The Ordered Weighted Geometric Operator: Properties and Application in MCDM Problems

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Abstract. The aim of this paper is to present the Ordered Weighted Geometric (OWG) operator. The OWG operator is based on the geometric mean and the OWA operator. It is a fuzzy majority guided aggregation operator proposed to aggregate information given on a ratio scale. Therefore, it allows us to incorporate the concept of fuzzy majority in problems where the information is provided using a ratio scale. Its properties are studied and an application for multicriteria decision making problems with multiplicative preference relations is presented [1].

Keywords: Aggregation, fuzzy majority, decision making, multiplicative preference relations.

1 Introduction

The measurement process for modeling problems consists in the construction of scales by mapping or transforming empirical results into numerical ones in such a way that the information is preserved. We can use different scales to represent that information. Scales which are available, in increasing order of strenght, are as follows [6]:

1. The *nominal scale,* unique up to any 1-1 transformation, which consists essentially of assigning labels to objects.
2. The *ordinal scale* which gives a rank order of objects and is invariant under monotone increasing transformations.
3. The *interval scale,* unique up to positive linear transformation of the form $y = ax + b,\ a > 0$.
4. The *difference scale,* invariant under a transformation of the form y=x+b.
5. The *ratio scale,* invariant under positive linear transformations of the form $y = ax,\ a > 0$.

[1] This research is supported by CICYT and by project PB-981305.

Particularly, in this contribution we are interested in problems where the ratio scale is used to model the information. This is the case, for example, in muticriteria decision making problems modelled by the *Analytic Hierarchy Process* (AHP) designed by Saaty [6,7], in which the performances on the alternatives are ratio-scale measurements provided on the pairwise comparisons of alternatives.

Multicriteria decision making problems are usually solved in a two-phase process [5]:

1. *Aggregation phase:* Using an aggregation operator the performance degrees of alternatives are aggregated for all criteria.
2. *Exploitation phase:* Applying a choice mechanism on the aggregated performance degrees a global ranking of alternatives is achieved.

In any multicriteria decision process the final solution must be obtained from the synthesis of performance degrees of the *majority* of criteria. The majority is traditionally defined as a threshold number of elements. This concept is not always included in the multicriteria decision process. The *fuzzy logic* provides one way to model it.

Fuzzy majority is a soft majority concept expressed by a *fuzzy quantifier* [9], which is manipulated via a fuzzy logic based calculus of linguistically quantified propositions. Therefore, using fuzzy majority guided aggregation operators we can incorporate the concept of majority in the computation of the solution. These operators reflect the fuzzy majority calculating their weights by means of the fuzzy quantifiers, as for example, the *Ordered Weighted Averaging (OWA)* [8].

As shown in [1,2], the proper aggregation operator of ratio-scale measurements is the *geometric mean* and is not the *arithmetic mean*. However, this operator does not allow to incorporate the concept of fuzzy majority in the decision processes. We could use the OWA operator, but this is not possible because it presents a similar behavior to the arithmetic mean.

In this paper, we present a fuzzy majority guided aggregation operator for synthesizing ratio-scale judgements. We define the *Ordered Weighted Geometric (OWG)* operator. It is based on the geometric mean and the OWA operator. It allows to incorporate the concept of fuzzy majority in the decision process when the information is provided using a ratio scale. We study its properties and present an application for multicriteria decision making processes with multiplicative preference relations.

In order to do this, the paper is set out as follows. The concept of fuzzy majority and the OWA and geometric mean operators are introduced in section 2. Section 3 is devoted to present the OWG operator and its properties. An example of its use in multicriteria decision making is given in section 4. Finally, some concluding remarks are pointed out in section 5.

2 Preliminaries

In this section we present the fuzzy quantifiers, used to representing the fuzzy majority, the OWA operator and the geometric mean.

2.1 Fuzzy Majority

Quantifiers can be used to represent the amount of items satisfying a given predicate. Classic logic is restricted to the use of only two quantifiers, *there exists* and *for all*, that are closely related respectively to the *or* and *and* connectives. However, human discourse is much richer and more diverse in its quantifiers, e.g. *about 5, almost all, a few, many, most, as many as possible, nearly half, at least half.* In an attempt to bridge the gap between formal systems and natural discourse and, in turn, to provide a more flexible knowledge representation tool, Zadeh introduced the concept of fuzzy quantifiers [9].

Zadeh suggested that the semantic of a fuzzy quantifier can be captured by using fuzzy subsets for its representation. He distinguished between two types of fuzzy quantifiers, *absolute* and *relative.* Absolute quantifiers are used to represent amounts that are absolute in nature such as *about 2* or *more than 5.* These absolute linguistic quantifiers are closely related to the concept of count or number of elements. He defined these quantifiers as fuzzy subsets of the non negative real numbers, $\mathcal{R}^+$. In this approach, an absolute quantifier can be represented by a fuzzy subset Q, such that for any $r \in \mathcal{R}^+$ the membership degree of r in Q, Q(r), indicates the degree to which the amount r is compatible with the quantifier represented by Q. Relative quantifiers, such as *most, at least half*, can be represented by fuzzy subsets of the unit interval, [0,1]. For any $r \in [0,1]$, Q(r) indicates the degree to which the proportion r is compatible with the meaning of the quantifier it represents. Any quantifier of natural language can be represented as a relative quantifier or given the cardinality of the elements considered, as an absolute quantifier. Functionally, fuzzy quantifiers are usually of one of three types, *increasing, decreasing,* and *unimodal.* An increasing type quantifier is characterized by the relationship $Q(r_1) \geq Q(r_2) \quad \text{if } r_1 > r_2$. These quantifiers are characterized by values such as *most, at least half.* A decreasing type quantifier is characterized by the relationship $Q(r_1) \leq Q(r_2) \quad \text{if } r_1 > r_2$.

An absolute quantifier $Q : \mathcal{R}^+ \rightarrow [0,1]$ satisfies:

$$Q(0) = 0, \text{and } \exists k \text{ such that } Q(k) = 1.$$

A relative quantifier, $Q : [0,1] \rightarrow [0,1]$, satisfies:

$$Q(0) = 0, \text{ and } \exists r \in [0,1] \text{ such that } Q(r) = 1.$$

The membership function of a non decreasing relative quantifier can be represented as

$$Q(r) = \begin{cases} 0 & \text{if } r < a \\ \frac{r-a}{b-a} & \text{if } a \leq r \leq b \\ 1 & \text{if } r > b \end{cases}$$

with $a, b, r \in [0, 1]$.

Some examples of relative quantifiers are shown in Figure 1, where the parameters, (a, b) are $(0.3, 0.8)$, $(0, 0.5)$ and $(0.5, 1)$, respectively.

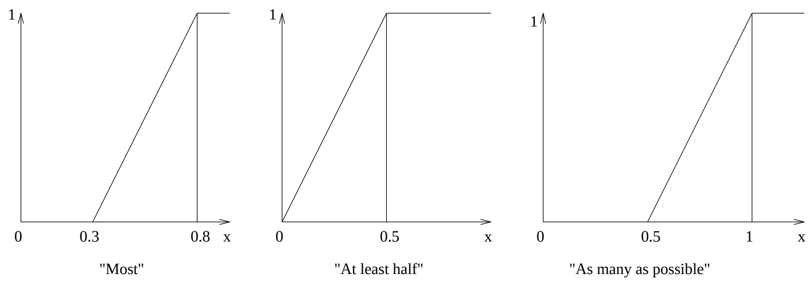

Fig. 1. Relative Fuzzy Quantifiers

2.2 The OWA Operator

The OWA operator was proposed by Yager in [8]. It provides a family of aggregation operators which have the "and" operator at one extreme and the "or" operator at the other extreme.

Definition 1. *An OWA operator of dimension m is a function ϕ,*

$$\phi : \mathcal{R}^m \rightarrow \mathcal{R},$$

that has associated a set of weights or weighting vector $W = [w_1, \ldots, w_m]$ such that,

1. $w_i \in [0, 1]$, *and*
2. $\Sigma_i w_i = 1$*;*

and is defined for aggregating a list of values $\{a_1, \ldots, a_m\}$ according to the following expression,

$$\phi(a_1, \ldots, a_m) = W \cdot B^T = \Sigma_{i=1}^m w_i \cdot b_i$$

where B is the associated ordered value vector, and each element $b_i \in B$ is the i-th largest value in the collection $a_1, \ldots, a_m$.

Proposition 1. The OWA operator satisfies the following properties:

1. *It is an or-and operator, i.e., it remains between the minimum and the maximum of the arguments:*

$$min(a_1, \ldots, a_m) \leq \phi(a_1, \ldots, a_m) \leq max(a_1, \ldots, a_m).$$

2. *It is commutative:*

$$\phi(a_1, \dots, a_m) = \phi(a_{\pi(1)}, \dots, a_{\pi(m)}) \ \forall \pi.$$

3. *It is idempotent:*

$$\phi(a_1, \dots, a_m) = a, \ \textit{if } a_i = a \ \forall i.$$

4. *It is increasing monotonous:*

$$\phi(a_1, \dots, a_m) \geq \phi(d_1, \dots, d_m), \ \textit{if } a_i \geq d_i \forall i.$$

5. *It leads to the arithmetic mean when* $w_i = \frac{1}{m}$ $\forall i$*:*

$$\phi(a_1, \dots, a_m) = \Sigma_{i=1}^{m} (\frac{1}{m}) \cdot b_i = (\frac{1}{m}) \Sigma_{i=1}^{m} b_i.$$

6. *It leads to maximum when* $W = [1, 0, \dots, 0]$.
7. *It leads to minimum when* $W = [0, \dots, 0, 1]$.

A natural question in the definition of the *OWA* operator is how to obtain the associated weighting vector. In [8], Yager proposed two ways to obtain it. The first approach is to use some kind of learning mechanism using some sample data; and the second approach is to try to give some semantics or meaning to the weights. The final possibility has allowed multiple applications on areas of fuzzy and multi-valued logics, evidence theory, design of fuzzy controllers, and the quantifier guided aggregations.

We are interested in the area of quantifier guided aggregations. The idea consists in calculating weights for the aggregation operations using fuzzy quantifiers representing the concept of *fuzzy majority*. In [8], Yager suggested an interesting way to compute the weights of the *OWA* aggregation operator using fuzzy quantifiers, which, in the case of a non decreasing relative quantifier Q, it is given by the following expression:

$$w_i = Q(i/m) - Q((i-1)/m), i = 1, \dots, m.$$

When a fuzzy quantifier Q is used to compute the weights of the *OWA* operator ϕ, it is symbolized by ϕ_Q.

2.3 Geometric Mean

As was aforementioned, the geometric mean operator is the traditional aggregation operator to combine ratio-scale judgements in the Saaty's multicriteria decision models. Its main characteristic is that in such decision context it guarantizes the reciprocity property of the multiplicative preference relations used to provide the ratio preferences. The geometric mean is defined as follows:

Definition 2. *A geometric mean operator of dimension* m *is a function* $g : \mathcal{R}^m \rightarrow \mathcal{R}$*, defined as*

$$g(a_1, \ldots, a_m) = \Pi_{i=1}^{m}(a_i)^{\frac{1}{m}}.$$

Proposition 2. The geometric mean satisfies the following properties:

1. *It is an or-and operator, i.e., it remains between the minimum and the maximum of the arguments.*
2. *It is commutative.*
3. *It is idempotent.*
4. *It is increasing monotonous.*

3 The OWG Operator

In this section, we present the OWG operator to aggregate ratio-scale judgements. It is based on the OWA operator [8] and on the geometric mean, and therefore, incorporates the advantage of the OWA operator to represent the concept of fuzzy majority and the advantage of geometric mean to deal with ratio-scale judgements. It is defined as follows.

Definition 3. *An OWG operator of dimension* m *is a function,* $\phi^G : \mathcal{R}^m \rightarrow \mathcal{R}$*, that has associated a set of weights or exponential weighting vector* $W = [w_1, \ldots, w_m]$ *such that,*

1. $w_i \in [0, 1]$*, and*
2. $\Sigma_i w_i = 1$*;*

and is defined for aggregating a list of values $\{a_1, \ldots, a_m\}$ *according to the following expression,*

$$\phi^G(a_1, \ldots, a_m) = \Pi_{i=1}^{m} c_i^{w_i}$$

where C *is the associated ordered value vector, and each element* $c_i \in C$ *is the i-th largest value in the collection* $a_1, \ldots, a_m$.

Proposition 3. The OWG operator satisfies the following properties:

1. *It is an or-and operator, i.e., it remains between the minimum and the maximum of the arguments:*
$$min(a_1, \ldots, a_m) \leq \phi^G(a_1, \ldots, a_m) \leq max(a_1, \ldots, a_m).$$
2. *It is commutative:*
$$\phi^G(a_1, \ldots, a_m) = \phi^G(a_{\pi(1)}, \ldots, a_{\pi(m)}),\ \forall \pi.$$

3. *It is idempotent:*

$$\phi^G(a_1, \ldots, a_m) = a, \text{ if } a_i = a \ \forall i.$$

4. *It is increasing monotonous:*

$$\phi^G(a_1, \ldots, a_m) \geq \phi^G(d_1, \ldots, d_m), \text{ if } a_i \geq d_i \forall i.$$

5. *It leads to the geometric mean when* $w_i = \frac{1}{m} \ \forall i$*:*

$$\phi^G(a_1, a_2, \ldots, a_m) = \Pi_{k=1}^m (c_k)^{\frac{1}{m}} = g(a_1, a_2, \ldots, a_m).$$

6. *It leads to maximum when* $W = [1, 0, \ldots, 0]$*:*

$$\phi^G(a_1, a_2, \ldots, a_m) = max_{i=1}^m (a_i).$$

7. *It leads to minimum when* $W = [0, \ldots, 0, 1]$*:*

$$\phi^G(a_1, a_2, \ldots, a_m) = min_{i=1}^m (a_i).$$

We can obtain W using the same method that in the OWA operator case, i.e., the weighting vector may be obtained using a fuzzy quantifier, Q, representing the concept of fuzzy majority. When a fuzzy quantifier Q is used to compute the weights of the OWG operator ϕ^G, then, it is symbolized by ϕ_Q^G.

In the following section, we present an example of the use of the OWG operator in a multicriteria decision making problem under multiplicative preference relations.

4 Solving a Multicriteria Decision Making Problem Using the OWG Operator

Let $X = \{x_1, x_2, \ldots, x_n, \ (n \geq 2)\}$ be a finite set of alternatives. The alternatives must be classified from best to worst (ordinal ranking), using the information known according to a finite set of general criteria or experts $E = \{e_1, e_2, \ldots, e_m, \ (m \geq 2)\}$. We assume that the experts' preferences over the set of alternatives, X, are represented by means of the multiplicative preference relations on X, i.e.,

$$A^k \subset X\text{x}X, \ A^k = [a_{ij}^k]$$

where a_{ij}^k indicates a ratio of preference intensity for alternative x_i to that of x_j, i.e., it is interpreted as x_i is a_{ij}^k times as good as x_j. Each a_{ij}^k is assessed using the ratio scale proposed by Saaty, that is, precisely the 1 to 9 scale [6]: $a_{ij}^k = 1$ indicates indifference between x_i and x_j, $a_{ij}^k = 9$ indicates that x_i is absolutely preferred to x_j, and $a_{ij}^k \in 2, 3, \ldots, 8$ indicates intermediate evaluations. In order to guarantee that A^k is "self-consistent", only some pairwise comparison statements are collected to construct it. The rest of the values are what satisfy the following conditions [6]:

1. Multiplicative Reciprocity Property: $a^k_{ij} \cdot a^k_{ji} = 1 \; \forall i, j$.
2. Saaty's Consistency Property: $a^k_{ij} = a^k_{it} \cdot a^k_{tj} \; \forall i, j, t$.

Then, we consider multiplicative preference relations assessed in Saaty's discrete scale, which has only the following set of values:

$$\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \ldots, \frac{1}{2}, 1, 2, \ldots, 7, 8, 9\}.$$

The multicriteria decision making problem when the experts express their preferences using multiplicative preference relations have been solved by Saaty using the decision AHP, which obtains the set of solution alternatives by means of the eigenvector method [6]. However, this decision process is not guided by the concept of majority.

In this paper, we present an alternative decision process to the AHP proposed by Saaty in order to show the application of the OWG operator. Following the choice scheme proposed in [5], i.e.,

Aggregation + Exploitation,

we design a selection process based on fuzzy majority to choose the best alternatives from multiplicative preference relations. This process is defined using the quantifier guided OWG operator to aggregate the preferences and to define a choice function of alternatives, called *multiplicative quantifier guided dominance degree*, which obtains the best alternatives from the aggregated information. This degree is defined as a multiplicative version of quantifier guided dominance degree proposed for fuzzy preference relations in [3,4].

Definition 4. Quantifier guided dominance degree for an alternative x_i, symbolized $MQGDD^k_i$, from a multiplicative preference relation, A^k, is defined according to the following expression:

$$MQGDD^k_i = \phi^G_Q(a^k_{i1}, \ldots, a^k_{in}).$$

In what follows, we present the phases of selection process based on fuzzy majority and designed to deal with multicriteria decision making problems under multiplicative preference relations.

1. Aggregation phase.

This phase defines a collective multiplicative preference relation, $A^c = [a^c_{ij}]$, which indicates the global preference according to the fuzzy majority of experts' opinions. A^c is obtained from $\{A^1, \ldots, A^m\}$ by means of the following expression:

$$a^c_{ij} = \phi^G_Q(a^1_{ij}, \ldots, a^m_{ij}),$$

where ϕ^G_Q is the OWG operator guided by the concept of fuzzy majority represented by the fuzzy quantifier Q.

2. Exploitation phase.

Using the quantifier guided choice degree defined for multiplicative preference relations, this phase transforms the aggregated or global information about the alternatives into a global ranking of them, supplying the set of solution alternatives.

Firstly, using the OWG operator ϕ_Q^G we obtain the choice degrees of alternatives from A^c:

$$[MQGDD_1, \dots, MQGDD_n],$$

with

$$MQGDD_i = \phi_Q^G(a_{i1}^c, \dots, a_{in}^c).$$

And secondly, the application of choice degree of alternatives over X allows us to obtain the following solution set of alternatives:

$$X^{sol} = \{x_i \,|\, x_i \in X, MQGDD_i = sup_{x_j \in X} MQGDD_j\}$$

whose elements are called maximum dominance ones.

4.1 Example

Consider the following illustrative example of the classification method of alternatives studied in this paper. Assume that we have a set of four experts, $E = \{e_1, e_2, e_3, e_4\}$, and a set of four alternatives, $X = \{x_1, x_2, x_3, x_4\}$. Suppose that experts supply their opinions by means of the following multiplicative preference relations:

$$A^1 = \begin{bmatrix} 1 & 3 & 5 \\ \frac{1}{3} & 1 & 2 \\ \frac{1}{5} & \frac{1}{2} & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 & 7 \\ \frac{1}{2} & 1 & 5 \\ \frac{1}{7} & \frac{1}{5} & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1 & 2 \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 5 & 3 \\ \frac{1}{5} & 1 & 9 \\ \frac{1}{3} & \frac{1}{9} & 1 \end{bmatrix}.$$

In the decision process we use the fuzzy majority criterion with the fuzzy quantifier *"at least half "*, with the pair $(0, 0.5)$, and the corresponding OWG operator with the weighting vector, $W = [\frac{1}{2}, \frac{1}{2}, 0, 0]$.

Selection Process Based on Fuzzy Majority

1. Aggregation phase

The collective multiplicative preference relation obtained in this phase is the following:

$$A^c = \begin{bmatrix} 1 & 15^{\frac{1}{2}} & 23^{\frac{1}{2}} \\ \frac{1}{2} & 1 & 45^{\frac{1}{2}} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{bmatrix}.$$

For example a^c_{21} is obtained as

$$a^c_{21} = \phi^G_Q(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}) = (\frac{1}{3})^{\frac{1}{2}} \cdot (\frac{1}{3})^{\frac{1}{2}} \cdot (\frac{1}{2})^0 \cdot (\frac{1}{2})^0 = \frac{1}{3}.$$

2. Exploitation phase

The quantifier guided choice degree of alternatives acting over the collective multiplicative preference relation and with the weighting vector $W = [\frac{2}{3}, \frac{1}{3}, 0]$ supplies the following values:

$$MQGDD_1 = 8.52, \ \ MQGDD_2 = 3.55, \ \ MQGDD_3 = 0.79.$$

These values represent the dominance that one alternative has over "at least half"of the alternatives according to "at least half" of the experts. For example the value $MQGDD_1$ is obtained as

$$MQGDD_1 = \phi^G_Q(1, 15^{\frac{1}{2}}, 23^{\frac{1}{2}}) = 23^{\frac{1}{2}^{\frac{2}{3}}} \cdot 15^{\frac{1}{2}^{\frac{1}{3}}} \cdot 1^0 = 8.52.$$

Clearly the solution set is:

$$X^{sol} = \{x_1\}.$$

5 Concluding Remarks

In this paper, we have presented a new aggregation operator for the synthesis of ratio judgements, called OWG operator. It has been designed incorporating the advantages of the geometric mean to deal with ratio judgements and the advantages of the OWA operator to represent the concept of fuzzy majority in the aggregation processes.

We have studied its properties and also have illustrated its use in a multicriteria decision making problem with multiplicative preference relations. Particularly, we have developed a selection process of alternatives based on a quantifier guided dominance degree.

In the future, we will research the use of the OWG operator for designing consensus processes in multicriteria decision making problems with multiplicative preference relations.

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