A Consistency-Based Procedure to Estimate Missing Pairwise Preference Values

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In this paper, we present a procedure to estimate missing preference values when dealing with pairwise comparison and heterogeneous information. This procedure attempts to estimate the missing information in an expert's incomplete preference relation using only the preference values provided by that particular expert. Our procedure to estimate missing values can be applied to incomplete fuzzy, multiplicative, interval-valued, and linguistic preference relations. Clearly, it would be desirable to maintain experts' consistency levels. We make use of the additive consistency property to measure the level of consistency and to guide the procedure in the estimation of the missing values. Finally, conditions that guarantee the success of our procedure in the estimation of all the missing values of an incomplete preference relation are given.

1. INTRODUCTION

Decision-making procedures, which try to find the best alternative(s) from a feasible set, are increasingly being used in various different fields for evaluation, selection, and prioritization purposes. Obviously, the comparison of different alternative actions according to their desirability in decision problems, in many cases, cannot be done using a single criterion or one person. Indeed, in the majority of decision-making problems, procedures have been established to combine opinions

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about alternatives related to different points of view.[1,2] These procedures are based on pair comparisons, in the sense that processes are linked to some degree of credibility of preference of one alternative over another. According to the nature of the information expressed for every pair of alternatives many different representation formats can be used to express preferences: fuzzy preference relations,[3,8] multiplicative preference relations,[9–13] interval-valued preference relations,[14–17] and linguistic preference relations.[18,19]

Since each expert is characterized by their own personal background and experience of the problem to be solved, experts' opinions may differ substantially (educational and cultural factors obviously influence an expert's preferences). This diversity of experts could lead to situations where some of them would not be able to efficiently express any kind of preference degree between two or more of the available options. Indeed, this may be due to an expert not possessing a precise or sufficient level of knowledge of part of the problem, or because that expert is unable to discriminate the degree to which some options are better than others. In these situations such an expert is forced to provide an incomplete fuzzy preference relation.[20]

Usual procedures for group decision-making problems correct this lack of knowledge of a particular expert using the information provided by the rest of the experts together with aggregation procedures.[21] These approaches have several disadvantages. Among them we can cite the following:

- The requirement of multiple experts to estimate the missing value of a particular expert.
- These procedures normally do not take into account the differences between experts' preferences, which could lead to the estimation of a missing value that would not naturally be compatible with the rest of the preference values given by that expert.
- Some of these missing information-retrieval procedures are interactive, that is, they need experts to collaborate in "real time," an option which is not always possible.

In this paper, we put forward a general procedure that attempts to estimate the missing information in any of the above formats of incomplete preference relations: fuzzy, multiplicative, interval valued and linguistic. Our proposal is different to the above procedures because the estimation of missing values in an expert's incomplete preference relation is done using only the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete preference relation is compatible with the rest of the information provided by that expert. In fact, because an important objective in the design of our procedure is to maintain experts' consistency levels, the procedure we propose in this paper is guided by a consistency property and only uses the known preference values. In particular, we use the additive consistency property of a fuzzy preference relation,[22] and its corresponding concept in the other preference relation formats, to define a consistency measure of the expert's information.[23]

To do this, the paper is set out as follows: Section 2 presents the four types of preference relations covered and the definition of an incomplete preference relation. Section 3 deals with the additive transitivity and consistency of preference relations to be used to guide the procedure in the estimation of the missing values. We also derive consistency measures for each one of the preference relation formats based on the additive consistency property. Both the estimation procedure, details of its implementation and examples of its application for each preference relation format are studied in Section 4. In Section 5, sufficient conditions to guarantee that all the missing values of an incomplete preference relation can be estimated are provided. Finally, our concluding remarks are pointed out in Section 6.

2. INCOMPLETE PREFERENCE RELATION

The intensity of preference between any two alternatives of a set of feasible ones $X = \{x_1, \ldots, x_n\}$, $(n \geq 2)$ may be adequately represented by means of a preference relation. Different types of preference relations can be defined according to the domain used to evaluate the intensity of preference. This is expressed in the following definition:

DEFINITION 1. *A preference relation P on a set of alternatives X is characterized by a function $\mu_P : X \times X \longrightarrow D$, where D is the domain of representation of preference degrees.*

When cardinality of X is small, the preference relation may be conveniently represented by an $n \times n$ matrix $P = (p_{ij})$, being $p_{ij} = \mu_P(x_i, x_j)\ \forall i, j \in \{1, \ldots, n\}$ interpreted as the preference degree or intensity of the alternative x_i over x_j measured in D.

The four main types of preference relations used in the literature are fuzzy preference relations, multiplicative preference relations, interval-valued preference relations, and linguistic preference relations.

1. *Fuzzy Preference Relations*. Fuzzy preference relations have been widely used to model preferences for decision-making problems. In this case, a difference scale [0, 1] is used to measure the intensity of preference of one alternative over another.[3,5,6]

   DEFINITION 2. *A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is it is characterized by a membership function*

   $$\mu_P : X \times X \longrightarrow [0, 1]$$

   Every value in the matrix P represents the preference degree or intensity of preference of the alternative x_i over x_j:

   - $p_{ij} = 1/2$ indicates indifference between x_i and x_j $(x_i \sim x_j)$.
   - $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j.
   - $p_{ij} > 1/2$ indicates that x_i is preferred to x_j $(x_i \succ x_j)$.

   On the basis of this interpretation, we have $p_{ii} = 1/2\ \forall i \in \{1, \ldots, n\}$ $(x_i \sim x_i)$.
2. *Multiplicative Preference Relations*. In this case, the intensity of preference represents the ratio of the preference intensity between the alternatives. According to Miller's study,[24] Saaty suggests measuring every value using a ratio scale, precisely the 1–9 scale.[11,12]

DEFINITION 3. *A multiplicative preference relation A on a set of alternatives X is characterized by a function*

$$\mu_A: X \times X \longrightarrow [1/9, 9]$$

The following meanings are associated with the numbers:

1	equally important
3	weakly more important
5	strongly more important
7	demonstrably or very strongly more important
9	absolutely more important
2,4,6,8	compromise between slightly differing judgments

3. *Interval-Valued Preference Relations.* Interval-valued preference relations are used as an alternative to fuzzy preference relations when there exists a difficulty in expressing the preferences with exact numerical values, but there is enough information to be able to estimate the intervals.[14–17]

DEFINITION 4. *An interval-valued preference relation P on a set of alternatives X is characterized by a membership function*

$$\mu_P: X \times X \longrightarrow \mathcal{P}[0, 1]$$

where $\mathcal{P}[0, 1] = \{[a, b] \mid a, b \in [0, 1],\ a \leq b\}$.

An interval-valued preference relation P can be seen as two "independent" fuzzy preference relations, the first one PL corresponding to the left extremes of the intervals and the second one PR to the right extremes of the intervals, respectively,

$$P = (p_{ij}) = ([pl_{ij}, pr_{ij}]) \text{ with } PL = (pl_{ij})\, PR = (pr_{ij}) \text{ and } pl_{ij} \leq pr_{ij}\ \forall i, j.$$

4. *Linguistic Preference Relations Based on the 2-Tuple Linguistic Model.* There are situations where it could be very difficult for the experts to provide precise numerical or interval-valued preferences, in which cases linguistic assessments may be used instead.[18,19,25] In this paper, we will make use of the 2-tuple linguistic model[26,27] to express expert's preferences. Different advantages of this representation to manage linguistic information over semantic and symbolic models were given in Herrera and Martínez[27]:

(a) The linguistic domain can be treated as continuous, whereas in the symbolic model it is treated as discrete.
(b) The linguistic computational model based on linguistic 2-tuples carries out processes of computing with words easily and without loss of information.

This linguistic model takes as a basis the symbolic representation model and in addition defines the concept of symbolic translation to represent the linguistic information by means of a pair of values called linguistic 2-tuple, (s, α), where s is a linguistic term and α is a numeric value representing the symbolic translation.

DEFINITION 5. *Let* $\beta \in [0, g]$ *be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set* $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$*, that is, the result of a* *symbolic aggregation operation. Let* $i = round(\beta)$ *and* $\alpha = \beta - i$ *be two values, such that,* $i \in [0, g]$ *and* $\alpha \in [-0.5, 0.5)$*, then* α *is called a symbolic translation.*

Based on the symbolic translation concept, a linguistic representation model to represent the linguistic information by means of 2-tuples (s_i, α_i), $s_i \in S$ and $\alpha_i \in [-0.5, 0.5)$ was developed. This model defines a set of transformation functions between linguistic terms and 2-tuples, and between numeric values and 2-tuples.

DEFINITION 6. *Let* $S = \{s_0, s_1, \ldots, s_{g-1}, s_g\}$ *be a linguistic term set and* $\beta \in [0, g]$ *a value supporting the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to* β *is obtained with the following function:*

$$\Delta : [0, g] \longrightarrow S \times [-0.5, 0.5)$$

$$\Delta(\beta) = (s_i, \alpha)$$

where $i = round(\beta)$*, that is,* s_i *is the closest index label to "*β*" and "*$\alpha = \beta - i$*" is the value of the symbolic translation.*

There exists a function, Δ^{-1}, such that given a 2-tuple it returns its equivalent numerical value $\beta \in [0, g] \subset \mathbb{R}$:

$$\Delta^{-1} : S \times [-0.5, 0.5) \longrightarrow [0, g]$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta$$

A linguistic term can be seen as a linguistic 2-tuple with a 0 symbolic translation value, that is, $s_i \in S \equiv (s_i, 0)$. Therefore, this linguistic model can be used to provide preference relations:

DEFINITION 7. *A linguistic preference relation* P *on a set of alternatives* X *is a set of 2-tuples on the product set* $X \times X$*, that is, it is characterized by a membership function*

$$\mu_P : X \times X \longrightarrow S \times [-0.5, 0.5).$$

Usual decision-making procedures assume that experts are capable of providing preference degrees between any pair of possible alternatives. However, this may not be always possible, which makes missing information a problem that has to be dealt with. A missing value in a fuzzy preference relation is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another, in which case he/she may decide not to "guess" to maintain the consistency of the values already provided. It must be clear that when an expert is not able to express a particular value p_{ij}, because he/she does not have a clear idea of how the alternative x_i is better than alternative x_j, this does not mean that he/she prefers both options with the same intensity.

To model these situations, in the following we introduce the *incomplete preference relation* concept:

DEFINITION 8. *A function $f: X \longrightarrow Y$ is partial when not every element in the set X necessarily maps onto an element in the set Y. When every element from the set X maps onto one element of the set Y then we have a total function.*

DEFINITION 9. A *preference relation P on a set of alternatives X with a partial membership function is an incomplete preference relation.*

As per this definition, a preference relation is complete when its membership function is a total one. Clearly, Definition 1 includes both definitions of complete and incomplete preference relations. However, as there is no risk of confusion between a complete and an incomplete preference relation, in this paper we refer to the first type as simply preference relations.

3. TRANSITIVITY AND CONSISTENCY OF PREFERENCE RELATIONS

The definition of a preference relation does not imply any kind of consistency property. In fact, the values of a preference relation may be contradictory. Obviously, an inconsistent source of information is not as useful as a consistent one, and thus, it would be quite important to be able to *measure* the consistency of the information provided by experts for a particular problem.

Consistency is usually characterized by *transitivity*, which represents the idea that the preference value obtained by directly comparing two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives. Clearly, different transitivity conditions can be used for different preference relations. In the following, we will introduce the transitivity conditions that will be used in this paper to measure the consistency for each one of the above preference relations.

3.1. Additive and Multiplicative Transitivity Properties

One of the properties suggested to model the concept of transitivity for fuzzy preference relations is the *additive transitivity* property[28]

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \;\; \forall i, j, k \in \{1, \ldots, n\}$$

or equivalently,

$$p_{ik} = p_{ij} + p_{jk} - 0.5 \;\; \forall i, j, k \in \{1, \ldots, n\} \tag{1}$$

This kind of transitivity has the following interpretation[22]: suppose we do want to establish a ranking between three alternatives x_i, x_j, and x_k, and that the information available about these alternatives suggests that we are in an indifference situation, that is, $x_i \sim x_j \sim x_k$. When giving preferences, this situation would be represented by $p_{ij} = p_{jk} = p_{ki} = 0.5$. Suppose now that we have a piece of information that says alternative $x_i \prec x_j$, i.e. $p_{ij} < 0.5$. This means that p_{jk} or p_{ik} have to change, otherwise there would be a contradiction, because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must then conclude that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ji} = p_{ki}$, and so $p_{ij} + p_{jk} + p_{ki} = p_{ij} + p_{jk} + p_{ji} = 1 + 0.5 = 1.5$. We have the same conclusion if $p_{ki} = 0.5$. In the case of being $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ki} has to be equal to or greater than p_{ji}, being equal only in the case of $p_{jk} = 0.5$ as we have already shown. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_k over x_i should be equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is, $p_{ki} - 0.5 = (p_{kj} - 0.5) + (p_{ji} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$.

We consider a fuzzy preference relation to be "additive consistent" when for every three options in the problem $x_i, x_j, x_k \in X$ their associated preference degrees p_{ij}, p_{jk}, p_{ik} fulfill expression (1). An additive consistent fuzzy preference relation will be referred to as *consistent* throughout this paper, as this is the only transitivity property we are considering.

In Chiclana et al.[29], we studied the transformation function between (reciprocal) multiplicative preference relations with values in the interval scale $[1/9, 9]$ and (reciprocal) fuzzy preference relations with values in $[0, 1]$. This study can be summarized in the following proposition:

PROPOSITION 1. *Suppose that we have a set of alternatives, $X = \{x_1, \ldots, x_n\}$, and associated with it a multiplicative reciprocal preference relation $A = (a_{ij})$, with $a_{ij} \in [1/9, 9]$ and $a_{ij} \cdot a_{ji} = 1 \quad \forall i, j$. Then, the corresponding fuzzy reciprocal preference relation, $P = (p_{ij})$, associated with A, with $p_{ij} \in [0, 1]$ and $p_{ij} + p_{ji} = 1 \quad \forall i, j$ is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij})$$

The above transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. Indeed, the additive transitivity condition for fuzzy preference relations corresponds to the following *multiplicative transitivity* condition for multiplicative preference relations[22]:

$$a_{ik} = a_{ij} \cdot a_{jk} \quad \forall i, j, k. \tag{2}$$

Expression (2) coincides with the original consistency property for multiplicative preference relations defined by Saaty.[11] This result supports the choice of the additive transitivity property to model consistency of fuzzy preference relations.

A multiplicative preference relation will be considered consistent when for every three alternatives, (x_i, x_j, x_k), their associated preference values verify (2).

3.2. Extending the Additive Transitivity Property to the Interval-Valued and Linguistic Cases

Additive transitivity property can be used to define a consistency property for both interval-valued preference relations and linguistic preference relations based on the 2-tuple linguistic model.

DEFINITION 10. *An interval-valued preference relation P is additive consistent if both left and right interval preference relations (PL, PR) are additive consistent, that is,*

$$pl_{ik} = pl_{ij} + pl_{jk} - 0.5 \ \ and \ \ pr_{ik} = pr_{ij} + pr_{jk} - 0.5 \ \forall i, j, k \in \{1, \ldots, n\}$$

Using the function Δ^{-1} that transforms 2-tuple values into numerical values in $[0, g]$, we adapt the definition of additive transitivity to linguistic preference relations as follows:

DEFINITION 11. *A linguistic preference relation will be considered consistent if for every three alternatives x_i, x_j, and x_k, the following condition holds:*

$$p_{ik} = \Delta\left(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \frac{g}{2}\right) \forall i, j, k \in \{1, \ldots, n\} \tag{3}$$

3.3. Consistency Measures for Preference Relations

The transitivity conditions presented in the previous sections allow us to find out whether a preference relation is consistent. However, they do not directly offer the possibility of measuring the "level of inconsistency." In Alonso et al.[23], we defined a consistency measure for fuzzy preference relations based on the additive transitivity property for fuzzy preference relations. In this section, we will extend this consistency measure to multiplicative, interval-valued, and linguistic preference relations.

For fuzzy preference relations, expression (1) can be used to obtain the following three estimates of the preference degree of alternative x_i over alternative x_k, p_{ik}, using an intermediate alternative x_j:

$$cp_{ik}^{j1} = p_{ij} + p_{jk} - 0.5\,; \ \ cp_{ik}^{j2} = p_{jk} - p_{ji} + 0.5; \ \ cp_{ik}^{j3} = p_{ij} - p_{kj} + 0.5 \tag{4}$$

Obviously, when the information provided in a fuzzy preference relation is completely consistent then cp_{ik}^{jl} $(\forall j \in \{1, \ldots, n\},\ \ \forall l \in \{1, 2, 3\})$ and p_{ik} coincide.

However, the information given in fuzzy preference relations does not usually fulfill (1). In such cases, the value

$$\varepsilon p_{ik} = \frac{1}{3}\left(\sum_{l\in\{1,2,3\}} \frac{\sum_{j=1,j\neq i,k}^{n} t_{ik}^{jl}}{(n-2)}\right), \text{ where } t_{ik}^{jl} = \left|cp_{ik}^{jl} - p_{ik}\right|$$

can be used to measure the error expressed in a preference degree between two options or alternatives. This error can also be interpreted as the consistency level between the preference value p_{ik} and the rest of the preference values of the fuzzy preference relation. Clearly, if $\varepsilon p_{ik} = 0$ then there is no inconsistency at all, and the higher the value of εp_{ik} the more inconsistent p_{ik} is with respect to the rest of the information.

The value

$$CL_P = \frac{\sum_{i,k=1,i\neq k}^{n} \varepsilon p_{ik}}{n^2 - n} \tag{5}$$

can be used to measure the *consistency level* of a fuzzy preference relation P. If $CL_P = 0$ then the preference relation P is fully consistent, otherwise, the higher CL_P the more inconsistent P is.

The measurement of consistency of multiplicative preference relations follows a similar process as mentioned above. Indeed, the estimate values ca_{ik}^{jl} are obtained using expression (2):

$$ca_{ik}^{j1} = a_{ij} \cdot a_{jk};\quad ca_{ik}^{j2} = \frac{a_{jk}}{a_{ji}};\quad ca_{ik}^{j3} = \frac{a_{ij}}{a_{kj}} \tag{6}$$

and the error between a_{ik} and every ca_{ik}^{jl} is defined as the following ratio:

$$\varepsilon a_{ik} = \left(\prod_{l\in\{1,2,3\}} \left(\prod_{j=1,j\neq i,k}^{n} t_{ik}^{jl}\right)^{1/(n-2)}\right)^{1/3}, \quad \text{where } t_{ik}^{jl} = \left\{\max\left(\frac{ca_{ik}^{jl}}{a_{ik}}, \frac{a_{ik}}{ca_{ik}^{jl}}\right)\right\}$$

Clearly, if $\varepsilon a_{ik} = 1$ then the preference degree a_{ik} is consistent with the rest of information in the multiplicative preference relation. Otherwise, the higher εa_{ik}, the more inconsistent a_{ik} is with respect to the rest of the information. Therefore, the value

$$CL_A = \left(\prod_{i,k=1,i\neq k}^{n} \varepsilon a_{ik}\right)^{1/(n^2-n)} \tag{7}$$

can be used to measure the consistency level of a multiplicative preference relation. If $CL_A = 1$ then the multiplicative preference relation is fully consistent, otherwise, the higher CL_A the more inconsistent A is.

The consistency level of an interval-valued preference relation is measured using the corresponding consistency levels of both PL and PR:

$$CL_P = (CL_{PL}, CL_{PR}) = \left(\frac{\sum_{i,k=1,i\neq k}^{n} \varepsilon pl_{ik}}{n^2 - n}, \frac{\sum_{i,k=1,i\neq k}^{n} \varepsilon pr_{ik}}{n^2 - n} \right)$$

When $CL_P = (0, 0)$, the interval-valued preference relation is completely consistent.

For linguistic preference relations, we use expression (5) in conjunction with (3) to define its consistency level as follows:

$$cp_{ik}^{j1} = \Delta\left(\Delta^{-1}(p_{ij}) + \Delta^{-1}(p_{jk}) - \frac{g}{2}\right);\ cp_{ik}^{j2} = \Delta\left(\Delta^{-1}(p_{jk}) - \Delta^{-1}(p_{ji}) + \frac{g}{2}\right);$$

$$cp_{ik}^{j3} = \Delta\left(\Delta^{-1}(p_{ij}) - \Delta^{-1}(p_{kj}) + \frac{g}{2}\right)$$

$$\varepsilon p_{ik} = \frac{1}{3}\left(\sum_{l\in\{1,2,3\}} \frac{\sum_{j=1,j\neq i,k}^{n} t_{ik}^{jl}}{(n-2)}\right), \quad \text{where} \quad t_{ik}^{jl} = \left|\Delta^{-1}\left(cp_{ik}^{jl}\right) - \Delta^{-1}(p_{ik})\right|$$

and

$$CL_P = \frac{\sum_{i,k=1,i\neq k}^{n} \varepsilon p_{ik}}{n^2 - n}$$

When $\varepsilon p_{ik} = 0$, the preference degree p_{ik} is consistent with respect to the rest of information in the preference relation. The linguistic preference relation is consistent when $CL_P = 0$.

When working with an incomplete preference relation, the above expressions for CL_P and CL_A cannot be used, and therefore have to be extended. To do this, the following sets are introduced:

$$B = \{(i, j) \mid i, j \in \{1, \ldots, n\} \wedge i \neq j\}$$

$$MV = \left\{(i, j) \in B \mid p_{ij} \text{ is unknown}\right\}$$

$$EV = B \setminus MV$$

MV is the set of pairs of alternatives for which the preference degree of the first alternative over the second one is unknown or missing; EV is the set of pairs of alternatives for which the expert provides preference values. Note that we do not take into account the preference value of one alternative over itself, as $x_i \sim x_i$ is always assumed.

We provide the necessary changes and the expression to compute the consistency level of an incomplete fuzzy preference relation:

$$H_{ik}^1 = \{j \neq i, k \mid (i, j), (j, k) \in EV\}\,; H_{ik}^2 = \{j \neq i, k \mid (j, k), (j, i) \in EV\};$$

$$H_{ik}^3 = \{j \neq i, k \mid (i, j), (k, j) \in EV\}$$

$$\mathcal{K} = \left\{l \in \{1, 2, 3\} \mid H_{ik}^l \neq \emptyset\right\}; \;\; CE_P = \left\{(i, k) \in EV \mid \exists j \in H_{ik}^1 \cup H_{ik}^2 \cup H_{ik}^3\right\}$$

$$\varepsilon p_{ik} = \frac{1}{\#\mathcal{K}} \left(\sum_{l \in \mathcal{K}} \frac{\sum_{j \in H_{ik}^l}^n t_{ik}^{jl}}{\#H_{ik}^l} \right); \;\; CL_P = \frac{\sum_{(i,k) \in CE_P} \varepsilon p_{ik}}{\#CE_P}$$

We call CE_P the *computable error* set because it contains all the elements for which we can compute every εp_{ik}. Clearly, this redefinition of CL_P is an extension of expression (5). Indeed, when a fuzzy preference relation is complete, both CE_P and B coincide and thus $\#CE_P = n^2 - n$.

4. ESTIMATION OF MISSING VALUES IN PREFERENCE RELATIONS

As we have already mentioned, missing information is a problem that has to be addressed because experts are not always able to provide preference degrees between every pair of possible alternatives. This section is devoted to the presentation of an iterative procedure to estimate missing values of incomplete preference relations and the sufficient conditions to guarantee the successful estimation of all the missing values. First, we will describe the general procedure and we will then point out the implementation details for each type of preference relations. Appropriate examples will be used to illustrate the application of the iterative procedure.

4.1. General Procedure

To develop the iterative procedure to estimate missing values, two different tasks have to be carried out: (A) establish the elements that can be estimated in each step of the procedure, and (B) produce the particular expression that will be used to estimate a particular missing value.

4.1.1. Elements to be Estimated in Every Iteration of the Procedure

The subset of missing values MV that can be estimated in step h of our procedure is denoted by EMV_h (*estimated missing values*) and obtained as follows:

$$KV_h = EV \cup \left(\bigcup_{l=0}^{h-1} EMV_l \right); \;\; UV_h = MV \setminus \left(\bigcup_{l=0}^{h-1} EMV_l \right);$$

$$EMV_h = \left\{(i, k) \in UV_h \;\middle|\; \exists j \in H_{ik}^1 \cup H_{ik}^2 \cup H_{ik}^3 \right\}$$

with $EMV_0 = \emptyset$, and where KV_h stands for *known values in iteration h* and UV_h means *unknown values in iteration h*.

When $EMV_{maxIter} = \emptyset$ with $maxIter > 0$, the procedure stops because there will not be any more missing values to be estimated. Furthermore, if $\bigcup_{l=0}^{maxIter} EMV_l = MV$ then all missing values are estimated and consequently the procedure is said to be successful in the completion of the fuzzy preference relation.

4.1.2. Expression to Estimate a Particular Missing Value

In iteration h, the following function is applied to obtain the estimate cp'_{ik} of a missing preference value with $(i, k) \in EMV_h$:

function estimate_p(i,k)

0. $\mathcal{K} = \emptyset$
1.1. $H_{ik}^1 = \{j \neq i, k \mid (i, j), (j, k) \in KV_h\}$; if $(H_{ik}^1 \neq \emptyset)$ then $\mathcal{K} = \mathcal{K} \cup \{1\}$
1.2. $H_{ik}^2 = \{j \neq i, k \mid (j, k), (j, i) \in KV_h\}$; if $(H_{ik}^2 \neq \emptyset)$ then $\mathcal{K} = \mathcal{K} \cup \{2\}$
1.3. $H_{ik}^3 = \{j \neq i, k \mid (i, j), (k, j) \in KV_h\}$; if $(H_{ik}^3 \neq \emptyset)$ then $\mathcal{K} = \mathcal{K} \cup \{3\}$
2. Calculate cp'_{ik}
3. Assure that cp'_{ik} belongs to the range of its corresponding type of preference relation

end function

The function $estimate\_p(i, k)$ computes the final estimate value of the missing preference degree of the alternative x_i over the alternative x_k as the corresponding average of all the estimate values that can be calculated using all the possible intermediate alternatives x_j for each one of the three possible expressions. Finally, the function checks if the estimated value is in the range of the particular type of preference relation. Steps 2 and 3 are dependent on the type of preference relation, and will be described later.

The general *iterative estimating procedure pseudocode* is

0. Initializations
1. $EMV_0 = \emptyset$
2. $h = 1$
3. while $EMV_h \neq \emptyset$ {
4.   for every $(i, k) \in EMV_h$ {
5.   function estimate_p(i,k)
6.   }
7.   h++
8. }
9. Post-processing operations

Both *initializations* and *postprocessing operations* steps are dependent on the type of preference relation.

4.2. Fuzzy Preference Relations: Implementation Details

In this case, steps 2 and 3 of the estimation function are

2. Calculate $cp'_{ik} = \frac{1}{\#\mathcal{K}} \left(\sum_{l \in \mathcal{K}} \frac{\sum_{j \in H^l_{ik}}^{n} cp^{jl}_{ik}}{\#H^l_{ik}} \right)$

3.1. if $cp'_{ik} < 0$ then $p_{ik} = 0$
3.2. else if $cp'_{ik} > 1$ then $p_{ik} = 1$
3.3. else $p_{ik} = cp'_{ik}$

There is no need to implement any special initialization nor any kind of post-processing operations in this case. The following example illustrates the application of the above procedure.

Example 1. Suppose that an expert provides the following incomplete fuzzy preference relation P over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$.

$$P = \begin{pmatrix} - & x & 0.4 & x \\ x & - & 0.7 & 0.85 \\ x & 0.4 & - & 0.75 \\ 0.3 & x & x & - \end{pmatrix}$$

With the application of only the first iteration of our procedure, all the missing values are successfully estimated:

$$\begin{aligned}
&H^1_{12} = \{3\}\ H^2_{12} = \emptyset \quad H^3_{12} = \{3\} \Rightarrow cp'_{12} = \tfrac{1}{2}((p_{13} + p_{32} - 0.5) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (p_{13} - p_{23} + 0.5)) = 0.25 \\
&H^1_{14} = \{3\}\ H^2_{14} = \emptyset \quad H^3_{14} = \emptyset \quad \Rightarrow cp'_{14} = p_{13} + p_{34} - 0.5 = 0.65 \\
&H^1_{21} = \{4\}\ H^2_{21} = \emptyset \quad H^3_{21} = \{3\} \Rightarrow cp'_{21} = \tfrac{1}{2}((p_{24} + p_{41} - 0.5) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (p_{23} - p_{13} + 0.5)) = 0.725 \approx 0.73 \\
&H^1_{31} = \{4\}\ H^2_{31} = \emptyset \quad H^3_{31} = \emptyset \quad \Rightarrow cp'_{31} = p_{34} + p_{41} - 0.5 = 0.55 \\
&H^1_{42} = \emptyset \quad H^2_{42} = \{2\}\ H^3_{42} = \emptyset \quad \Rightarrow cp'_{42} = p_{32} - p_{34} + 0.5 = 0.15 \\
&H^1_{43} = \{1\}\ H^2_{43} = \{2\}\ H^3_{43} = \emptyset \quad \Rightarrow cp'_{43} = \tfrac{1}{2}((p_{41} + p_{13} - 0.5) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (p_{23} - p_{24} + 0.5)) = 0.275 \approx 0.28
\end{aligned}$$

Therefore, we have

$$\begin{pmatrix} - & x & 0.4 & x \\ x & - & 0.7 & 0.85 \\ x & 0.4 & - & 0.75 \\ 0.3 & x & x & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.25 & 0.4 & 0.65 \\ 0.73 & - & 0.7 & 0.85 \\ 0.55 & 0.4 & - & 0.75 \\ 0.3 & 0.15 & 0.27 & - \end{pmatrix}$$

The consistency levels for the incomplete fuzzy preference relation is

$$CE_P = \{(2,3),(2,4),(3,2),(3,4)\} \;\Rightarrow\; \epsilon p_{23} = 0.1;\;\; \epsilon p_{24} = 0.05;$$

$$\epsilon p_{32} = 0;\;\; \epsilon p_{34} = 0.05 \;\;\Rightarrow$$

$$CL_P = (\epsilon p_{23} + \epsilon p_{24} + \epsilon p_{32} + \epsilon p_{34})/4 = 0.05$$

The consistency level for the complete fuzzy preference relation is 0.054, which is quite similar to the above one.

4.3. Multiplicative Preference Relations: Implementation Details

For incomplete multiplicative preference relations, we adapt the estimation function in the following way:

2. Calculate $cp'_{ik} = \left(\prod_{l \in \mathcal{K}} \left(\prod_{j \in H^l_{ik}}^{n} ca^{jl}_{ik} \right)^{1/\#H^l_{ik}} \right)^{1/\#\mathcal{K}}$

3.1. if $cp'_{ik} < 1/9$ then $a_{ik} = 1/9$
3.2. else if $cp'_{ik} > 9$ then $a_{ik} = 9$
3.3. else $a_{ik} = cp'_{ik}$

Example 2. Suppose that we have the following incomplete multiplicative preference relation over a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$:

$$A = \begin{pmatrix} - & 0.80 & 1.55 & 1 \\ 1.25 & - & x & 3.74 \\ 0.65 & x & - & 1.93 \\ 1 & 0.33 & 0.52 & - \end{pmatrix}$$

The complete multiplicative preference relation obtained after just one iteration is

$$\begin{pmatrix} - & 0.80 & 1.55 & 1 \\ 1.25 & - & x & 3.74 \\ 0.65 & x & - & 1.93 \\ 1 & 0.33 & 0.52 & - \end{pmatrix} \to \begin{pmatrix} - & 0.80 & 1.55 & 1 \\ 1.25 & - & 1.87 & 3.74 \\ 0.65 & 0.56 & - & 1.93 \\ 1 & 0.33 & 0.52 & - \end{pmatrix}$$

4.4. Interval-Valued Preference Relations: Implementation Details

Interval-valued preference relations need some initialization steps to create the left and right interval fuzzy preference relations, PL and PR, as well as postprocessing operations to unify the estimated left and right interval fuzzy preference relations into the final estimated interval-valued preference relation. Also, steps 2 and 3 are

2. Calculate $cp'_{ik} = (cpl'_{ik}, cpr'_{ik})$

$$= \left(\frac{1}{\#\mathcal{K}} \left(\sum_{l \in \mathcal{K}} \frac{\sum_{j \in H^l_{ik}}^{n} cpl^{jl}_{ik}}{\#H^l_{ik}} \right), \frac{1}{\#\mathcal{K}} \left(\sum_{l \in \mathcal{K}} \frac{\sum_{j \in H^l_{ik}}^{n} cpr^{jl}_{ik}}{\#H^l_{ik}} \right) \right)$$

3.1. if $cpl'_{ik} < 0$ then $pl_{ik} = 0$ else $pl_{ik} = cpl'_{ik}$
3.2. if $cpr'_{ik} > 1$ then $pr_{ik} = 1$ else $pr_{ik} = cpr'_{ik}$
3.3. $p_{ik} = (pl_{ik}, pr_{ik})$

Example 3. Suppose a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$, and the following incomplete interval-valued preference relation:

$$P = \begin{pmatrix} - & (0.45, 0.60) & (0.55, 0.75) & (0.30, 0.40) \\ (0.40, 0.55) & - & (0.45, 0.80) & x \\ (0.25, 0.45) & (0.20, 0.55) & - & x \\ (0.60, 0.70) & x & x & - \end{pmatrix}$$

In this case, our procedure is capable of estimating the missing values in just one iteration:

$$\begin{pmatrix} - & (0.45, 0.60) & (0.55, 0.75) & (0.30, 0.40) \\ (0.40, 0.55) & - & (0.45, 0.80) & x \\ (0.25, 0.45) & (0.20, 0.55) & - & x \\ (0.60, 0.70) & x & x & - \end{pmatrix} \to \begin{pmatrix} - & (0.45, 0.60) & (0.55, 0.75) & (0.30, 0.40) \\ (0.40, 0.55) & - & (0.45, 0.80) & (0.28, 0.37) \\ (0.25, 0.45) & (0.20, 0.55) & - & (0.15, 0.25) \\ (0.60, 0.70) & (0.63, 0.72) & (0.75, 0.85) & - \end{pmatrix}$$

4.5. Linguistic Preference Relations: Implementation Details

In the initialization step for linguistic preference relations, we apply the transformation function Δ^{-1} to obtain a numeric preference relation. As a postprocessing operation, the estimated numeric preference relation is transformed back into a linguistic preference relation by applying the inverse transformation function, Δ. Steps 2 and 3 for this case are

2. Calculate $cp'_{ik} = \frac{1}{\#\mathcal{K}} \left(\sum_{l \in \mathcal{K}} \frac{\sum_{j \in H^l_{ik}}^{n} \Delta^{-1}\left(cp^{jl}_{ik}\right)}{\#H^l_{ik}} \right)$

3.1. if $cp'_{ik} < 0$ then $cp'_{ik} = 0$

3.2. else if $cp'_{ik} > g$ then $cp'_{ik} = g$

3.3. $p_{ik} = \Delta(cp'_{ik})$

Example 4. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives and $S = \{MW, W, E, B, MB\}$ the set of linguistic labels used to provide preferences, with the following meaning:

MW = Much Worse W = Worse E = Equally Preferred

B = Better MB = Much Better

Suppose the following incomplete linguistic preference relation

$$P = \begin{pmatrix} - & x & W & x \\ x & - & x & E \\ B & x & - & W \\ x & B & B & - \end{pmatrix}$$

Note that the expert did not provide any α values, which is a common practice when expressing preferences with linguistic terms. In these cases, we set $\alpha = 0$

$$P = \begin{pmatrix} - & x & (W, 0) & x \\ x & - & x & (E, 0) \\ (B, 0) & x & - & (W, 0) \\ x & (B, 0) & (B, 0) & - \end{pmatrix}$$

First, we apply Δ^{-1} to obtain the corresponding β values in [0, 4]:

$$P = \begin{pmatrix} - & x & (W,0) & x \\ x & - & x & (E,0) \\ (B,0) & x & - & (W,0) \\ x & (B,0) & (B,0) & - \end{pmatrix} \xrightarrow{\Delta^{-1}} \begin{pmatrix} - & x & 1 & x \\ x & - & x & 2 \\ 3 & x & - & 1 \\ x & 3 & 3 & - \end{pmatrix}$$

Applying the estimation procedure, we have:

$$\begin{pmatrix} - & x & 1 & x \\ x & - & x & 2 \\ 3 & x & - & 1 \\ x & 3 & 3 & - \end{pmatrix} \rightarrow \begin{pmatrix} - & x & 1 & 0 \\ x & - & 2.67 & 2 \\ 3 & 1.67 & - & 1 \\ 4 & 3 & 3 & - \end{pmatrix} \rightarrow \begin{pmatrix} - & 0.61 & 1 & 0 \\ 3.61 & - & 2.67 & 2 \\ 3 & 1.67 & - & 1 \\ 4 & 3 & 3 & - \end{pmatrix}$$

Finally, the application of Δ produces the final 2-tuple linguistic preference relation:

$$\begin{pmatrix} - & 0.61 & 1 & 0 \\ 3.61 & - & 2.67 & 2 \\ 3 & 1.67 & - & 1 \\ 4 & 3 & 3 & - \end{pmatrix} \xrightarrow{\Delta} \begin{pmatrix} - & (W,-0.39) & (W,0) & (MW,0) \\ (MB,-0.39) & - & (B,-0.33) & (W,0) \\ (B,0) & (E,-0.33) & - & (W,0) \\ (MB,0) & (B,0) & (B,0) & - \end{pmatrix}$$

5. SUFFICIENT CONDITIONS TO ESTIMATE ALL MISSING VALUES

It is very important to establish conditions that guarantee that all the missing values of an incomplete preference relation can be estimated. In the following, we provide sufficient conditions to guarantee the success of the above procedure.

1. If for all $(i,k) \in MV$ $(i \neq k)$, there exists at least a $j \in H_{ik}^1 \cup H_{ik}^2 \cup H_{ik}^3$, then all missing preference values can be estimated in the first iteration of the procedure ($EMV_1 = MV$).
2. Under the assumption of the additive consistency property, a different sufficient condition was given in Herrera-Viedma et al.[22] This condition states that any incomplete preference relation can be completed when the preference values of the first alternative over the second one is known for the following set of $n-1$ pairs of alternatives $\{(x_1,x_2),(x_2,x_3),\ldots,(x_{n-1},x_n)\}$.
3. In Herrera-Viedma et al.[30], a more general condition than the previous one is that of having a set of $n-1$ nonleading diagonal preference values, where each one of the alternatives is compared at least once. This general case includes the one when a complete row or column of preference values is known.

As a consequence of this last condition, the only cases where an incomplete preference relation cannot be completed using the above procedure are those where

Table I. Pairs of values that allow the estimation of missing values in iteration 1.

Missing value (i, k)	Pairs of values to estimate p_{ik}
$(1, 4)$	$(1, 2), (2, 4);\ (2, 4), (2, 1)$
$(2, 5)$	$(2, 4), (5, 4)$
$(4, 2)$	$(4, 1), (1, 2);\ (4, 1), (2, 1)$
$(5, 1)$	$(5, 4), (4, 1)$
$(5, 2)$	$(5, 4), (2, 4)$

there is one or more alternative for which no preference degree is known. This is illustrated in the following example:

Example 5. Suppose an expert provides the following incomplete preference relation over a set of five different alternatives, $X = \{x_1, x_2, x_3, x_4, x_5\}$,

$$P = \begin{pmatrix} - & e & x & x & x \\ e & - & x & e & x \\ x & x & - & x & x \\ e & x & x & - & x \\ x & x & x & e & - \end{pmatrix}$$

where x means "a missing value" and e means "a value is known." We note that the actual values of the known preference values or even the type of preference relation are not relevant for the purpose of this example.

At the beginning of our iterative procedure, we have

$$EMV_1 = \{(1, 4), (2, 5), (4, 2), (5, 1), (5, 2)\}$$

and Table I shows all the pairs of alternatives that are available to estimate each one of the above missing values.

At this point, the preference relation is

$$P = \begin{pmatrix} - & e & x & 1 & x \\ e & - & x & e & 1 \\ x & x & - & x & x \\ e & 1 & x & - & x \\ 1 & 1 & x & e & - \end{pmatrix}$$

where 1 values mean the ones that have been reconstructed in the first iteration.

In iteration 2, the estimated values of iteration 1 are added to the values expressed directly by the expert to construct the set EMV_2. In our case, we have $EMV_2 = \{(1, 5), (4, 5)\}$ and Table II.

Table II. Pairs of values that allow the estimation of missing values in iteration 2.

Missing value (i, k)	Pairs of values to estimate p_{ik}
(1, 5)	(1, 2), (2, 5); (2, 5), (2, 1); (1, 2), (5, 2); (1, 4), (5, 4)
(4, 5)	(4, 2), (2, 5); (2, 5), (2, 4); (4, 1), (5, 1); (4, 2), (5, 2)

The incomplete fuzzy preference relation obtained is

$$P = \begin{pmatrix} - & e & x & 1 & 2 \\ e & - & x & e & 1 \\ x & x & - & x & x \\ e & 1 & x & - & 2 \\ 1 & 1 & x & e & - \end{pmatrix}$$

where numbers 1 and 2 indicate the steps in which the missing values were estimated.

In iteration 3, $EMV_3 = \emptyset$, and thus, the procedure ends and fails in the completion of the preference relation. The reason for this failure is that the expert did not provide any information for the alternative x_3 ($p_{3i}, p_{i3} \in MV,\ \forall i, 1 \leq i \leq n, i \neq 3$). Fortunately, this kind of situation is not very common in real problems, and therefore the procedure will usually succeed in estimating all the missing values of an incomplete preference relation.

6. CONCLUSIONS

We have looked at the issue of incomplete preference relations, that is, preference relations with some values missing or not known. We have proposed an iterative procedure to estimate missing preference values in different types of incomplete preference relations: fuzzy, multiplicative, interval valued, and linguistic preference relations. Our proposal estimates the missing information in an expert's incomplete preference relation using only the preference values provided by that particular expert. By doing this, we assure that the reconstruction of the incomplete preference relation is compatible with the rest of the information provided by that expert. Because an important objective in the design of our procedure was to maintain experts' consistency levels, the procedure is guided by the additive consistency property. Also, measures of consistency based on the additive consistency property have been introduced for each one of the four types of preference relations.

We have provided conditions under which all the missing values on incomplete preference relations can be estimated using the proposed iterative procedure. However, there may still be cases in which not every missing value in an incomplete preference relation can be estimated using this procedure. This is a problem that was not covered in this paper, being an issue for further research in the near future.

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