework of Example 2 with $\mathcal{S} = \{F, D, C, B, A\}$, $\text{density}_R = \{\text{extreme}\}$ and the assignable sets shown in Fig. 5 $\mathcal{S}_R = \{C, B, A\}, \mathcal{S}_{RE} = \{B, A\}, \mathcal{S}_{RC} = \{C\}, \mathrm{AS}_R^5 = \{s_3^5, s_4^5\}, \mathrm{AS}_R^9 = \{s_5^9, s_6^9, s_7^9, s_8^9\}, \mathcal{S}_{RE} \subset \mathrm{AS}_R^9$, and $\mathcal{S}_{RC} \subset \mathrm{AS}_R^5$. We can find out a priori the cardinality and labels of $\mathcal{S}_{RE} = \{A, B\}$ and $\mathcal{S}_{RC} = \{C\}$ because $\text{lab}_2 = ((9-1)/2) - 3 = 1$ and $\text{lab}_3 = 3 - 1 = 2$. So we know that $\{A, B\}$ will be represented with semantics from AS_R^9 and $\{C\}$ from AS_R^5 according to the decision rule.

4) Representing the Central Set: Finally, we have to establish the representation associated with the central label of $\mathcal{S}$, i.e., the representation of $\mathcal{S}_C = \{s_C\}$. In our case, given that we are representing $\mathcal{S}_R$, we establish the representation of $\underline{s_C}$. The representation of $\underline{s_C}$ depends on the value of the variable $\text{density}_{\mathcal{S}_R}$. Then, it will be represented according to Table III.

IV. Unbalanced Linguistic Representation Model

In this section, we formalize the ideas introduced in the above section. Then we develop a semantic representation algorithm for unbalanced linguistic term sets that provides a semantics to the linguistic terms belonging to an unbalanced linguistic term set. First, we define several representation functions that control the semantic assignment to each linguistic term according to several parameters. Afterwards we introduce some additional

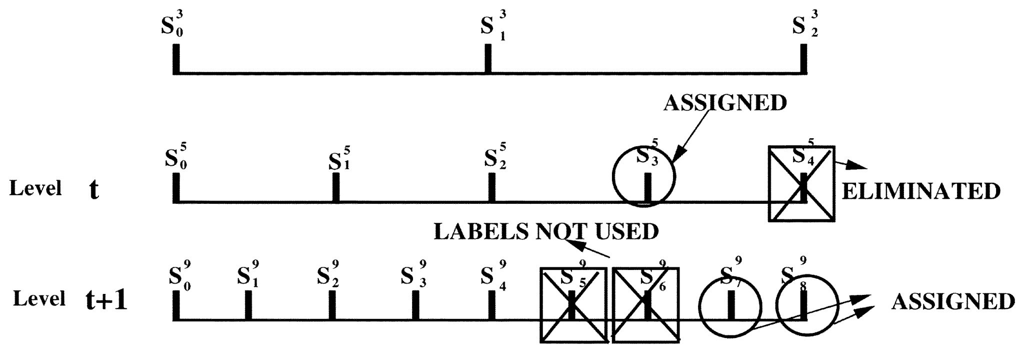

Fig. 8. Representation of labels A, B, C of Fig. 1.

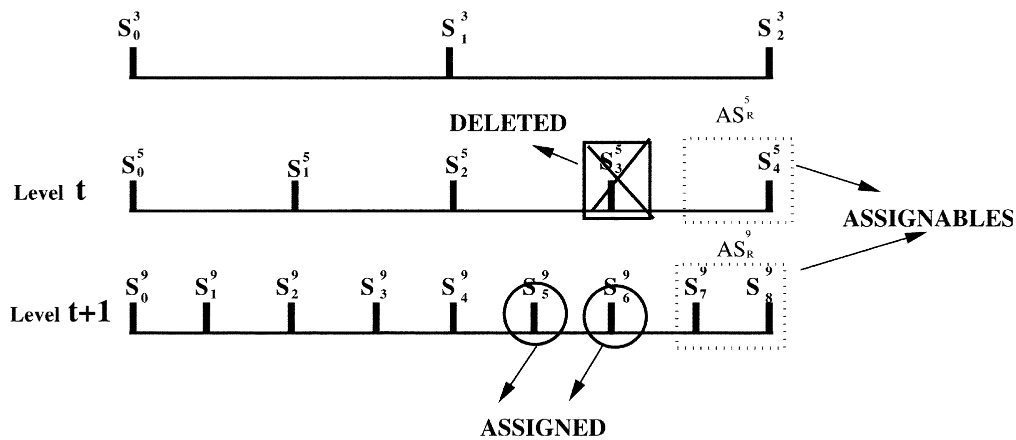

Fig. 9. $\mathrm{R}^{\mathrm{Rep}}$ if density$_{\mathcal{S}_R}$ = ``middle" (after two assignments).

TABLE III
REPRESENTING $\underline{s_C}$

IF $density_{\mathcal{S}_R}$= "extreme"
THEN
$\underline{s_C} \leftarrow \underline{s_C^{n(t)}}$.
ELSE (maximum density in the middle of $\mathcal{S}$)
$\underline{s_C} \leftarrow \underline{s_C^{n(t+1)}}$.

necessary steps that bridge some gaps in the current representation in order to guarantee that the representation of the unbalanced term set will support processes of CW without loss of information. Finally, we present the formal semantic representation algorithm that assigns the semantics to the unbalanced linguistic term set.

A. Representation Functions

According to the basic ideas of the representation process, the semantics assigned to each term depends on the density of the lateral set density$_{\mathcal{S}_R} \in \{\text{extreme}, \text{middle}\}$ and on the level of the LH used to assign the semantics, t or $t{+}1$. Therefore, we can infer that we need different representation functions in accordance with the parameters mentioned above. Here we present four different representation functions that cover the different possibilities and their role in relation to the value of their parameters.

1) Representation Function of $\mathcal{S}_R$ in Level $t+1$ of LH: $\mathbf{assign}^{\mathbf{R}}_{\mathbf{t+1}}(\mathbf{density})$: This function carries out the representation of unbalanced linguistic terms in the right lateral set $\mathcal{S}_R$ from the assignable set $\mathrm{AS}_R^{n(t+1)}$ of level $t{+}1$ of LH. It acts depending on the value of parameter $\mathbf{density} \in \{\text{middle}, \text{extreme}\}$.

1) If $\mathbf{density}$ = middle, then the lab_{t+1} labels contained in $\mathcal{S}_{RC} \subset \mathcal{S}_R$ are represented by means of the lab_{t+1} smallest labels contained in $\mathrm{AS}_R^{n(t+1)}$ following the representation rule $\mathrm{R}^{\mathrm{Rep}}$ and beginning by the label following to the middle label, i.e., $s_{C+1}^{n(t+1)}$.
2) If $\mathbf{density}$ = extreme: the lab_{t+1} labels contained in $\mathcal{S}_{RE} \subset \mathcal{S}_R$ are represented by means of the lab_{t+1} largest labels contained in $\mathrm{AS}_R^{n(t+1)}$ following the representation rule $\mathrm{R}^{\mathrm{Rep}}$ and beginning by the highest label $s_{n(t+1)-1}^{n(t+1)}$.

2) Representation Function of $\mathcal{S}_R$ in Level t of an LH: $\mathbf{assign}^{\mathbf{R}}_{\mathbf{t}}(\mathbf{density})$: This function carries out the representation of unbalanced linguistic labels of $\mathcal{S}_R$ in the subset of assignable labels $\mathrm{AS}_R^{n(t)}$ of level t of LH. Similarly, it acts depending on the value of parameter $\mathbf{density} \in \{\text{middle}, \text{extreme}\}$.

1) If $\mathbf{density}$ = middle: the lab_t labels contained in $\mathcal{S}_{RE} \subset \mathcal{S}_R$ are represented by means of the lab_t highest labels contained in $\mathrm{AS}_R^{n(t)}$ beginning by the label $s_{C+1+\delta}^{n(t)}$, with $\delta = \text{round}((\text{lab}_{t+1})/2)$.
2) If $\mathbf{density}$ = extreme: the lab_t labels contained in $\mathcal{S}_{RC} \subset \mathcal{S}_R$ are represented by means of the lab_t smallest labels contained in $\mathrm{AS}_R^{n(t)}$ beginning by the label $s_{n(t)-1-\delta}^{n(t)}$.

3) Representation Function of $\mathcal{S}_L$ in the Level $t+1$ of an LH: $\mathbf{assign}^{\mathbf{L}}_{\mathbf{t+1}}(\mathbf{density})$: This function carries out the representation of unbalanced linguistic labels of $\mathcal{S}_L$ in

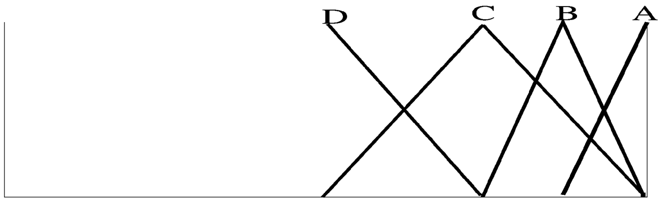

Fig. 10. Initial representation for $\mathcal{S}_R$.

the subset of assignable labels $\mathrm{AS}_L^{n(t+1)}$ of level $t+1$ of LH. As above, it acts depending on the value of parameter $\mathbf{density} \in \{\text{middle}, \text{extreme}\}$.

1) If $\mathbf{density}$ = middle: the lab_{t+1} labels contained in $\mathcal{S}_{LC} \subset \mathcal{S}_L$ are represented by means of the lab_{t+1} largest labels contained in $\mathrm{AS}_L^{n(t+1)}$ following the representation rule $\mathrm{R}^{\mathrm{Rep}}$ and beginning by the label previous to the middle label, i.e., $s_{C-1}^{n(t+1)}$.
2) If $\mathbf{density}$ = extreme: the lab_{t+1} labels contained in $\mathcal{S}_{LE} \subset \mathcal{S}_L$ are represented by means of the lab_{t+1} smallest labels contained in $\mathrm{AS}_L^{n(t+1)}$ following the representation rule $\mathrm{R}^{\mathrm{Rep}}$ and beginning by the smallest label $s_0^{n(t+1)}$.

4) Representation Function of $\mathcal{S}_L$ in the Level t of an LH, $\mathbf{assign_t^L}(\mathbf{density})$*:* This function carries out the representation of unbalanced linguistic labels of $\mathcal{S}_L$ in the subset of assignable labels $\mathrm{AS}_L^{n(t)}$ of the level t of LH. It also acts depending on the value of parameter $\mathbf{density} \in \{\text{middle}, \text{extreme}\}$.

1) If $\mathbf{density}$ = middle: the lab_t labels contained in $\mathcal{S}_{LE} \subset \mathcal{S}_L$ are represented by means of the lab_t smallest labels contained in $\mathrm{AS}_L^{n(t)}$ beginning by the label $s_{C-1-\delta}^{n(t)}$, with $\delta = \mathrm{round}((\text{lab}_{t+1})/2)$.
2) If $\mathbf{density}$ = extreme: the lab_t labels contained in $\mathcal{S}_{LC} \subset \mathcal{S}_L$ are represented by means of the lab_t highest labels contained in $\mathrm{AS}_L^{n(t)}$ beginning by the label $s_\delta^{n(t)}$.

B. Bridging Representation Gaps

In [23], several conditions were studied that must be satisfied by the semantics of a linguistic term set S in order to guarantee that the processes of CW using the linguistic 2-tuple computational model are carried out in a precise way. Such conditions are the following.

1) *S is a fuzzy partition.* According to Ruspini [44], a finite family $\{s_0, \ldots, s_g\}$ of fuzzy subsets in the universe X (in our case $X = [0,1]$) is called a fuzzy partition if $\sum_{i=0}^{g} \mu_{s_i}(x) = 1, \forall x \in X$.
2) *The membership functions of its terms are triangular,* i.e., $s_i = (a_i, b_i, c_i)$. Then $\exists! x / \mu_{s_i}(x) = 1, s_i \in S$.

Our aim is to represent the unbalanced term sets so that we can operate with them in a precise way, but following the basic ideas exposed to represent an unbalanced linguistic term set $\mathcal{S}$ and the above representation functions. The semantics obtained for the terms of $\mathcal{S}$ satisfies the second condition but not the first one. Therefore, we provide some additional steps that our representation algorithm must carry out to represent $\mathcal{S}$ as a fuzzy partition.

We can observe this in Example 2, where the initial proposal for representing the unbalanced information of the right lateral set $\mathcal{S}_R = \{C, B, A\}$ shown in Fig. 1 is used. The labels from the LH implied in their representation are $\{s_3^5\}$ and $\{s_7^9, s_8^9, \}$, respectively (graphically see Fig. 10).

TABLE IV
BRIDGING THE LABELS BETWEEN LEVELS

IF $density_{\mathcal{S}_R}$= "extreme"		
THEN		
	$\overline{s_{jump}} \leftarrow \overline{s_i^{n(t)}}$,	$\underline{s_{jump}} \leftarrow \underline{s_k^{n(t+1)}}, k = 2*i$
ELSE		
	$\underline{s_{jump}} \leftarrow \underline{s_i^{n(t)}}$,	$\overline{s_{jump}} \leftarrow \overline{s_k^{n(t+1)}}, k = 2*i$

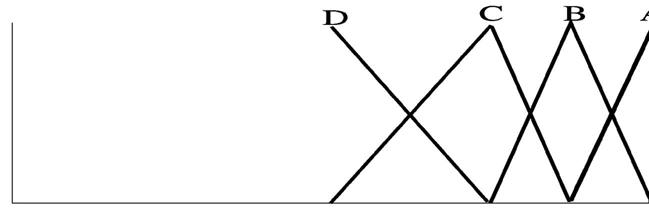

Fig. 11. Fuzzy partition representation for $\mathcal{S}_R$.

In such a situation, the semantics associated with $\mathcal{S}_R$ cannot form a fuzzy partition because of the representation of the downside of the label $\underline{C}$. We can see that label C represents the jump between levels t and $t+1$, noted as s_{jump}, and whenever this jump occurs there the same problem appears regarding the fuzzy partition. Therefore in such jumps we have to *bridge the unbalanced term,* in a way similar to the central label s_C, to obtain a fuzzy partition. It means that its representation will be assigned splitting the upside and the downside. The representation will depend on the density of the lateral set (see Table IV).

Therefore, to represent $\mathcal{S}_R$ shown in Fig. 10 as a fuzzy partition, we bridge the jump representing C with the following semantics: $\underline{C} = \underline{s_{\mathrm{jump}}} \leftarrow \underline{s_6^9}, \bar{C} = \overline{s_{\mathrm{jump}}} \leftarrow \overline{s_3^5}$. The Fig. 11 shows the new representation for $\mathcal{S}_R$.

C. Output: Semantics and Additional Information

The representation algorithm provides the semantics for the unbalanced linguistic term set and the following additional information, in order to control and manage the modeling of linguistic information in any unbalanced linguistic term set $\mathcal{S}$.

1) *A hierarchical semantic representation* LH($\mathcal{S}$): For an unbalanced linguistic term set $\mathcal{S} = \{s_i, i = 0, \ldots, g\}$, we obtain its representation in the LH, i.e., LH($\mathcal{S}$) $= \{s_{I(i)}^{G(i)}, i = 0, \ldots, g\}$, such that $\forall s_i \in \mathcal{S}\ \exists l(t, n(t)) \in$ LH that contains a label $s_k^{n(t)} \in S^{n(t)}$, in such a way that $I(i) = k$ and $G(i) = n(t)$, with I and G being functions that assign to each unbalanced label $s_i \in \mathcal{S}$ the index of the label that represents it in LH and the granularity of label set of LH in which it is represented, respectively. This representation will be generated by the representation functions.
2) *A bridge mark Brid*: We define a boolean function Brid: $\mathcal{S} \rightarrow \{\text{False}, \text{True}\}$ for those $s_i \in \mathcal{S}$ that are considered s_{jump}, i.e., labels whose semantic representation is achieved from two levels in LH (including the central label s_C).
3) *Subsets ordering:* The five subsets of the unbalanced linguistic term set $\mathcal{S}$ $\mathcal{S}_{LE}, \mathcal{S}_{LC}, \mathcal{S}_C, \mathcal{S}_{RC}, \mathcal{S}_{RE}$ are ordered in increasing order.
4) *Set of levels of LH*, T_{LH}: It contains those levels used in the representation of $\mathcal{S}$ $T_{\mathrm{LH}} = \{t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$, where t_{LE} is the level of LH used to represent $\mathcal{S}_{LE}$, t_{LC} is the

level of LH used to represent $\mathcal{S}_{LC}$, and so on. We should point out that if $\exists l(t,n(t)), ((n(t)-1)/2) = \#(\mathcal{S}_R)$, then $t_{RC} = t_{RE} = t$ and $\mathcal{S}_{RC} = \mathcal{S}_{RE} = \mathcal{S}_R$. Similarly it happens for $\mathcal{S}_L$.

TABLE V
REPRESENTATION ALGORITHM OF UNBALANCED LINGUISTIC TERM SETS

INPUTS:
$\{(\#(\mathcal{S}_L), density_{\mathcal{S}_L}), \#(\mathcal{S}_C), (\#(\mathcal{S}_R), density_{\mathcal{S}_R})\}$,
$\mathcal{S} = \{s_0, ..., s_n\}$,
$LH = \bigcup_t l(t, n(t))$
BEGIN
IF $\exists l(t, n(t)), \frac{n(t)-1}{2} = \#(\mathcal{S}_R)$
THEN
To represent $\mathcal{S}_R$ by means of $S_R^{n(t)}$: $\mathcal{S}_R \leftarrow S_R^{n(t)}$
To represent $\underline{s_C} \in \mathcal{S}$ as $\underline{s_C^{n(t)}}$: $\underline{s_C} \leftarrow \underline{s_C^{n(t)}}$
$t_{RE} = t_{RC} = t$
ELSE
To look for t and $t+1$, such that:
$\frac{n(t)-1}{2} < \#(\mathcal{S}_R) < \frac{n(t+1)-1}{2}$
$lab_t = \frac{n(t+1)-1}{2} - \#(\mathcal{S}_R)$
$lab_{t+1} = \#(\mathcal{S}_R) - lab_t$
$assign_{t+1}^R(density_{\mathcal{S}_R})$
$assign_t^R(density_{\mathcal{S}_R})$
IF $density_{\mathcal{S}_R}$ = "extreme"
THEN
$\underline{s_{C+lab_t}} \leftarrow \underline{s_{n(t+1)-1-lab_{t+1}}^{n(t+1)}}$; $Brid \leftarrow$ "$True$" $(**)$
$\underline{s_C} \leftarrow \underline{s_C^{n(t)}}$; $Brid \leftarrow$ "$True$" $(**)$
$t_{RE} = t,\ t_{RC} = t+1$
ELSE
$\overline{s_{C+1+lab_{t+1}}} \leftarrow \overline{s_{C+1+lab_{t+1}}^{n(t+1)}}$; $Brid \leftarrow$ "$True$" $(**)$
$\underline{s_C} \leftarrow \underline{s_C^{n(t+1)}}$; $Brid \leftarrow$ "$True$" $(**)$
$t_{RE} = t+1,\ t_{RC} = t$
END-IF
END-IF
IF $\exists l(t, n(t)), \frac{n(t)-1}{2} = \#(\mathcal{S}_L)$
THEN
To represent $\mathcal{S}_L$ by means of $S_L^{n(t)}$: $\mathcal{S}_L \leftarrow S_L^{n(t)}$
To represent $\overline{s_C} \in \mathcal{S}$ as $\overline{s_C^{n(t)}}$: $\overline{s_C} \leftarrow \overline{s_C^{n(t)}}$
$t_{LE} = t_{LC} = t$
ELSE
Look for t and $t+1$, such that:
$\frac{n(t)-1}{2} < \#(\mathcal{S}_L) < \frac{n(t+1)-1}{2}$
$lab_t = (n(t+1)-1)/2 - \#(\mathcal{S}_L)$
$lab_{t+1} = \#(\mathcal{S}_L) - lab_t$
$assign_{t+1}^L(density_{\mathcal{S}_L})$
$assign_t^L(density_{\mathcal{S}_L})$
IF $density_{\mathcal{S}_L}$ = "extreme"
THEN
$\overline{s_{C-lab_t}} \leftarrow \overline{s_{lab_{t+1}+1}^{n(t+1)}}$; $Brid \leftarrow$ "$True$" $(**)$
$\overline{s_C} \leftarrow \overline{s_C^{n(t)}}$; $Brid \leftarrow$ "$True$" $(**)$
$t_{LE} = t+1,\ t_{LC} = t$
ELSE
$\underline{s_{C-lab_{t+1}-1}} \leftarrow \underline{s_{C-lab_{t+1}-1}^{n(t+1)}}$; $Brid \leftarrow$ "$True$" $(**)$
$\overline{s_C} \leftarrow \overline{s_C^{n(t+1)}}$; $Brid \leftarrow$ "$True$" $(**)$
$t_{LE} = t,\ t_{LC} = t+1$
END-IF
END-IF
END
OUTPUTS:
$S = \{s_0 = (a_0, b_0, c_0), ..., s_n = (a_n, b_n, c_n)\}$
$LH(\mathcal{S}) = \{s_{I(i)}^{G(i)},\ i = 0, ..., g\}$
T_{LH}
$Brid$

TABLE VI
LH($\mathcal{S}$) AND BRID($\mathcal{S}$)

$\mathcal{S}$	$LH(\mathcal{S})$	$Brid(\mathcal{S})$
$s_0 = N$	$s_{I(0)}^{G(0)} = s_0^5$	False
$s_1 = L$	$s_{I(1)}^{G(1)} = s_1^5$	False
$s_2 = M$	$s_{I(2)}^{G(2)} = s_2^5$ **or** s_4^9	True
$s_3 = AH$	$s_{I(3)}^{G(3)} = s_5^9$	False
$s_4 = H$	$s_{I(4)}^{G(4)} = s_6^9$ **or** s_{12}^{17}	True
$s_5 = QH$	$s_{I(5)}^{G(5)} = s_{13}^{17}$	False
$s_6 = VH$	$s_{I(6)}^{G(6)} = s_{14}^{17}$	False
$s_7 = AT$	$s_{I(7)}^{G(7)} = s_{15}^{17}$	False
$s_8 = T$	$s_{I(8)}^{G(8)} = s_{16}^{17}$	False

D. Algorithm

Using the initial ideas, the above representation functions, and the bridging process, we present in Table V the semantic representation algorithm for unbalanced linguistic term sets that represents the unbalanced terms by means of triangular membership functions using the linguistic hierarchies.

Remark 6: Those steps of the algorithm signed with $(**)$ have been included to accomplish the bridging processes. The *Brid* term assigned to *True* corresponds to the labels assigned in the same line.

E. Using the Representation Algorithm

In this section, we apply the representation algorithm to an unbalanced linguistic term set. We shall use the LH shown in Fig. 4. Let us suppose that we want to manage linguistic information assessed on the unbalanced linguistic term set shown in Fig. 2, i.e., $\mathcal{S} = \{N, L, M, AH, H, QH, VH, AT, T\}$. Then, the description of S in the algorithm is $\{(\mathbf{2}, \mathbf{extreme}), \mathbf{1}, (\mathbf{6}, \mathbf{extreme})\}$, with $\mathcal{S}_L = \{N, L\}$, $\mathcal{S}_C = \{M\}$, and $\mathcal{S}_R = \{AH, H, QH, VH, AT, T\}$. Then, the representation of $\mathcal{S}$ according to the above representation algorithm runs as follows.

1) *Representation of* $\mathcal{S}_R$: As in LH, the condition $\exists l(t, n(t))/((n(t)-1)/2) = \#(\mathcal{S}_R)$ is not satisfied. Then we have to look for two levels t and $t+1$ such that $((n(t)-1)/2) < \#(\mathcal{S}_R) < ((n(t+1)-1)/2)$. The levels that satisfy the above condition are $t = 3$ and $t+1 = 4$ because their respective cardinalities in LH are $n(t) = 9$ and $n(t+1) = 17$. Therefore, $\text{lab}_t = 2$ and $\text{lab}_{t+1} = 4$. As $\text{density}_{\mathcal{S}_R} = \text{extreme}$, the representation functions assign_{t+1}^R and assign_t^R represent four labels $\mathcal{S}_{RE} = \{T, AT, VH, QH\}$ in level 4 and two labels $\mathcal{S}_{RC} = \{H, AH\}$ in level 3, respectively. Applying both functions, we obtain the following representation of $\mathcal{S}_{RE}$ and $\mathcal{S}_{RC}$ in LH (see Fig. 12):

$$S_{RE} = \{T \leftarrow s_{16}^{17}, AT \leftarrow s_{15}^{17}, VH \leftarrow s_{14}^{17}, QH \leftarrow s_{13}^{17}\}$$
$$S_{RC} = \{H \leftarrow s_6^9, AH \leftarrow s_5^9, \}.$$

The current representation is not a fuzzy partition because we did not bridge the s_{jump} yet. The label H represents s_{jump} between level 3 and level 4 in this example, as can be seen in Fig. 13.

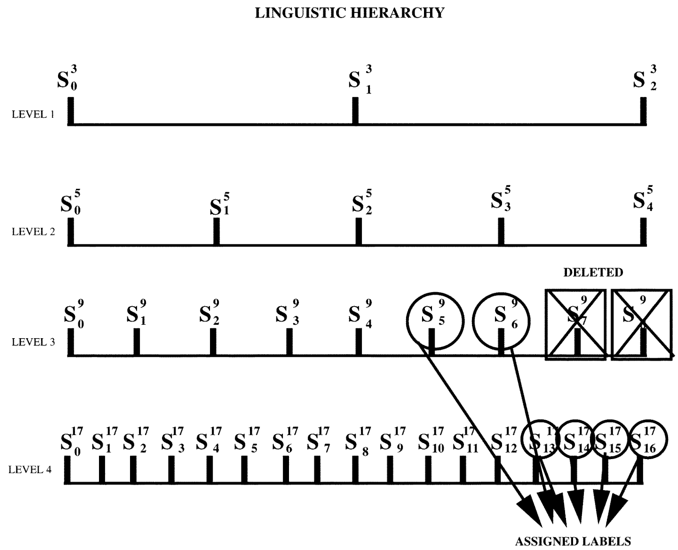

Fig. 12. Labels used to represent $\mathcal{S}_R$.

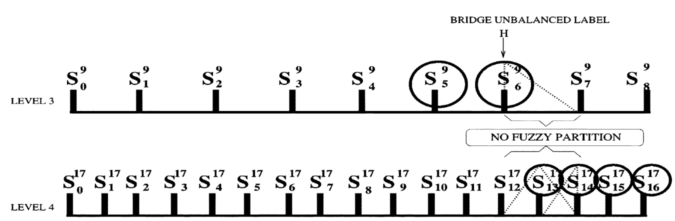

Fig. 13. Semantics: no fuzzy partition.

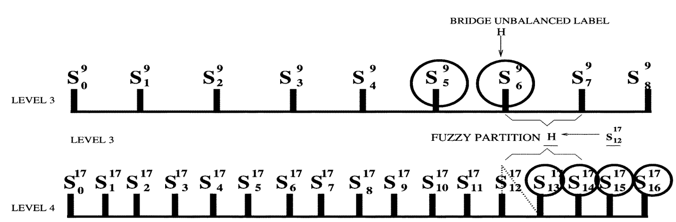

Fig. 14. Semantics: fuzzy partition.

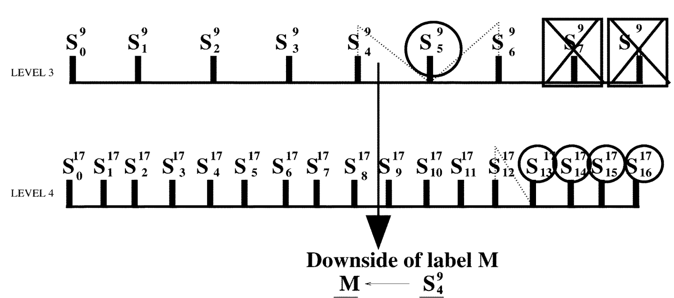

Fig. 15. Representation of $\underline{M}$.

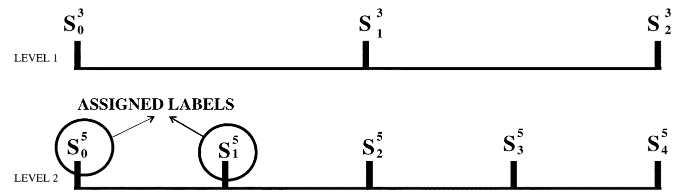

Fig. 16. Labels used to represent $\mathcal{S}_L$.

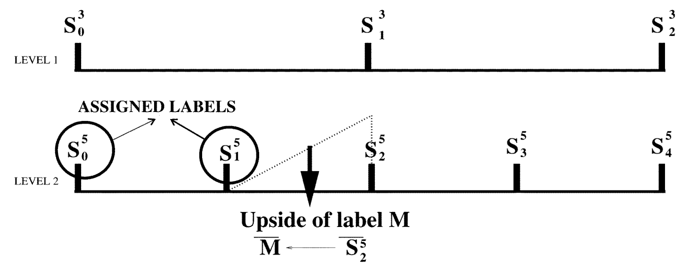

Fig. 17. Representation of $\bar{M}$.

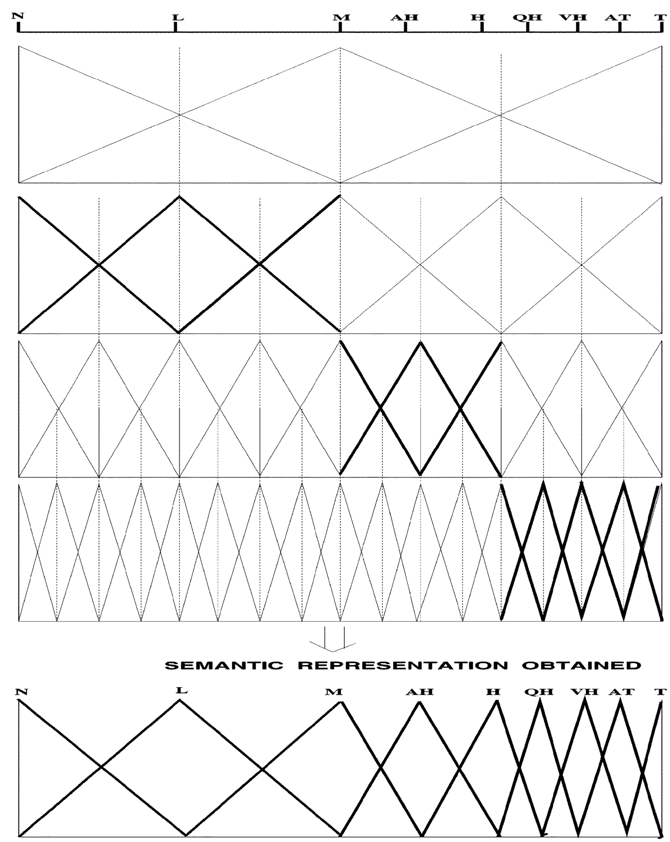

Fig. 18. Semantics of $\mathcal{S} = \{N, L, M, AH, H, QH, VH, AT, T\}$ in LH.

Then we have to bridge the label s_{jump}, representing its semantics by splitting upside and downside semantic representation. According to Table IV, the representation must be $\bar{H} \leftarrow \overline{s_6^9}$ and $\underline{H} \leftarrow \underline{s_{12}^{17}}$ (see Fig. 14).

2) *Representation of $\underline{\mathcal{S}_C}$*: As density$_{\mathcal{S}_R}$ = extreme, then following the algorithm, the downside of the central label $\underline{M}$ is represented in level 3 of LH by means of $\underline{s_4^9}$, as shown in Fig. 15.

3) *Representation of $\mathcal{S}_L$*: In this case, in LH the following condition $\exists t \in LH/((n(t)-1)/2) = \#(\mathcal{S}_L)$ is satisfied with $t = 2$ because $n(t) = 5$. Then, the representation of $\mathcal{S}_L$ is obtained from level 2 of LH as follows (see Fig. 16): $\{L \leftarrow s_1^5, N \leftarrow s_0^5\}$.

4) *Representation of $\overline{\mathcal{S}_C}$*: The upside of the central label $\bar{M}$ is represented in level 2 of LH by means of $\overline{s_2^5}$, as shown in Fig. 17.

Consequently, at the end of the representation algorithm $\mathcal{S} = \{N, L, M, AH, H, QH, VH, AT, T\}$, the semantics obtained are (graphically in Fig. 18):

- $\mathcal{S}_L : \{N \leftarrow \underline{s_0^5}, L \leftarrow s_1^5\}$;
- $\mathcal{S}_C : \{M \leftarrow \overline{s_2^5} \bigcup \underline{s_4^9}\}$;
- $\mathcal{S}_R : \{AH \leftarrow s_5^9, H \leftarrow \overline{s_6^9} \bigcup \underline{s_{12}^{17}}, QH \leftarrow s_{13}^{17}, VH \leftarrow s_{14}^{17}, AT \leftarrow s_{15}^{17}, T \leftarrow s_{16}^{17}\}$.

To control such representation of $\mathcal{S}$ in the processes of CW, the algorithm provides the following information:

1) LH($\mathcal{S}$) and Brid($\mathcal{S}$), which are given in Table VI.

Remark 7: As can be observed in Table VI when Brid($\mathcal{S}$) =True, then there exist two representation possibilities in LH. In such case, we have to use one of them in order to facilitate and simplify the processes of CW. To do so, we propose to use the linguistic assessment defined in the lower level, i.e., $s_{I(2)}^{G(2)} = s_2^5$ and $s_{I(4)}^{G(4)} = s_6^9$.

2) The following five subsets of unbalanced linguistic labels ordered in increasing order:
   - $\mathcal{S}_{LE} = \mathcal{S}_{LC} = \mathcal{S}_L = \{N, L\}$;
   - $\mathcal{S}_C = \{M\}$;
   - $\mathcal{S}_{RC} = \{AH, H\}$;
   - $\mathcal{S}_{RE} = \{QH, VH, AT, T\}$.
3) The set of levels of LH used in the representation of $\mathcal{S}$ $\{t_{LE}, t_{LC}, t_{RC}, t_{RE}\} = \{2, 2, 3, 4\}$.

V. Unbalanced Linguistic Computational Model

So far, we have developed a method to provide semantics to the terms of unbalanced linguistic term sets but our aim is to operate with unbalanced linguistic information in processes of CW without loss of information.

The semantics provided by the semantic representation algorithm satisfies the conditions imposed in [23] to accomplish processes of CW in a precise way using the 2-tuple linguistic representation model. Consequently, the proposal of an unbalanced linguistic computational model will be based on:

1) the 2-tuple fuzzy linguistic representation model [22], which provides a model to operate with unbalanced linguistic information without loss of information whenever its semantics is obtained by means of the algorithm proposed in Table V;
2) the representation of the unbalanced linguistic term set on an LH, which provides a reference framework to manage unbalanced linguistic information in the computational operations.

Therefore, to develop the unbalanced linguistic computational model using an LH as the semantic representation framework, we define two transformation functions to convert unbalanced terms into terms in the LH and vice versa. Once these functions have been defined, we present the unbalanced linguistic computational model defining different operators to deal with this type of information, such as aggregation, negation or comparison operators.

A. Unbalanced Linguistic Transformation Functions

The semantics of the unbalanced linguistic terms is defined on linguistic terms of different levels from an LH, and the linguistic information is modelled by means of the linguistic 2-tuple representation. Hence, to facilitate the definition of the unbalanced linguistic computational model, we introduce two unbalanced linguistic transformation functions that convert an unbalanced linguistic term $s_i \in \mathcal{S}$ into the linguistic term in the LH $s_k^{n(t)} \in LH = \bigcup_t l(t, n(t))$ and vice versa.

1) $\mathcal{LH}$: Transformation function that associates with each unbalanced linguistic 2-tuple (s_i, α), $s_i \in \mathcal{S}$ its respective linguistic 2-tuple in LH $(s_k^{n(t)}, \alpha), s_k^{n(t)} \in LH. \mathcal{LH} : (\mathcal{S} \times [-0.5, 0.5)) \rightarrow (LH \times [-0.5, 0.5))$ such that $\forall (s_i, \alpha_i) \in (\mathcal{S} \times [-0.5, 0.5)) \exists \mathcal{LH}(s_i, \alpha_i) = (s_{I(i)}^{G(i)}, \alpha_i), s_{I(i)}^{G(i)} \in LH$.
2) $\mathcal{LH}^{-1}$: Transformation function that associates with each linguistic 2-tuple expressed in LH its respective unbalanced linguistic 2-tuple. $\mathcal{LH}^{-1} : (LH \times [-0.5, 0.5)), \rightarrow (\mathcal{S} \times [-0.5, 0.5)), \forall (s_k^{n(t)}, \alpha_k) \in (LH \times [-0.5, 0.5)) \,|\, s_k^{n(t)} \in S^{n(t)}$, with t being a level of LH. Then it is defined by cases as follows.

Case 1) When we have an unbalanced label represented directly with $s_k^{n(t)}$ according to $LH(\mathcal{S})$. If the following condition is satisfied:

$$\exists s_i \in \mathcal{S} \,|\, G(i) = n(t) \quad \textbf{and} \quad I(i) = k \tag{10}$$

then we can ensure that $\mathcal{LH}^{-1}(s_k^{n(t)}, \alpha_k) = (s_i, \lambda)$, where λ, which is the symbolic translation, is unknown. Therefore, to determine its value, we have to consider two possible situations depending on the semantic representation of s_i as shown in (11) at the bottom of the page.

Case 1.1) If Brid(s_i) =False, then the semantics of s_i is represented with only one label in LH, and therefore $\lambda = \alpha_k$, with $\mathcal{LH}^{-1}(s_k^{n(t)}, \alpha_k) = (s_i, \alpha_k)$.

Case 1.2) If Brid(s_i) =True, then the semantics of s_i is represented with two labels in LH from levels, t and t+1. In such case, the definition of $\mathcal{LH}^{-1}$ depends on the localization of s_i in $\mathcal{S}$.

a) If $s_i \in \mathcal{S}_{RE}$ **or** $s_i \in \mathcal{S}_{LC}$ then we know that $\overline{s_i}$ is defined from $TF_{t+1}^t(s_k^{n(t)}, 0) = s_{2k}^{n(t+1)}$ while $\underline{s_i}$ is defined from $s_k^{n(t)}$. Therefore, we have two possibilities.

$$\lambda = \left(\frac{\Delta_t^{-1}\left(s_k^{n(t)}, \alpha_k\right) \cdot (n(t+1) - 1)}{n(t) - 1} \right) - \text{round}\left(\frac{\Delta_t^{-1}\left(s_k^{n(t)}, \alpha_k\right) \cdot (n(t+1) - 1)}{n(t) - 1} \right). \tag{11}$$

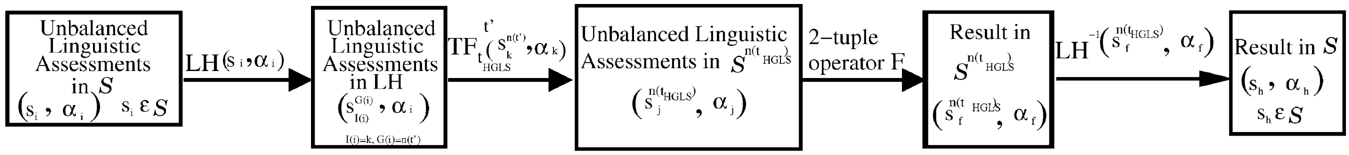

Fig. 19. Scheme of an aggregation operator of unbalanced linguistic information.

i) If α_k represents a symbolic translation on $\overline{s_k^{n(t)}}$ (upside part of the membership function) then

$$\alpha_k < 0 \Rightarrow \lambda \in [-0.5, 0).$$

Taking into account that the semantics of $\overline{s_k^{n(t)}}$ belongs to the level t, λ is computed using (5), as shown (11).

ii) If α_k represents a symbolic translation on $\underline{s_k^{n(t)}}$ (downside part of the membership function), then

$$\alpha_k > 0 \Rightarrow \lambda \in [0, 0.5)$$

with

$$\lambda = \alpha_k.$$

c) If $s_i \in \mathcal{S}_{RC}$ **or** $s_i \in \mathcal{S}_{LE}$, then we know that $\overline{s_i}$ is defined from $s_k^{n(t)}$ while $\underline{s_i}$ is defined from $\underline{TF_{t+1}^t(s_k^{n(t)}, 0)} = \underline{s_{2k}^{n(t+1)}}$. Therefore, we have two possibilities.

i) If α_k represents a symbolic translation on $\overline{s_k^{n(t)}}$ then

$$\alpha_k < 0 \Rightarrow \lambda \in [-0.5, 0)$$

where

$$\lambda = \alpha_k.$$

ii) If α_k represents a symbolic translation on $\underline{s_k^{n(t)}}$, then

$$\alpha_k > 0 \Rightarrow \lambda \in [0, 0.5)$$

and λ is computed using (11).

c) If s_i is the central label of $\mathcal{S}$, i.e., if $s_i \in \mathcal{S}_C$, then depending on the levels of LH used to represent the semantics of s_i, t_{LC} and t_{RC}, we find three possibilities.

i) If $t_{LC} = t_{RC}$, then $\mathcal{LH}^{-1}(s_k^{n(t)}, \alpha_k) = (s_i, \alpha_k)$.

ii) If $t_{LC} > t_{RC}$, then

$$\mathcal{LH}^{-1}\left(s_k^{n(t)}, \alpha_k\right) = (s_i, \lambda), \quad \begin{cases} \lambda = \alpha_k & \text{if } \alpha_k > 0 \\ \lambda = \text{eq. (11)} & \text{if } \alpha_k < 0. \end{cases}$$

iii) If $t_{LC} < t_{RC}$, then

$$\mathcal{LH}^{-1}\left(s_k^{n(t)}, \alpha_k\right) = (s_i, \lambda), \quad \begin{cases} \lambda = \alpha_k & \text{if } \alpha_k < 0 \\ \lambda = \text{eq. (11)} & \text{if } \alpha_k > 0. \end{cases}$$

Case 2) If (10) is not satisfied, then $\mathcal{LH}^{-1}(s_k^{n(t)}, \alpha_k) = \mathcal{LH}^{-1}(TF_{t'}^t(s_k^{n(t)}, \alpha_k))$, with $t' \in \{t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$ being a level such that if $TF_{t'}^t(s_k^{n(t)}, \alpha_k) = (s_{k'}^{n(t')}, \alpha_{k'})$. Then $\exists s_i \in \mathcal{S} \,|\, G(i) = n(t')$ **and** $I(i) = k'$

B. Computational Model

As in the 2-tuple fuzzy linguistic computational model, we present for the unbalanced linguistic computational model a comparison operator, a negation operator, and a tool for aggregating unbalanced linguistic information. We define these operators using the transformation functions $\mathcal{LH}$ and $\mathcal{LH}^{-1}$.

1) *An unbalanced linguistic comparison operator*. The comparison of linguistic information represented by unbalanced linguistic 2-tuples is carried out according to an ordinary lexicographic order defined as in the 2-tuple fuzzy linguistic computational model shown in Section II.

2) *An unbalanced 2-tuple negation operator*

$$\mathcal{NEG}(s_i, \alpha_i) = \mathcal{LH}^{-1}(\text{Neg}(\mathcal{LH}(s_i, \alpha_i))) \qquad (12)$$

$s_i \in \mathcal{S}$ and Neg being the 2-tuple negation operator.

3) *An unbalanced linguistic aggregation operator*. As we have shown in order to deal with unbalanced linguistic information, we represent it in an LH. Therefore, any unbalanced linguistic aggregation operator must aggregate unbalanced linguistic information by means of its representation in an LH. The labels of an unbalanced linguistic term set are represented in an LH using labels from different levels, i.e., labels assessed on label sets with a different granularity associated with the levels. Consequently, to define an unbalanced linguistic aggregation operator consists in defining an aggregation operator of multigranular linguistic information [19], [25]. In this situation, an unbalanced linguistic aggregation operator needs to develop the following steps to process the unbalanced linguistic information (see graphically in Fig. 19).

a) *Represent the unbalanced linguistic assessments to be aggregated in an LH*. The first step of the operator must be the transformation of the unbalanced linguistic information expressed in $\mathcal{S}$ into an LH in order to manage it. This step is carried out applying the unbalanced linguistic transformation function $\mathcal{LH}$ to the unbalanced linguistic assessments.

b) *Choose a level of LH to compute the unbalanced linguistic information.* With the function $\mathcal{LH}$, we obtain the unbalanced linguistic information represented in different levels of LH. That is, in order to aggregate $\mathcal{LH}$ unbalanced linguistic assessments, these will be transformed into linguistic 2-tuples expressed in those different label sets that compose the hierarchical structure of LH. In this context, we cannot process the information directly because it is expressed in different expression linguistic domains. To overcome this problem, we propose to choose a level of LH, called *basic representation level* (t_{HGLS}), which will support the computation processes of unbalanced linguistic assessments. As in [19] and [25], we choose t_{HGLS} as the level of LH used in the representation algorithm, which is associated with the highest granularity label set (HGLS), i.e., $t_{\text{HGLS}} = \max\{t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$. Then, we transform into t_{HGLS} the different linguistic 2-tuples associated with the unbalanced linguistic assessments by means of the set of transformation functions between levels of LH, $\{TF^{t'}_{t_{\text{HGLS}}}, t' = t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$. The use of these transformation functions depends on the semantic representation of $\mathcal{S}$ on LH. The application of these transformation functions is carried out by means of a special transformation function $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}$ defined in LH for unbalanced linguistic 2-tuples.

Definition 4: Let (s_i, α_i) be an unbalanced linguistic 2-tuple ($s_i \in \mathcal{S}$) and let $(s_k^{n(t')}, \alpha_k) = \mathcal{LH}(s_i, \alpha_i)$ be its respective representation in a level of LH $t' \in \{t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$, i.e., a linguistic 2-tuple expressed in $S^{n(t')}$. With a basic representation level $t_{\text{HGLS}} = \max\{t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$ fixed, then the transformation function between the levels t' of LH and t_{HGLS} for the representation of (s_i, α_i) in LH $(s_k^{n(t')}, \alpha_k)$, is defined by cases as follows.

Case 1) If s_i is not a bridge unbalanced label, i.e., Brid$(s_i) =$ false, then the semantic representation of s_i is associated with only one label in LH, and therefore, $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) = TF^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k), \forall t'$.

Case 2) If s_i is a bridge unbalanced label, i.e., if Brid(s_i) =true, then the semantic representation of s_i is associated with two labels in LH, and in such case, the definition of $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}$ depends on the localization of s_i in $\mathcal{S}$.

Case 2.1) If $s_i \in \mathcal{S}_{RE}$ **or** $s_i \in \mathcal{S}_{LC}$ (i.e., ($t' = t_{RE}$ **and** $t' + 1 = t_{RC}$) **or** ($t' = t_{LC}$ **and** $t' + 1 = t_{LE}$), then we know that $\overline{s_i}$ is defined from $TF^{t'}_{t'+1}(s_k^{n(t')}, 0)$ while $\underline{s_i}$ is defined from $s_k^{n(t')}$. Therefore, we have two possibilities.

i) If α_k represents a symbolic translation on the upside of $s_k^{n(t')}$, i.e., if $\alpha_k < 0$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) = TF^{t'+1}_{t_{\text{HGLS}}}(s_{2k}^{n(t'+1)}, \alpha_k)$.

ii) If α_k represents a symbolic translation on the downside of $s_k^{n(t')}$, i.e., if $\alpha_k \geq 0$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) = TF^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k)$.

Case 2.2) If $s_i \in \mathcal{S}_{RC}$ **or** $s_i \in \mathcal{S}_{LE}$ (i.e., ($t' = t_{RC}$ **and** $t' + 1 = t_{RE}$) **or** ($t' = t_{LE}$ **and** $t' + 1 = t_{LC}$)) then we know that $\overline{s_i}$ is defined from $s_k^{n(t')}$ while $\underline{s_i}$ is defined from $TF^{t}_{t+1}(s_k^{n(t)}, 0)$. Therefore, we have two possibilities.

i) If α_k represents a symbolic translation on the upside of $s_k^{n(t')}$, i.e., if $\alpha_k \leq 0$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) = TF^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k)$.

ii) If α_k represents a symbolic translation on the downside of $s_k^{n(t')}$, i.e., if $\alpha_k > 0$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) = TF^{t'+1}_{t_{\text{HGLS}}}(s_{2k}^{n(t'+1)}, \alpha_k)$.

Case 2.3) If s_i is the middle label of $\mathcal{S}$, i.e., if $s_i \in \mathcal{S}_C$, then depending on the levels of LH used to represent the semantics of s_i, t_{LC}, and t_{RC}, we find three possibilities.

i) If $t_{LC} = t_{RC}$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) = TF^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k)$.

ii) If $t_{LC} > t_{RC}$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) =$

$$\begin{cases} TF^{t'}_{t_{\text{HGLS}}}\left(s_k^{n(t')}, \alpha_k\right) & \text{if } \alpha_k \geq 0 \\ TF^{t'+1}_{t_{\text{HGLS}}}(s_{2k}^{n(t'+1)}, \alpha_k) & \text{if } \alpha_k < 0. \end{cases}$$

iii) If $t_{LC} < t_{RC}$, then $\mathcal{TF}^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) =$

$$\begin{cases} TF^{t'}_{t_{\text{HGLS}}}(s_k^{n(t')}, \alpha_k) & \text{if } \alpha_k \leq 0 \\ TF^{t'+1}_{t_{\text{HGLS}}}(s_{2k}^{n(t'+1)}, \alpha_k) & \text{if } \alpha_k > 0. \end{cases}$$

Once unbalanced linguistic assessments are represented in t_{HGLS}, then the computation of unbalanced linguistic information is developed in the expression domain associated with the level t_{HGLS}, i.e., the label set of LH $S^{n(t_{\text{HGLS}})}$.

c) *Compute or aggregate the unbalanced linguistic information by means of the the 2-tuple fuzzy linguistic computational model.*

When we have represented all the unbalanced linguistic assessments to be aggregated by means of linguistic 2-tuples expressed in the same linguistic expression domain $S^{n(t_{\text{HGLS}})}$, then we carry out the CW process of unbalanced linguistic information using any aggregation operator of linguistic 2-tuples F, such as arithmetic mean, weighted average, OWA operators, etc. [22], [24]. An example of operator F is the arithmetic mean operator for linguistic 2-tuples.

Definition 5 [22]: Let $x = \{(r_1, \alpha_1), \ldots, (r_n, \alpha_n)\}$ be a set of linguistic 2-tuples. The 2-tuple arithmetic mean

TABLE VII
ASSESSMENTS OBTAINED IN EACH TEST

	T1	T2	T3	T4	T5	T6
John Smith	D	C	B	C	C	C
Martina Grant	A	D	D	C	B	A

$\bar{x}^e$ is computed as $\bar{x}^e = \Delta(\sum_{i=1}^{n}(1/n)\Delta^{-1}(r_i,\alpha_i)) = \Delta((1/n)\sum_{i=1}^{n}\beta_i)$.

d) *Express the final result in the unbalanced linguistic term set.*

The aggregation operators of linguistic 2-tuples are homogeneous. In this case, this means that if we aggregate linguistic 2-tuples expressed in $S^{n(t_{\mathrm{HGLS}})}$, the aggregation result is also expressed in $S^{n(t_{\mathrm{HGLS}})}$. Therefore, if we want the aggregation operator of unbalanced linguistic information to be homogeneous, we have to require that it returns the aggregation result expressed in $\mathcal{S}$. This is achieved by applying the transformation function $\mathcal{LH}^{-1}$ on the result obtained by F.

According to the steps above and once an LH has been fixed, we define a generic aggregation operator of unbalanced linguistic information.

Definition 6: Let $A = \{(a_1,\alpha_1),\ldots,(a_p,\alpha_p), a_i \in \mathcal{S}, \alpha_i \in [-0.5, 0.5)\}$ be a set of unbalanced linguistic assessments to be aggregated. Then a generic aggregation operator of unbalanced linguistic information $\Lambda^F : (\mathcal{S}\times[-0.5,0.5))^p \rightarrow \mathcal{S}\times[-0.5,0.5)$ is defined according to the following expression: $\Lambda^F[(a_1,\alpha_1),\ldots,(a_p,\alpha_p)] = \mathcal{LH}^{-1}(s_k^{n(t_{\mathrm{HGLS}})},\lambda)$, with $(s_k^{n(t_{\mathrm{HGLS}})},\lambda)$ being the linguistic 2-tuple obtained as $(s_k^{n(t_{\mathrm{HGLS}})},\lambda) = F(\mathcal{TF}_{t_{\mathrm{HGLS}}}^{t'}(\mathcal{LH}(a_1,\alpha_1)),\ldots,\mathcal{TF}_{t_{\mathrm{HGLS}}}^{t'}(\mathcal{LH}(a_p,\alpha_p)))$, $t', t_{\mathrm{HGLS}} = \max\{t_{LE}, t_{LC}, t_{RC}, t_{RE}\}$, and F any aggregation operator of linguistic 2-tuples.

In the following section, we present an example of the application of this unbalanced linguistic computational model.

VI. EXAMPLE ON EDUCATIONAL EVALUATION BASED ON SEVERAL TESTS

A usual problem in education is to evaluate students' knowledge from different tests to obtain a global evaluation.

Let us suppose that two students, John Smith and Martina Grant, have completed six different tests to demonstrate their knowledge and those tests are equally important. The evaluations of tests are assessed using the grading system shown in Fig. 1, which, as we said at the beginning, is an unbalanced linguistic term set $\mathcal{S} = \{F, D, C, B, A\}$. Let us suppose that assessments obtained by pupils in each test are those shown in Table VII. Then, to obtain a final evaluation for each student taking into account all test assessments, we apply our methodology to deal with unbalanced linguistic information.

A. Applying the Representation Algorithm of Unbalanced Linguistic Information

First, we apply the representation algorithm of unbalanced linguistic information to represent the unbalanced labels of $\mathcal{S}$ using the LH shown in Fig. 4. The description of $\mathcal{S}$ in the algorithm is $\{(\mathbf{1}, \mathbf{extreme}), \mathbf{1}, (\mathbf{3}, \mathbf{extreme})\}$. Therefore, $\mathcal{S}_L = \{F\}$, $\mathcal{S}_C = \{D\}$, and $\mathcal{S}_R = \{C, B, A\}$. Then, the representation of $\mathcal{S}$ according to the representation algorithm runs as follows.

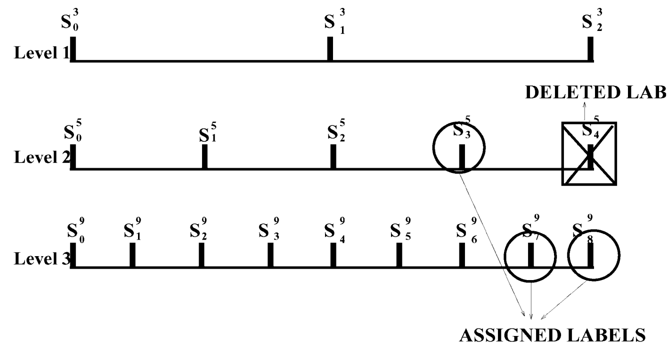

Fig. 20. Labels used to represent $\mathcal{S}_R$.

1) *Representation of* $\mathcal{S}_R$: As in LH, the condition $\exists l(t, n(t))/((n(t) - 1)/2) = \#(\mathcal{S}_R)$ is not satisfied; then we have to look for two levels t and $t+1$ such that $((n(t) - 1)/2) < \#(\mathcal{S}_R) < ((n(t+1) - 1)/2)$. The levels that satisfy the above condition are $t = 2$ and $t + 1 = 3$ because their respective cardinalities in LH are $n(t) = 5$ and $n(t+1) = 9$. Therefore, $\mathrm{lab}_t = 1$ and $\mathrm{lab}_{t+1} = 2$. As $\mathrm{density}_{\mathcal{S}_R} = \mathrm{extreme}$, the representation functions assign_{t+1}^R and assign_t^R represent two labels $\mathcal{S}_{RE} = \{B, A\}$ in level 3 and one label $\mathcal{S}_{RC} = \{C\}$ in level 2, respectively. Applying both functions, we obtain the representation of $\mathcal{S}_{RE}$ and $\mathcal{S}_{RC}$ in LH shown in Fig. 20, i.e., $\{A \leftarrow s_8^9, B \leftarrow s_7^9\}$ and $\{C \leftarrow s_3^5\}$. We point out that the unbalanced label C is a bridge label, i.e., $\mathrm{Brid}(s_2 = C) = $True. Furthermore, in this case we know that the semantics associated with C is obtained as $\underline{C} \leftarrow \underline{s_6^9}$ and $\bar{C} \leftarrow \overline{s_3^5}$.
2) *Representation of* $\underline{\mathcal{S}_C}$: As $\mathrm{density}_{\mathcal{S}_R} = \mathrm{extreme}$, then following the algorithm, the downside of the central label $\underline{D}$ is represented in level 2 of LH by means of $\underline{s_2^5}$.
3) *Representation of* $\mathcal{S}_L$: In this case, in LH the following condition $\exists t \in LH/((n(t) - 1)/2) = \#(\mathcal{S}_L)$ is satisfied with $t = 1$ because $n(1) = 3$. Then, the representation of $\mathcal{S}_L$ is obtained from level 1 of LH as $\{F \leftarrow s_0^3\}$.
4) *Representation of* $\overline{\mathcal{S}_C}$: Therefore, the upside of the central label $\bar{D}$ is represented in level 1 of LH by means of $\overline{s_1^3}$.

Consequently, at the end of the representation algorithm, $\mathcal{S} = \{F, D, C, B, A\}$ is represented in LH using the labels of different levels shown in Fig. 21 and with the semantic representation shown in Fig. 22, that is, $\mathcal{S}_L : \{F \leftarrow s_0^3\}$, $\mathcal{S}_C : \{D \leftarrow \overline{s_1^3} \bigcup \underline{s_2^5}\}$, and $\mathcal{S}_R : \{C \leftarrow \overline{s_3^5} \bigcup \underline{s_6^9}, B \leftarrow s_7^9, A \leftarrow s_8^9\}$.

Furthermore, we require the following information to control the representation of $\mathcal{S}$.

1) $LH(\mathcal{S})$ and $\mathrm{Brid}(\mathcal{S})$, which are given in Table VIII.
2) The following five subsets of unbalanced linguistic labels ordered in increasing order: $\mathcal{S}_{LE} = \mathcal{S}_{LC} = \mathcal{S}_L = \{F\}$, $\mathcal{S}_C = \{D\}$, $\mathcal{S}_{RC} = \{C\}$, $\mathcal{S}_{RE} = \{B, A\}$.

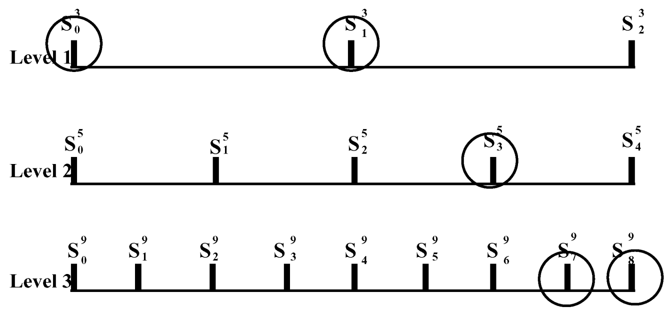

Fig. 21. Labels used in LH to represent $\mathcal{S}$.

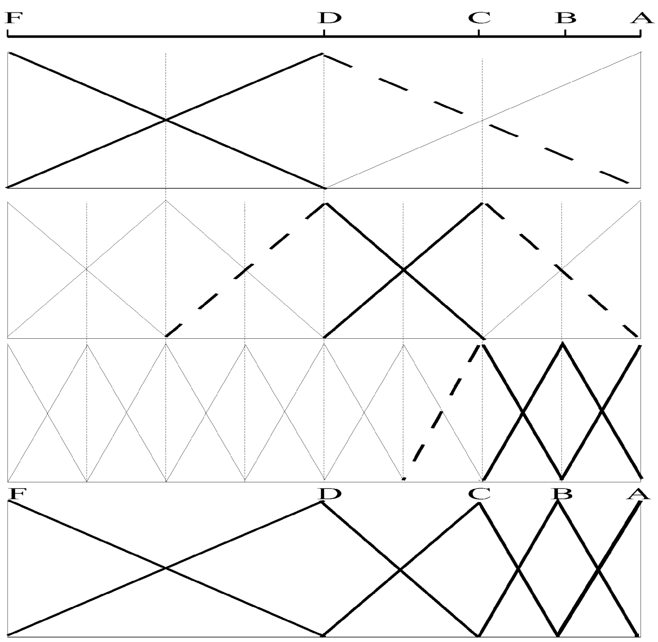

Fig. 22. Semantic representation of the grading system in LH.

TABLE VIII
LH($\mathcal{S}$) AND BRID($\mathcal{S}$)

$\mathcal{S}$	$LH(\mathcal{S})$	$Brid(\mathcal{S})$
$s_0 = F$	$s_{I(0)}^{G(0)} = s_0^3$	False
$s_1 = D$	$s_{I(1)}^{G(1)} = s_1^3$	True
$s_2 = C$	$s_{I(2)}^{G(2)} = s_3^5$	True
$s_3 = B$	$s_{I(3)}^{G(3)} = s_7^9$	False
$s_4 = A$	$s_{I(4)}^{G(4)} = s_8^9$	False

TABLE IX
UNBALANCED LINGUISTIC ASSESSMENTS EXPRESSED IN 2-TUPLES

	T1	T2	T3	T4	T5	T6
J. Smith	(D,0)	(C,0)	(B,0)	(C,0)	(C,0)	(C,0)
M. Grant	(A,0)	(D,0)	(D,0)	(C,0)	(B,0)	(A,0)

3) The set of levels of LH used in the representation of $\mathcal{S}$, $\{t_{LE}, t_{LC}, t_{RC}, t_{RE}\} = \{1, 1, 2, 3\}$.

TABLE X
GLOBAL EVALUATIONS IN $\mathcal{S}$

John Smith	$(C, -.08)$
Martina Grant	$(C, .16)$

B. Obtaining the Global Evaluations by Means of Λ^F

Once the grading system is represented in LH, then we obtain the global evaluations that qualify the pupils' knowledge using an aggregation operator of unbalanced linguistic information Λ^F, with $F = \bar{x}^e$ i.e., the arithmetic mean for 2-tuples given in Definition 5.

First, we transform the partial unbalanced linguistic evaluations into 2-tuple representation (see Table IX). From the 2-tuple unbalanced linguistic assessments shown in Table IX, we obtain the global evaluations for each pupil shown in Table X.

For example, Martina's evaluation is computed using our methodology as follows:

$$\begin{aligned}&(C, .16)\\ &\quad = \Lambda^{\bar{x}^e}[(A,0),(D,0),(D,0),(C,0),(B,0),(A,0)]\\ &\quad = \mathcal{LH}^{-1}\Big(\bar{x}^e\Big(\mathcal{TF}_{t_{\text{HGLS}}}^{t'}(\mathcal{LH}(A,0))\\ &\qquad \mathcal{TF}_{t_{\text{HGLS}}}^{t'}(\mathcal{LH}(D,0))\\ &\qquad \mathcal{TF}_{t_{\text{HGLS}}}^{t'}(\mathcal{LH}(D,0)), \mathcal{TF}_{t_{\text{HGLS}}}^{t'}(\mathcal{LH}(C,0))\\ &\qquad \mathcal{TF}_{t_{\text{HGLS}}}^{t'}(\mathcal{LH}(B,0)), \mathcal{TF}_{t_{\text{HGLS}}}^{t'}(\mathcal{LH}(A,0))\Big)\Big).\end{aligned}$$

As

$$\begin{aligned}t_{\text{HGLS}} &= \max\{1,1,2,3\} = 3\\ \mathcal{LH}(A,0) &= s_8^{n(3)} = s_8^9\\ \mathcal{LH}(D,0) &= s_1^{n(1)} = s_1^3\\ \mathcal{LH}(C,0) &= s_3^{n(2)} = s_3^5\\ \mathcal{LH}(B,0) &= s_7^{n(3)} = s_7^9\end{aligned}$$

then

$$\begin{aligned}&\Lambda^{\bar{x}^e}[(A,0),(D,0),(D,0),(C,0),(B,0),(A,0)]\\ &\quad = \mathcal{LH}^{-1}\left(\bar{x}^e\left(\mathcal{TF}_3^3\left(s_8^9,0\right), \mathcal{TF}_3^1\left(s_1^3,0\right)\right.\right.\\ &\qquad \mathcal{TF}_3^1\left(s_1^3,0\right), \mathcal{TF}_3^2\left(s_3^5,0\right), \mathcal{TF}_3^3\left(s_7^9,0\right)\\ &\qquad \left.\left.\mathcal{TF}_3^3\left(s_8^9,0\right)\right)\right).\end{aligned}$$

On the one hand, $\text{Brid}(s_8^9) = \text{Brid}(s_7^9)$ =false; then $\mathcal{TF}_3^3(s_8^9,0) = TF_3^3(s_8^9,0)$ and $\mathcal{TF}_3^3(s_7^9,0) = TF_3^3(s_7^9,0)$, respectively.

On the other hand, although $\text{Brid}(s_1^3) = \text{Brid}(s_2^5) =$true, as all symbolic translation values are zero then $\mathcal{TF}_3^1(s_1^3,0) = TF_3^1(s_1^3,0)$ and $\mathcal{TF}_3^2(s_3^5,0) = TF_3^2(s_3^5,0)$, respectively. Therefore

$$\begin{aligned}
&\Lambda^{\bar{x}^e}[(A,0),(D,0),(D,0),(C,0),(B,0),(A,0)]\\
&\quad= \mathcal{LH}^{-1}\left(\bar{x}^e\left(TF_3^3\left(s_8^9,0\right),TF_3^1\left(s_1^3,0\right)\right.\right.\\
&\qquad\left.\left. TF_3^1\left(s_1^3,0\right),TF_3^2\left(s_3^5,0\right),TF_3^3\left(s_7^9,0\right),TF_3^3\left(s_8^9,0\right)\right)\right)\\
&\quad= \mathcal{LH}^{-1}\left(\bar{x}^e\left(\left(s_8^9,0\right),\left(s_4^9,0\right),\left(s_4^9,0\right),\left(s_6^9,0\right)\right.\right.\\
&\qquad\left.\left.\left(s_7^9,0\right),\left(s_8^9,0\right)\right)\right)\\
&\quad= \mathcal{LH}^{-1}\left(\Delta\left(\frac{1}{6}\left(\Delta^{-1}\left(s_8^9,0\right)+\Delta^{-1}\left(s_4^9,0\right)\right.\right.\right.\\
&\qquad+\Delta^{-1}\left(s_4^9,0\right)+\Delta^{-1}\left(s_6^9,0\right)\\
&\qquad\left.\left.\left.+\ \Delta^{-1}\left(s_7^9,0\right)+\Delta^{-1}\left(s_8^9,0\right)\right)\right)\right)\\
&\quad= \mathcal{LH}^{-1}\left(\Delta\left(\frac{1}{6}(8+4+4+6+7+8)\right)\right)\\
&\quad= \mathcal{LH}^{-1}\left(\Delta\left(\frac{1}{6}(37)\right)\right) = \mathcal{LH}^{-1}(\Delta(6.16))\\
&= \mathcal{LH}^{-1}\left(s_6^9,.16\right).
\end{aligned}$$

Now, we explain how to apply the transformation function $\mathcal{LH}^{-1}$. Condition 10 given in the definition of $\mathcal{LH}^{-1}$ is not satisfied, and therefore we apply case 2 of its definition, that is, we have to look for a 2-tuple linguistic assessment in $\text{LH}(\mathcal{S})$ that represents the same information as the linguistic 2-tuple $(s_6^9,.16)$. To do that, first we look for the level of LH t' where $(s_6^9,.16)$ should be represented. This is made by calculating $TF_{t'}^3(s_6^9,.16), \forall t' \in \{1,2\}$, i.e., $TF_1^3(s_6^9,.16) = \Delta((\Delta^{-1}(s_6^9,.16)\cdot 2)/8)) = \Delta(1.54) = (s_2^3,-.46)$, and $TF_2^3(s_6^9,.16) = \Delta((\Delta^{-1}(s_6^9,.16)\cdot 4)/8)) = \Delta(3.08) = (s_3^5,.08)$. As $s_3^5 \in LH(\mathcal{S})$, then $t' = 2$. To definitively obtain the 2-tuple linguistic assessment equivalent to $(s_6^9,.16)$, we have to apply case 1 of the definition of $\mathcal{LH}^{-1}$. As the unbalanced label associated with s_3^5 is C and it is a bridge unbalanced label, then we apply concretely the case 1.2(b) of the definition of $\mathcal{LH}^{-1}$, i.e., as $C \in \mathcal{S}_{\mathcal{RC}}$ and 16 represents a symbolic translation value on the downside of s_6^9; then $\mathcal{LH}^{-1}(s_6^9,.16) = (C,.16)$.

VII. Concluding Remarks

In this paper, we have developed a methodology to deal with unbalanced linguistic information, that is, linguistic information assessed in linguistic term sets whose labels are neither uniformly distributed nor symmetric. This methodology is based on the concept of linguistic hierarchy and on the 2-tuple fuzzy linguistic representation model. This methodology is composed of a representation algorithm and a computational approach for unbalanced linguistic information.

This methodology is very useful to model different real world problems dealing with linguistic terms assessed in unbalanced linguistic term sets, such as evaluation processes, decision making, and information retrieval.

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