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Theory and Methodology

Three models of fuzzy integer linear programming ☆

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Abstract

In this paper we study some models for dealing with Fuzzy Integer Linear Programming problems which have a certain lack of precision of a vague nature in their formulation. We present methods to solve them with either fuzzy constraints, or fuzzy numbers in the objective function or fuzzy numbers defining the set of constraints. These methods are based on the representation theorem and on fuzzy number ranking methods.

Keywords: Integer Programming; Fuzzy constraints; Fuzzy numbers

1. Introduction

Integer Linear Programming (ILP) problems have an outstanding relevance in many fields, such as those related to artificial intelligence, operations research, etc. They are especially important for representing and reasoning with propositional knowledge. Thus, the use of Mathematical Programming (MP) techniques for treating propositional logic is useful. In particular, several research efforts have involved the use of MP as a tool for modeling and performing deductive reasoning. An arbitrary system of rules can be represented and solved as an Integer Linear Program. The applications of Integer Programming to logic lead to new algorithms for inference in Knowledge-Based-Systems [13,15,20,23].

A classical ILP problem can be written as follows:

$$\begin{aligned} \max \quad & z = cx \\ \text{s.t.} \quad & \sum_{j \in N} a_{ij} x_j \le b_i, \quad i \in M = \{1, \dots, m\}, \\ & x_j \ge 0, \quad j \in N = \{1, \dots, n\}, \\ & x_j \in \mathbb{N}, \quad j \in N, \end{aligned} \tag{1}$$

where $\mathbb{N}$ is the set of integer numbers, $c \in \mathbb{R}^n$ and a_{ij}, $b_i \in \mathbb{R}$, $i \in M$, $j \in N$.

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In real situations however the information available in the system under consideration not of an exact nature. The aim of this paper is to study different problems, where some lack of precision of a vague nature may be assumed on their formulations, providing a tool helping reasoning in imprecise Knowledge-Based-Systems. This kind of problems will be called fuzzy integer linear programming (FILP) problems.

In [12] a classification of them was shown, along with a description of each of the possible problems, and an initial study of the fuzzy boolean linear programming problems with fuzzy constraint was carried out. In view of this classification we will study the FILP models to ascertain whether there are either fuzzy constraints, or fuzzy numbers as coefficients in the objective function or fuzzy numbers defining the set of constraints. In the following section we will firstly discuss the FILP problem with fuzzy constraints, in section three the FILP problem with fuzzy numbers in the objective function and in section four the FILP problem with fuzzy numbers in the technological matrix. Finally, some conclusions will be pointed out.

2. FILP problems with fuzzy constraints

This problem can be written as

$$\begin{aligned} Max \quad & z = cx \\ \text{s.t.} \quad & \sum_{j \in N} a_{ij} x_j \lesssim b_i, \quad i \in M, \\ & x_j \geq 0, \quad j \in N, \\ & x_j \in \mathbb{N}, \quad j \in N. \end{aligned} \tag{2}$$

The symbol $\lesssim$ means that the decision-maker is willing to permit some violations in the accomplishment of the constraints, that is, he considers fuzzy constraints defined by membership functions

$$\mu_i : \mathbb{R}^n \to (0,1], \quad i \in M. \tag{3}$$

Each of these gives the degree to which each $x \in \mathbb{R}^n$ accomplishes the respective constraint.

This problem was studied in [8], where an auxiliary ILP problem was presented as a transformation of the former FILP problem into a deterministic model with linear constraints, a modified objective function and some supplementary constraints and variables.

Next, following the ideas expressed in [18,12] we will present an alternative model which allows a fuzzy solution of the problem to be obtained according to the use of the representation theorem of fuzzy sets.

Consider a linear membership function for the i-th constraint,

$$\mu_i(x) = \begin{cases} 1 & \text{if } a_i x \leq b_i, \\ [(b_i + d_i) - a_i x]/d_i & \text{if } b_i \leq a_i x \leq b_i + d_i, \\ 0 & \text{if } a_i x \geq b_i + d_i, \end{cases}$$

and denote for each constraint

$$X_i = \{ x \in \mathbb{R}^n \mid a_i x \lesssim b_i,\ x_j \geq 0,\ x_j \in \mathbb{N} \}, \quad i \in M.$$

If $X = \bigcap_{r \in M} X_i$ then (2) can be rewritten as

$$\max \{ z = cx \mid x \in X \}. \tag{4}$$

It is clear, $\forall\alpha \in (0,1]$, an α-cut of the constraint set will be the classical set $X(\alpha) = \{x \in \mathbb{R}^n \mid \mu_X(x) \geq \alpha\}$ where $\forall x \in \mathbb{R}^n$, $\mu_X(x) = \inf\{\mu_i(x),\ i \in M\}$. In this way, $X_i(\alpha)$ will denote an α-cut of the i-th constraint, $i \in M$.

Then, denoting $\forall\alpha \in (0,1]$,

$$S(\alpha) = \{x \in \mathbb{R}^n \mid cx = \max\ cy,\ y \in X(\alpha)\},$$

the fuzzy set defined by the membership function

$$\lambda(x) = \begin{cases} \sup\limits_{x \in S(\alpha)} \alpha & x \in \bigcup\limits_{\alpha} S(\alpha), \\ 0 & \text{elsewhere}, \end{cases} \tag{5}$$

is the fuzzy solution of the problem (2) (Orlovski, [16]).

As $\forall\alpha \in (0,1]$,

$$X(\alpha) = \bigcap_{i \in M} \{x \in \mathbb{R}^n \mid a_i x \leq r_i(\alpha),\ x_j \geq 0,\ x_j \in \mathbb{N}\}$$

with $r_i(\text{a}) = b_i + d_i(1 - \alpha)$, (3) can be written as the following auxillary parametric ILP problem:

$$\begin{aligned} \max\quad & z = cx \\ \text{s.t.}\quad & a_i x \leq b_i + d_i(1-\alpha), \quad i \in M, \\ & x_j \geq 0, \\ & x_j \in \mathbb{N}, \quad \alpha \in (0,1], \quad j \in N. \end{aligned} \tag{6}$$

In [1] an approach was shown to solve (6), and by means of the parametric solution of (6) the fuzzy solution to (2) is obtained with a membership function like (5).

The initial problem (2) may be presented with nonlinear membership functions for the constraints. As was shown in [7], the form of the membership functions does not make the use of the representation theorem complicated. In [9] it was shown that in all cases the objective function associated to the fuzzy solution is included into the same interval. In [7] a method was presented which allows us to obtain the fuzzy solution to a fuzzy linear programming problem with nonlinear membership functions from the fuzzy solution associated to the fuzzy linear programming problem with linear membership functions and same right margins [7,292–293, Proposition 2]. These results may be applied directly to the FILP problems. Therefore, the use of nonlinear membership functions do not interfere in the computational efficiency of the solution method.

2.1. Numerical example

Consider the following problem:

$$\begin{aligned} \max\quad & z = 2x_1 + 5x_2 \\ \text{s.t.}\quad & 2x_1 - x_2 \lesssim 9, \\ & 2x_1 + 8x_2 \lesssim 31, \\ & x_1,\ x_2 \geq 0, \\ & x_1,\ x_2 \in \mathbb{N}, \end{aligned}$$

with $d_1 = 3$ and $d_2 = 4$ the right margins allowed by the decision-maker. The auxiliary parametric integer programming problem is

$$\begin{aligned}\max\quad & 2x_1 + 5x_2 \\ \text{s.t.}\quad & 2x_1 - x_2 \leq 9 + 3(1-\alpha), \\ & 2x_1 + 8x_2 \leq 31 + 4(1-\alpha), \\ & x_1,\ x_2 \geq 0, \\ & x_1,\ x_2 \in \mathbb{N}, \\ & 0 < \alpha \leq 1.\end{aligned}$$

Applying the resolution method the following solution is obtained:

$$\begin{aligned} x(\alpha) &= (5,3), & z(\alpha) &= 25 \quad \forall \alpha \in (0, 0.25], \\ x(\alpha) &= (4,3), & z(\alpha) &= 23 \quad \forall \alpha \in (0.25, 0.75], \\ x(\alpha) &= (3,3), & z(\alpha) &= 21 \quad \forall \alpha \in (0.75, 1], \end{aligned}$$

and finally, the fuzzy solution is the following fuzzy set:

$$\underset{\sim}{S} = \{(5,3)/0.25,\ (4,3)/0.75,\ (3,3)/1\}.$$

3. FILP problems with imprecise costs

Here we study FILP problems with imprecise coefficients in the objective function, that is, with coefficients defined by fuzzy numbers. The problem can be written as

$$\begin{aligned}\max\quad & z = \sum_{j \in N} \underset{\sim}{c}_j x_j \\ \text{s.t.}\quad & \sum_{j \in N} a_{ij} x_j \leq b_i, \quad i \in M, \\ & x_j \geq 0, \quad j \in N, \\ & x_j \in \mathbb{N}, \quad j \in N,\end{aligned} \tag{7}$$

where a_{ij}, $b_i \in \mathbb{R}$ are real coefficients, and the costs in the objective are fuzzy numbers, that is, $\underset{\sim}{c}_j \in F(\mathbb{R})$, $F(\mathbb{R})$ being the set of real fuzzy numbers, $i \in M$, $j \in N$.

Thus, one has the membership functions

$$\mu_j : \mathbb{R} \to [0,1], \quad j \in N, \tag{8}$$

expressing the lack of precision on the values of the coefficients that the decision-maker has.

For each feasible solution, there is a fuzzy number which is obtained by means of the fuzzy objective function. Hence, in order to solve the optimization problem, obtaining both the optimal solution and the corresponding fuzzy value of the objective, methods ranking the fuzzy numbers obtained from this function may be considered. From this point of view two ways to solve (7) can be approached. The first will consist of the use of several well known ranking fuzzy numbers methods, each of which will provide a different auxiliary conventional optimization model solving the former problem. The second approach will explore the behavior of the representation theorem for fuzzy sets when it is used as tool to solve the proposed problem.

3.1. The use of fuzzy number ranking methods

In this section, let X be the set of feasible alternatives of (7), and g a function mapping the set of feasible alternatives of (7) into the set of fuzzy numbers,

$$g : X \to F(\mathbb{R}), \qquad g(x) = \underset{\sim}{c}x = \sum_{j \in N} \underset{\sim}{c}_j x_j, \quad \underset{\sim}{c}_j \in F(\mathbb{R}), \tag{9}$$

where extended sum and product by positive real numbers have been considered defined in $F(\mathbb{R})$ by means of the Zadeh's Extension Principle.

Consider the set of fuzzy numbers $A = \{g(x) \mid x \in X\}$. Then $x^* \in X$ will be said to be an optimal alternative if the fuzzy number $g(x^*)$ is greatest in A. Hence, the problem now is how to determine the greatest in A.

The problem of comparison of fuzzy numbers has been widely investigated in publications. Many fuzzy numbers ranking methods (FNRM) can be found for instance in [2] and [10]. This paper will focus on those FNRMs which are defined by means of a ranking function, and particularly by means of a linear ranking function (LRF), which is not too restrictive because many well known FNRM can be formulated by using linear ranking functions in some way.

Consider A, $B \in F(\mathbb{R})$. A simple method of comparison between them consists of the definition of a certain function $f : F(\mathbb{R}) \to \mathbb{R}$. If this function $f(\cdot)$ is known, then $f(A) > f(B)$, $f(A) = f(B)$, $f(A) < f(B)$ are equivalent to $A > B$, $A = B$, $A < B$ respectively. Usually, f is called an LRF if

$$\forall A, B \in F(\mathbb{R});\ \forall r \in \mathbb{R} \quad r > 0; \quad f(A + B) = f(A) + f(B) \text{ and } f(rA) = rf(A). \tag{10}$$

As it is well known, from this definition several FNRPs may be considered. In [3] an extensive study of these LRF can be found.

To simplify, triangular fuzzy numbers will be considered. They will be denoted $\underset{\sim}{c}_j = (r_j, c_j, R_j)$, and their membership functions supposed in the form

$$\forall u \in \mathbb{R},\ j \in N, \quad \mu_{c_j}(u) = \begin{cases} (u - r_j)/(c_j - r_j) & \text{if } r_j \le u \le c_j, \\ (R_j - u)/(R_j - c_j) & \text{if } c_j \le u \le R_j, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then the following result holds [11]. Let us assume a linear expression $\underset{\sim}{y} = \sum_j \underset{\sim}{c}_j x_j$ in which the $\underset{\sim}{c}_j$'s are fuzzy numbers with membership functions similar to the ones given by (11), and $x_j \ge 0$, $j \in N$. Then the membership function of the fuzzy number $\underset{\sim}{y}$ is given by

$$\mu(z) = \begin{cases} h_j(z) = (z - rx)/(cx - rx) & \text{if } x > 0,\ rx \le z \le cx, \\ g_j(z) = (Rx - z)/(Rx - cx) & \text{if } x > 0,\ cx \le z \le Rx, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where $r = (r_1, \dots, r_n)$, $c = (c_1, \dots, c_n)$ and $R = (R_1, \dots, R_n)$. If it is denoted that $d = R - c$ and $d' = c - r$, then $d \cdot x$ and $d' \cdot x$ will be the lateral margins (right and left respectively) of the fuzzy number $\underset{\sim}{c}x$.

If we apply different methods of ranking fuzzy numbers to (7) then it is interesting to observe how the optimal solution to it will be an optimal solution of a conventional Programming problem with similar constraints and a nonfuzzy objective function. This nonfuzzy objective reflects, by means of the ranking functions, the preference of the decision-maker.

Consider a ranking function f mapping each fuzzy set into the real line, $f: A \to \mathbb{R}$. Then a solution for (7) can be found from

$$\begin{aligned} \max \quad & f(\underset{\sim}{c}x) \\ \text{s.t.} \quad & Ax \le b, \\ & x_j \in \mathbb{N}, \quad j \in N. \end{aligned} \tag{13}$$

Therefore according to the ranking function f used, different auxillary models solving (7) can be obtained. Clearly, if we use LRF then the auxiliary problem obtained in (13) will be the following ILP problem:

$$\max \left\{ \sum_{j \in N} f(\underset{\sim}{c}_j) x_j \mid j \in N,\ x \in X \right\}. \tag{14}$$

Some of these auxiliary models are shown in the following

a) The use of the Index of Chang [4] provides the problem

$$\max \left\{ (dx + d'x) \cdot (3cx + dx - d'x)/6 \mid Ax \le b,\ x_j \in \mathbb{N} \right\} \tag{15}$$

which is nonlinear.

b) The use of first, second and third Index of Yager [21,22] produces respectively the models

$$\max \left\{ [c + (d - d')/3] x \mid Ax \le b,\ x_j \in \mathbb{N} \right\}, \tag{16}$$

$$\max \left\{ (cx + dx)/(dx + 1) \mid Ax \le b,\ x_j \in \mathbb{N} \right\}, \tag{17}$$

$$\max \left\{ [c + (d - d')/4] x \mid Ax \le b,\ x_j \in \mathbb{N} \right\}. \tag{18}$$

3.2. The use of the representation theorem

Consider $\forall c \in \mathbb{R}^n$, $c = (c_1, \ldots, c_n)$, the membership function

$$\mu(c) = \inf_j \mu_j(c_j), \quad j \in N.$$

It is clear, as was shown in [19], that $\mu(\cdot)$ defines a fuzzy objective which induces a fuzzy preorder in X. Consequently a fuzzy solution to (7) can be found from the solution of the Multiobjective Parametric Integer Linear Programming problem

$$\max_{x \in X} \left\{ cx \mid \forall c \in \mathbb{R}^n\!: \mu(c) \ge 1 - \alpha \right\}. \tag{19}$$

But, taking into account that

$$\mu(c) \ge 1 - \alpha \Leftrightarrow \inf_j \mu_j(c_j) \ge 1 - \alpha \Leftrightarrow \mu_j(c_j) \ge 1 - \alpha, \quad j \in N, \quad \alpha \in [0,1],$$

from (12),

$$\mu_j(c_j) \ge 1 - \alpha \Leftrightarrow h_j^{-1}(1 - \alpha) \le c_j \le g_j^{-1}(1 - \alpha), \quad j \in N,$$

is obtained, and denoting $\phi_j \equiv h_j^{-1}$, $\Psi_j \equiv g_j^{-1}$, $j \in N$, the problem (19) can be written as

$$\max \left\{ cx \mid x \in X, \quad \Phi(1 - \alpha) \le c \le \Psi(1 - \alpha),\ \alpha \in [0,1] \right\} \tag{20}$$

where $\Phi(\cdot) = [\phi_1(\cdot), \ldots, \phi_n(\cdot)]$ and $\Psi = [\Psi_1(\cdot), \ldots, \Psi_n(\cdot)]$.

Moreover, if $\Gamma(1 - \alpha)$, $\alpha \in [0,1]$, denotes the set of vectors $c \in \mathbb{R}^n$ with all of their components c_j in the interval $[\phi_j(1 - \alpha), \Psi_j(1 - \alpha)]$, $j \in N$, (20) can be finally rewritten as

$$\max \left\{ cx \mid x \in X,\ c \in \Gamma(1 - \alpha),\ \alpha \in [0,1] \right\} \tag{21}$$

which for each $\alpha \in [0,1]$ is a Multiobjective ILP problem denoted $M(\alpha)$ and having in its objective function costs that can assume values in the respective intervals. Different alternatives can be considered here: first, following [5], the resolution of all problems in the family $\{M(\alpha),\ \alpha \in [0,1]\}$ where the fuzzy solution for (7) will be obtained from the solution of the following Multiobjective Integer Programming problem:

$$\begin{array}{ll} \max & (c^1x, c^2x,, \ldots, x^{2^n}x) \\ \text{s.t.} & Ax \leq b, \\ & x \geq 0, \quad c^k \in E(1-\alpha), \\ & \alpha \in [0,1], \quad k = 1, 2, \ldots, 2^n, \end{array} \tag{22}$$

where $E(1-\alpha) \subset \Gamma(1-\alpha)$ is the subset constituted by vectors whose j-th component is equal to either the upper or the lower bound of c_j, $\phi_j(1-\alpha)$ or $\Psi_j(1-\alpha)$, $j \in N$.

On the other hand, according to some results by Ishibuchi and Tanaka, [14], on the use of interval arithmetic for solving LP problems with interval objective functions, the fuzzy solution for (7) can be found from the parametric solution of the following biobjective parametric problem, P(α):

$$\begin{array}{ll} \max & z'(\alpha) = (z^1(x, \alpha), z^c(x, \alpha)) \\ \text{s.t.} & Ax \leq b, \\ & x_j \in \mathbb{N}, \quad j \in N, \\ & \alpha \in [0,1], \end{array} \tag{23}$$

where $z^1(x, \alpha)$ and $z^c(x, \alpha)$ in the case of triangular fuzzy numbers are defined by

$$z^1(x, \alpha) = \sum_{j=1}^{n} \left[c_j - \alpha(c_j - r_j)\right] x_j \text{ and } z^c(x, \alpha) = \tfrac{1}{2} \sum_{j=1}^{n} \left[2c_j + \alpha(R_j + r_j - 2c_j)\right] x_j.$$

Now, in accordance with the representation theorem for fuzzy sets, one can define

$$\underset{\sim}{S} = \bigcup_{\alpha} \alpha S(1-\alpha)$$

which is a fuzzy set giving the fuzzy solution to the former problem, in which $S(1-\alpha)$ is defined as the set of solutions of the auxiliary problem considered according to the two approaches, (22) or (23), for every $\alpha \in [0,1]$.

Concretely, a decision-maker may be able to assign weights $\beta_k \in [0,1]$ to each of the objectives taking part in (22) or (23), such that $\Sigma_k \beta_k = 1$. Then conventional parametric LP problems are obtained.

Let us assume (22) and (23) and consider $\beta = \omega = (\omega_1, \ldots, \omega_t)$ and $\beta = \nu = (\nu_1,\ \nu_2)$ then these problems are denoted $M_\beta(\alpha)$ and $P_\beta(\alpha)$ respectively. If the set of optimal points of these is defined as $S_\beta(1-\alpha)$ for every $\alpha \in [0,1]$, then the fuzzy solution with weight β will be given by the fuzzy set

$$\underset{\sim}{S}_\beta = \bigcup_{\alpha} S_\beta(1-\alpha).$$

In [11] it was shown that using weight vectors then $P_\beta(\alpha)$ is a particular case of $M_\beta(\alpha)$. It is enough to take $\omega_1 = (\nu_1 + \frac{1}{2}\nu_2)$, $\omega_t = \frac{1}{2}\nu_2$, and $\omega_i = 0$ for $0 < i < t$, and the equivalence is obtained.

3.3. Numerical example

Consider the following problem:

$$\begin{aligned} \max \quad & z = \underset{\sim}{c}_1 x_1 + 5x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \le 12, \\ & 2x_1 + 8x_2 \le 35, \\ & x_j \ge 0, \quad x_j \in \mathbb{N}, \quad j \in N, \end{aligned}$$

where $\underset{\sim}{c} = (1, 3, 5)$.
We have the following functions:

$$\phi_1(1-\alpha) = 3 - 2\alpha, \qquad \Psi_1(1-\alpha) = 3 + 2\alpha,$$

and the associated interval parametric problem is:

$$\begin{aligned} \max \quad & z = c_1 x_1 + 5x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \le 12, \\ & 2x_1 + 8x_2 \le 35, \\ & 3 - 2\alpha \le c_1 \le 3 + 2\alpha, \\ & x_j \ge 0, \quad x_j \in \mathbb{N}, \quad j \in N, \quad \alpha \in [0,1]. \end{aligned}$$

From (23) the auxiliary multiobjective ILP problem is:

$$\begin{aligned} \max \quad & \{(3-2\alpha)x_1 + 5x_2, (3+2\alpha)x_1 + 5x_2\} \\ \text{s.t.} \quad & 2x_1 - x_2 \le 12, \\ & 2x_1 + 8x_2 \le 35, \\ & x_j \ge 0, \quad x_j \in \mathbb{N}, \quad j \in N, \quad \alpha \in [0,1]. \end{aligned}$$

Next, we solve the above auxiliary multiobjective ILP problem for the following weight vectors: $\beta = (1,0)$ and $\beta = (0.5,0.5)$

For $\beta = (1,0)$ the auxiliary parametric problem is:

$$\begin{aligned} \max \quad & z = (3-2\alpha)x_1 + 5x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \le 12, \\ & 2x_1 + 8x_2 \le 35, \\ & x_j \ge 0, \quad x_j \in \mathbb{N}, \quad j \in N, \quad \alpha \in [0,1], \end{aligned}$$

the optimal solution of which is

$$\begin{aligned} & x(\alpha) = (7,2), \qquad z(\alpha) = 31 - 14\alpha \quad \forall \alpha \in [0, 0.25], \\ & x(\alpha) = (5,3), \qquad z(\alpha) = 30 - 10\alpha \quad \forall \alpha \in [0.25, 0.875], \\ & x(\alpha) = (1,4), \qquad z(\alpha) = 23 - 2\alpha \quad \forall \alpha \in [0.875, 1], \\ & \underset{\sim}{S}_\beta = \{(7,2)/0.25, (5,3)/0.875, (1,4)/1\}. \end{aligned}$$

For $\beta = (0.5,0.5)$:

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \le 12, \\ & 2x_1 + 8x_2 \le 35, \\ & x_j \ge 0, \quad x_j \in \mathbb{N}, \quad j \in N, \quad \alpha \in [0,1], \end{aligned}$$

and the corresponding optimal solution

$$x(\alpha) = (7,2), \qquad z(\alpha) = 31 \quad \forall \alpha \in [0,1], \qquad \underset{\sim}{S}_\beta = \{(7,2)/1\}.$$

Using Ranking Function the auxiliary problem obtained is

a) Using the ranking function of Chang,

$$\max \left\{ 2x_1^2 + 3.33x_1x_2 \,|\, 2x_1 - x_2 \le 12,\, 2x_1 + 8x_2 \le 35,\, x_j \ge 0,\, x_j \in \mathbb{N},\, j \in N \right\},$$

the optimal solution of which is $x^* = (7,2)$.

b) Using the ranking functions of Yager,

$$\max \left\{ 3x_1 + 5x_2 \,|\, 2x_1 - x_2 \le 12,\, 2x_1 + 8x_2 \le 35,\, x_j \ge 0,\, x_j \in \mathbb{N},\, j \in N \right\},$$

$$\max \left\{ \frac{5x_1 + 5x_2}{2x_1 + 1} \,|\, 2x_1 - x_2 \le 12,\, 2x_1 + 8x_2 \le 35,\, x_j \ge 0,\, x_j \in \mathbb{N},\, j \in N \right\},$$

$$\max \left\{ 3x_1 + 5x_2 \,|\, 2x_1 - x_2 \le 12,\, 2x_1 + 8x_2 \le 35,\, x_j \ge 0,\, x_j \in \mathbb{N},\, j \in N \right\},$$

and the corresponding optimal solutions are $x^* = (7,2)$, $x^* = (0,4)$ and $x^* = (7,2)$ respectively.

Remark. As we can see according to the solution method used we have different solutions, which is in accordance with the imprecise raising of the problem. When the representation theorem is used then a fuzzy solution is obtained, which contains good alternatives, and hence the decision maker eventually makes the final choice himself.

4. FILP problems with fuzzy numbers as coefficients of the technological matrix

Now, we consider FILP problems with fuzzy numbers defining the set of constraints. These can be formulated as follows:

$$\begin{aligned} \max \quad & z = cx \\ \text{s.t.} \quad & \sum_{j \in N} \underset{\sim}{a}_{ij} x_j \lesssim \underset{\sim}{b}_i, \quad i \in M, \\ & x_j \ge 0, \quad j \in N, \\ & x_j \in \mathbb{N}, \quad j \in N, \end{aligned} \tag{24}$$

where $\underset{\sim}{a}_{ij}$, $\underset{\sim}{b}_i \in F(\mathbb{R})$. The symbol $\lesssim$ means, as in (2), that the decision-maker permits certain flexible accomplishment for the constraints. Thus, the following membership functions are considered:
For each row (constraint) in (24),

$$\exists \mu_i \in F(\mathbb{R}) \text{ such that } \mu_i : \mathbb{R} \to [0,1], \quad i \in M, \tag{25}$$

which defines the fuzzy number on the right-hand side.
For each $i \in N$ and $j \in N$,

$$\exists \mu_{ij} \in F(\mathbb{R}) \text{ such that } \mu_{ij} : \mathbb{R} \to [0,1], \tag{26}$$

defining the fuzzy numbers in the technological matrix.
For each row of (24),

$$\exists \mu^i \in F[F(\mathbb{R})], \quad \mu^i : F(\mathbb{R}) \to (0,1], \tag{27}$$

giving, for every $x \in \mathbb{R}^n$, the accomplishment degree of the fuzzy number

$$\underset{\sim}{a}_{i1} x_1 + \underset{\sim}{a}_{i2} x_2 + \cdots + \underset{\sim}{a}_{in} x_n, \quad i \in N,$$

with respect to the i-th constraint, that is, the adequacy between this fuzzy number and the corresponding one $\underset{\sim}{b}_i$ with respect to the i-th constraint.

Let $\underset{\sim}{t}_i$ be a fuzzy number, fixed by the decision maker, giving his allowed maximum violation in the accomplishment of the i-th constraint. Then as an auxiliary problem to solve (24), one can propose the following one:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} \underset{\sim}{a}_{ij} x_j \boxed{<} \underset{\sim}{b}_i + \underset{\sim}{t}_i (1-\alpha), \quad i \in M, \\ & x_j \geq 0, \quad a \in (0,1] \quad j \in N, \end{aligned} \tag{28}$$

where $\boxed{<}$ represents a relation between fuzzy numbers. Moreover, according to the characteristics of the relation $\boxed{<}$, different models of conventional LP problems are obtained. Considering this relation as a ranking function, the auxiliary model obtained is the following:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & f\left(\sum_{i=1}^{n} \underset{\sim}{a}_{ij} x_j \right) \leq f(\underset{\sim}{b}_i + \underset{\sim}{t}_i (1-\alpha)), \quad i \in M, \\ & x_j \geq 0, \quad \alpha \in (0,1], \quad j \in N, \end{aligned} \tag{28}$$

and, if we use LRF, then (28) becomes the following parametric LP problem:

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} f(\underset{\sim}{a}_{ij}) x_j \leq f(\underset{\sim}{b}_i) + f(\underset{\sim}{t}_i)(1-\alpha), \quad i \in M, \\ & x_j \geq 0, \quad \alpha \in (0,1], \quad j \in N. \end{aligned} \tag{29}$$

4.1. Numerical example

Consider the following problem:

$$\begin{aligned} \max \quad & z = 2x_1 + 5x_2 \\ \text{s.t.} \quad & \underset{\sim}{2} x_1 - \underset{\sim}{1} x_2 \lesssim \underset{\sim}{9}, \\ & \underset{\sim}{2} x_1 + \underset{\sim}{8} x_2 \lesssim \underset{\sim}{31}, \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{N}, \end{aligned}$$

with
$\underset{\sim}{a}_{11} = (1, 2, 3)$, $\underset{\sim}{a}_{12} = (0.5, 1, 2)$, $\underset{\sim}{b}_1 = (7, 9, 10)$, $\underset{\sim}{t}_1 = (2.5, 3, 4,)$ and
$\underset{\sim}{a}_{21} = (1.5, 2, 3.5)$, $\underset{\sim}{a}_{22} = (7, 8, 10)$, $\underset{\sim}{b}_2 = (29, 31, 35)$, $\underset{\sim}{t}_2 = (3, 4, 6)$.

By means of (28) the auxiliary problem is written as

$$\begin{aligned} \max \quad & z = 2x_1 + 5x_2 \\ \text{s.t.} \quad & \underset{\sim}{2}x_1 - \underset{\sim}{1}x_2 \boxed{<} \underset{\sim}{9} + \underset{\sim}{3}(1-\alpha), \\ & \underset{\sim}{2}x_1 + \underset{\sim}{8}x_2 \boxed{<} \underset{\sim}{31} + \underset{\sim}{4}(1-\alpha), \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{N}, \quad \alpha \in (0,1], \end{aligned}$$

and applying the ranking function for fuzzy numbers, the auxiliary parametric models which represent the preferences according to the ranking method are obtained. In this example, we apply a linear ranking function, the first index of Yager.

a) Using the first index of Yager,

$$\begin{aligned} \max \quad & z = 2x_1 + 5x_2 \\ \text{s.t.} \quad & 2x_1 - 1.166x_2 \leq 8.666 + 3.166(1-\alpha), \\ & 2.333x_1 + 8.333x_2 \leq 31.666 + 4.333(1-\alpha), \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{N}, \quad \alpha \in (0,1], \end{aligned}$$

whose optimal solution is

$$\underset{\sim}{S} = \{(7,2)/0.052, (4,3)/0.384, (6,2)/0.683, (3,3)/0.923, (5,2)/1\}.$$

Note how different this solution is from the corresponding one in the parallel model without fuzzy numbers in the technological matrix shown in Section 2.1.

If we use nonlinear ranking functions then nonlinear parametric programming problems are obtained, which makes its solution more complicated. For example, if we use the index of Chang, [4], we obtain the next auxiliary problem.

b) Using the index of Chang,

$$\begin{aligned} \max \quad & z = 2x_1 + 5x_2, \\ \text{s.t.} \quad & (2x_1 + 1.5x_2)(x_1 - 0.583x_2) \leq [3 + 1.5(1-\alpha)][4.333 + 1.583(1-\alpha)], \\ & (2x_1 + 3x_2)(1.666x_1 + 4.166x_2) \leq [6 + 3(1-\alpha)][15.833 + 2.166(1-\alpha)], \\ & x_1, x_2 \geq 0, \\ & x_1, x_2 \in \mathbb{N}, \quad \alpha \in (0,1], \end{aligned}$$

which is a nasty nonlinear parametric programming problem.

5. Conclusions

In this paper we study three models for dealing with the lack of precision of a vague nature in the formulation of ILP problems, with either fuzzy constraints, or fuzzy numbers in the objective function or fuzzy numbers defining the set of constraints. Some approaches based on the representation theorem and on FNRP have been provided to solve them.

The computational burden of the auxiliary method is in relation to the number of auxiliary integer linear programming problems that we must solve.

All the models obtained in the paper have expressions for triangular fuzzy numbers, but all of them may be easily rewritten for the case of fuzzy numbers of LR kind.

As we have already said, the use of the representation theorem gives us integer parametric auxiliary problems, the solutions of which are used for building the fuzzy solution of the models. The fuzzy solutions are in accordance with the imprecise raising of the problems, and contain good alternatives, and hence the decision maker eventually makes the final choice himself. On the other hand, when we use ranking methods, each method has its own advantages over the others in some particular situations, the choice of a ranking method has effects on the results as we can see in the examples, and also the decision maker must choose the final ranking method according to his preferences.

Finally, we must also point out the necessity of developing an interactive decision support system in fuzzy integer programming problems, which would allow intelligent decisions according to the actuation preferences of the decision makers. This problem will be dealt with in future papers.

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References

[1] Bailey, M.G., and Gillett, B.E., "Parametric integer programming analysis: A contraction approach", *Journal of the Operational Research Society* 31 (1980) 253–262.
[2] Bortolan, G., and Degani, R., "A review of some methods for ranking fuzzy subsets", *Fuzzy Sets and Systems* 15 (1985) 1–20.
[3] Campos, L., Gonzalez, A., and Vila, M.A., "On the use of the ranking function approach to solve fuzzy matrix games in a direct way", *Fuzzy Sets and Systems* 49 (1992) 1–11.
[4] Chang, W., "Ranking of fuzzy utilities with triangular membership functions", in: *Proc. Int. Conf. on Policy Anal. and Inf. Systems*, 1981, 263–272.
[5] Delgado, M. Verdegay, J.L., and Vila, M.A., "Imprecise costs in mathematical programming problems", *Control and Cybernetics* 16 (1987) 113–121.
[6] Delgado, M., Verdegay, J.L., and Vila, M.A., "A general model for fuzzy linear programming", *Fuzzy Sets and Systems* 29 (1989) 21–29.
[7] Delgado, M., Herrera, F., Verdegay, J.L., and Vila, M.A., "Post-optimality analysis on the membership functions of a fuzzy linear programming problem", *Fuzzy Sets and Systems* 53 (1993) 280–297.
[8] Fabian, C., and Stoica, M., "Fuzzy integer programming", in: H.J. Zimmermann, L.A. Zadeh and B.R. Games (eds), *Fuzzy Sets and Decision Analysis*, North-Holland, (Amsterdam), 1984, 123–131.
[9] Garcia-Aguado, C., and Verdegay, J.L., "On the sensitivity of membership functions for fuzzy linear programming problems", *Fuzzy Sets and Systems* 56 (1993) 47–49.
[10] Gonzalez, A., "A study of the ranking function approach through mean values", *Fuzzy Sets and Systems* 35 (1990) 29–41.
[11] Herrera, F., "Problems and algorithms in fuzzy integer programming", Ph.D. Thesis, University of Granada, 1991.
[12] Herrera, F., and Verdegay, J.L., "Approaching fuzzy integer linear programming problems", in: M. Fedrizzi, J. Kacprzyk and M. Roubens (eds.), *Interactive Fuzzy Optimization*, Springer-Verlag, Berlin, 1991.
[13] Hooker, J.N., "A quantitative approach to logical inference", *Decision Support Systems* 4 (1988) 45–69.
[14] Ishibuchi, H., and Tanaka, H., "Multiobjective programming in optimization of the interval objective function", *European Journal of Operational Research* 48 (1990) 219–225.
[15] Jeroslow, R.G., *Logic-Based Decision Support Mixed Integer Model Formulation*, North-Holland, Amsterdam, 1989.
[16] Orlovski, S.A., "On programming with fuzzy constraint sets", *Kybernetes* 6 (1977) 197–201.
[17] Taha, H.A., *Integer Programming. Theory, Applications and Computations*, Academic Press, New York, 1975.

[18] Verdegay, J.L., "Fuzzy mathematical programming", in: M.M. Gupta and E. Sanchez (eds.), *Fuzzy Information and Decision Processes*, North-Holland, Amsterdam, 1982, 231–237.
[19] Verdegay, J.L., "A dual approach to solve the fuzzy linear programming problem", *Fuzzy Sets and Systems* 14 (1984) 131–141.
[20] Williams, H.P., "Linear and integer programming applied to the propositional calculus", *International Journal of Systems Research and Information Science* 2 (1987) 81–100.
[21] Yager, R.R., "Ranking fuzzy subsets over the unit interval", in: *Proc. 1978 CDC*, 1978, 1435–1437.
[22] Yager, R.R., "A procedure for ordering fuzzy subsets of the unit interval". *Information Science* 24 (1981) 143–161.
[23] Yager, R.R., "A mathematical programming approach to inference with the capability of implementing default rules", *International Journal of Man–Machine Studies* 29 (1988) 685–714.
[24] Zimmermann, H.J., and Pollatschek, M.A., "Fuzzy 0–1 linear programs", in: H.J. Zimmermann, L.A. Zadeh and B.R. Gaines (eds.), *Fuzzy Sets and Decision Analysis*, North-Holland, (Amsterdam), 1984, 133–145.