Aggregation Operators for Linguistic Weighted Information

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***Abstract*—The aim of this paper is to model the processes of the aggregation of weighted information in a linguistic framework. Three aggregation operators of weighted linguistic information are presented: linguistic weighted disjunction (LWD) operator, linguistic weighted conjunction (LWC) operator, and linguistic weighted averaging (LWA) operator. A study of their axiomatics is presented to demonstrate their rational aggregation.**

***Index Terms*— Aggregation operators, fuzzy linguistic quantifier, linguistic modeling.**

I. Introduction

SOME situations present precise and rigorous quantitative aspects as well as fuzzy and unrigorous qualitative aspects. Therefore, phenomena must be defined using quantitative concepts as well as qualitative concepts. Dealing with quantitative concepts is easy, and it may be done by means of the numerical variables. The problem is how to deal with qualitative concepts. The use of fuzzy set theory, proposed and developed by Zadeh [34], has given very good results modeling the qualitative aspects [35]. Fuzzy set theory provides a flexible framework, where it is possible to satisfactorily solve many of the obstacles of lack of precision. It is an approximate technique in its nature, which represents the qualitative aspects in qualitative terms (linguistic terms) by means of *linguistic variables*, that is, variables whose values are not numbers but words or sentences in a natural or artificial language. The use of words or sentences rather than numbers is, in general, a less specific, more flexible, direct, realistic, and adequate form to express the qualitative aspects and is very widespread, as may be seen in [1], [4], [6], [13], [22]–[24], [31], [33].

On the other hand, we can find situations where the information handled is not equally important, i.e., the framework is heterogeneous. For example, when a group of medical experts expresses its opinions on the possible illness that a patient presents, on the one hand, its diagnostics must not be considered with equal relevance, given that, there will be medical experts with more experience or with more study years than others, and therefore, all the opinions shall not be equally reliable; but, on the other hand, a final and global diagnostic must be made using the initial and individual diagnostics. This heterogeneous framework has been considered by various authors in opinions aggregation operators [2], [10], [20], [32]; consensus models in group decision making [14], [16]; fuzzy pattern matching [9]; and knowledge systems [21].

One way of modeling the first aspect is by assigning a weight to each medical expert. The weights are quantitative or qualitative values, which may be interpreted in at least two different ways [9], [10].

1) Each medical expert is viewed as a subgroup and the weight reflects the relative size of this subgroup.
2) The weight may reflect the relevance of the medical expert in the group. This level of relevance may act as a constraint on the opinions that a medical expert may express.

One way of dealing with the second aspect, in general, is to use adequate operators for combining information, usually called information aggregation operators, before reaching a final decision or action. Issues of weighted aggregation operators have been studied in a quantitative setting in [2], [8], [9], [20], [21], [25], [27], and [32], and in a qualitative setting in [3] and [31]–[33].

In short, we can find situations where the information handled is imprecise by nature and is not equally important, and where some appropriate aggregation operators of weighted information are required. According to this idea, in this paper, we will present three aggregation operators for linguistic[1]weighted information (linguistic variables for expressing experts' opinions and linguistic weights on the experts)

1) *Linguistic Weighted Disjunction (LWD)*;
2) *Linguistic Weighted Conjunction (LWC)*; and
3) *Linguistic Weighted Averaging (LWA)*.

They are defined using *the LOWA operator* [12], [15], *the weighted minimum and maximum operators* [8], two families of connectives [11], and the concept of *fuzzy majority* represented by a *fuzzy linguistic quantifier* [36]. In order to demonstrate the good performance of these operators, we shall study some postulated axiomatics and properties of an intuitively acceptable weighted aggregation operator.

In order to do so, the paper is structured as follows. Section II shows the considered linguistic framework and the LOWA operator. Section III presents the linguistic weighted aggregation operators and some of their properties. Section IV contains an example of the application of these operators in decision making for nonhomogeneous groups. Finally, some conclusions are discussed.

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[1] The word "linguistic" is related to the concept of "linguistic variables" in a formal way, and it does not imply some connections to linguistics.

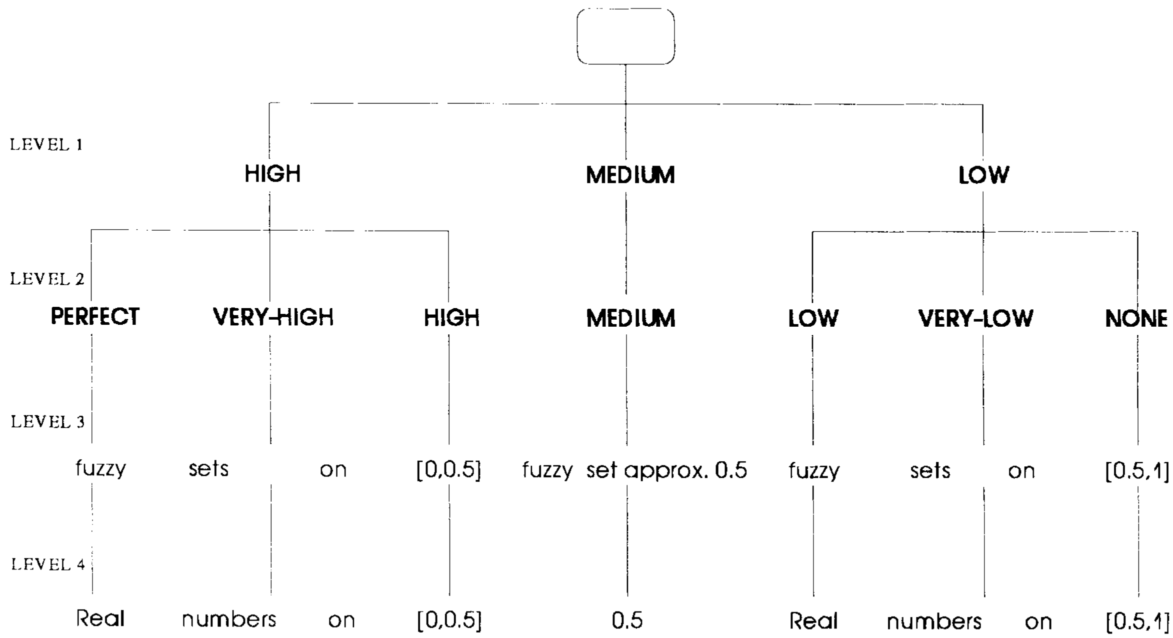

Fig. 1. Hierarchy of labels.

II. Preliminaries

Here, before defining our aggregation operators, we shall present the work hypothesis. We will specify a concrete linguistic model to represent the information and the LOWA operator to aggregate linguistic information.

A. Linguistic Approach

Usually, in a quantitative setting, the information is expressed by means of numerical values. However, when we work in a qualitative setting, that is, with vague or imprecise knowledge, the information cannot be estimated with an exact numerical value. In that case, a more realistic approach may be to use linguistic assessments instead of numerical values [35], that is, to suppose that the variables which participate in the problem are assessed by means of linguistic terms [5], [6], [12], [13], [22], [31], [35]. This approach is appropriate for a lot of problems, since it allows a representation of the information in a more direct and adequate form if we are unable to express it with precision.

A linguistic variable differs from a numerical one in that its values are not numbers, but words or sentences in a natural or artificial language. Since words, in general, are less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means of approximated characterization of phenomena, which are too complex, or too ill-defined, to be amenable to their description in conventional quantitative terms.

Definition [35]: A linguistic variable is characterized by a quintuple $(H, T(H), U, G, M)$. H is the name of the variable; $T(H)$ (or simply T) denotes the term set of H, i.e., the set of names of linguistic values of H, with each value being a fuzzy variable denoted generically by X and ranging across a universe of discourse U which is associated with the base variable u. G is a syntactic rule (*which usually takes the form of a grammar*) for generating the names of values of H, and M is a semantic rule for associating its meaning with each $H, M(X)$, which is a fuzzy subset of U.

Usually, depending on the problem domain, an appropriate linguistic term set is chosen and used to describe the vague or imprecise knowledge. The elements in the term set will determine the granularity of the uncertainty, that is the level of distinction among different countings of uncertainty. In [1] the use of term sets with an odd cardinal was studied; the mid term represents an assess of "approximately 0.5"; the rest of the terms are placed symmetrically around it, and the limit of granularity 11 or no more than 13.

For instance, Fig. 1 shows a hierarchical structure of linguistic values or labels. Clearly, level 1 provides a granularity containing three labels, level 2 a granularity with nine labels, and of course, different granularity levels could be presented. In fact, in Fig. 1, level 4 presents the finest granularity in a decision process—the numerical values.

On the other hand, the semantic of the elements in the term set is given by fuzzy numbers defined in the [0, 1] interval, which are described by membership functions. Because the linguistic assessments are just approximate ones given by the individuals, we can consider that linear trapezoidal membership functions are good enough to capture the vagueness of those linguistic assessments, since it may be impossible or unnecessary to obtain more accurate values. This representation is achieved by the four-tuple $(a_i, b_i, \alpha_i, \beta_i)$. The first two parameters indicate the interval in which the membership value is 1; the third and fourth parameters indicate the left and right width. Formally speaking, it seems difficult to accept that all individuals should agree on the same membership function associated to linguistic terms, and therefore, there are not any universality distribution concepts.

It is well known and accepted that the tuning of membership functions is a crucial issue in control processes with linguistic rules. In our context, we consider an environment where individuals can discriminate perfectly the same term set under a similar conception, taking into account that the concept of a linguistic variable serves the purpose of providing a means of approximated characterization of imprecise preference information. Moreover, in our development, we do not use the membership functions for aggregating the labels; we define aggregation operators for linguistic labels by direct computation on labels.

B. Characterization of the Linguistic Label Set

Accordingly, to establish what kind of label set to use ought to be the first priority. Then, let $S = \{s_i\}, i \in H = \{0, \cdots, T\}$ be a finite and totally ordered term set on [0,1] in the usual sense [1], [4]. Any label, s_i, represents a possible value for a linguistic variable, that is, a vague property or constraint on [0,1]. We consider a term set, S, as in [1] with its semantic given by linear trapezoidal membership functions. Moreover, it must have the following characteristics:

1) the set is ordered: $s_i \geq s_j$ if $i \geq j$;
2) there is the negation operator: $\text{Neg}(s_i) = s_j$ such that $j = T - i$;
3) maximization operator: $\text{Max}(s_i, s_j) = s_i$ if $s_i \geq s_j$;
4) minimization operator: $\text{Min}(s_i, s_j) = s_i$ if $s_i \leq s_j$.

For example, this is the case of the following term set of the level 2 of the Fig. 1:

$$\begin{aligned}
P &= \text{Perfect} = (1, 1, .25, 0)\\
VH &= \text{Very_High} = (.75, .75, .15, .25)\\
H &= \text{High} = (.6, .6, .1, .15)\\
M &= \text{Medium} = (.5, .5, .1, .1)\\
L &= \text{Low} = (.4, .4, .15, .1)\\
VL &= \text{Very_Low} = (.25, .25, .25, .15)\\
N &= \text{None} = (0, 0, 0, .25).
\end{aligned}$$

C. The LOWA Operator

Assuming the proposed linguistic approach, two main approaches can be found in order to aggregate linguistic values: the first acts by direct computation on labels [5], and the second uses the associated membership functions [1], [22], [35].

Most available techniques belong to the latter. However, the final results of those methods are fuzzy sets which do not correspond to any label in the original term set. If one finally wants to have a label, then a "linguistic approximation" is needed [1], [22], [23], [35]. The process of linguistic approximation consists of finding a label whose meaning is the same or the closest (according to some metric) to the meaning of an unlabeled membership function generated by some computational model.

In this context, to manipulate the linguistic information, we shall work with operators for combining the linguistic values (nonweighted and weighted) by direct computation on labels. Specifically, in this section we shall present the nonweighted operator of combination of the linguistic values based on direct computation, the LOWA operator [12], [15], which will be used latter in the definition of the three weighted operators of combination of linguistic values by direct computation that we propose here.

The *linguistic ordered weighted averaging* (LOWA) operator, defined in [12] and [15], is based on the *ordered weighted averaging* (OWA) operator defined by Yager [28], and on the *convex combination of linguistic labels* defined by Delgado *et al*. [5].

Definition 1: Let $A = \{a_1, \cdots, a_m\}$ be a set of labels to be aggregated, then the LOWA operator, ϕ, is defined as

$$\begin{aligned}
&\phi(a_1, \cdots, a_m)\\
&\quad = W \cdot B^T = \boldsymbol{C}^m\{w_k, b_k, k = 1, \cdots, m\}\\
&\quad = w_1 \odot b_1 \oplus (1 - w_1) \odot \boldsymbol{C}^{m-1}\{\beta_h, b_h, h = 2, \cdots, m\}
\end{aligned}$$

where $W = [w_1, \cdots, w_m]$, is a weighting vector, such that,

1) $w_i \in [0, 1]$ and,
2) $\Sigma_i w_i = 1$.

$\beta_h = w_h / \Sigma_2^m w_k, h = 2, \cdots, m$, and $B = \{b_1, \cdots, b_m\}$ is a vector associated to A, such that,

$$B = \sigma(A) = \{a_{\sigma(1)}, \cdots, a_{\sigma(n)}\}$$

where,

$$a_{\sigma(j)} \leq a_{\sigma(i)} \quad \forall i \leq j,$$

with σ being a permutation over the set of labels A. $\boldsymbol{C}^m$ is the convex combination operator of m labels and if $m = 2$, then it is defined as

$$\begin{aligned}
&\boldsymbol{C}^2\{w_i, b_i, i = 1, 2\}\\
&\quad = w_1 \odot s_j \oplus (1 - w_1) \odot s_i = s_k, s_j, s_i\\
&\quad \in S, (j \geq i)
\end{aligned}$$

such that

$$k = \min\{T, i + \text{round}(w_1 \cdot (j - i))\}$$

where "round" is the usual round operation, and $b_1 = s_j, b_2 = s_i$.

If $w_j = 1$ and $w_i = 0$ with $i \neq j \quad \forall i$, then the convex combination is defined

$$\boldsymbol{C}^m\{w_i, b_i, i = 1, \cdots, m\} = b_j.$$

In [15], we demonstrated that the LOWA operator presents some evidence of rational aggregation, because, on the one hand, it verifies the following properties:

- the LOWA operator is *increasing monotonous* with respect to the argument values;
- the LOWA operator is *commutative*; and
- the LOWA operator is an *"orand"* operator.

And on the other hand, it verifies these axioms: *unrestricted domain*, *unanimity or idempotence*, *positive association of social and individual values*, *independence of irrelevant alternatives*, *citizen sovereignty*, and *neutrality*.

Here, we present an extension of the LOWA operator, an *inverse LOWA operator,* that will be used in the definition of some weighted operators.

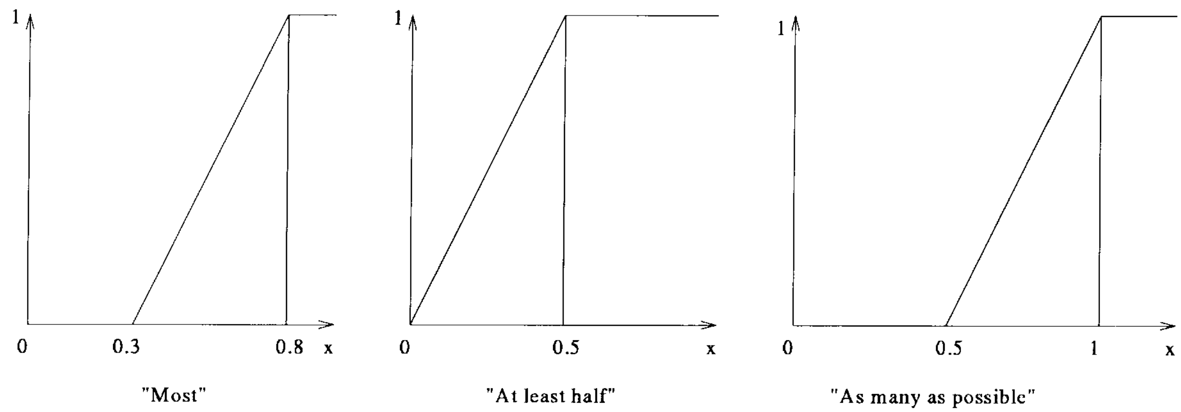

Fig. 2. Proportional fuzzy linguistic quantifiers.

Definition 2: An I-LOWA (Inverse-Linguistic Ordered Weighted Averaging) operator, ϕ^I, is a type of LOWA operator, in which

$$B = \sigma^I(A) = \{a_{\sigma(1)}, \cdots, a_{\sigma(n)}\}$$

where,

$$a_{\sigma(i)} \leq a_{\sigma(j)} \quad \forall i \leq j.$$

If $m = 2$, then it is defined as

$$\begin{aligned} \boldsymbol{C}^2\{w_i, b_i, i = 1, 2\} &= w_1 \odot s_j \oplus (1 - w_1) \odot s_i = s_k, s_j, s_i \\ &\in S, (j \leq i) \end{aligned}$$

such that

$$k = \min\{T, i + \text{round}(w_1 \cdot (j - i))\}.$$

If the definition of the LOWA operator is compared to the definition of the I-LOWA operator, it is possible to observe that in the first one the large values are more estimated than low values, unlike in the second one. Therefore, from this viewpoint, the LOWA operator presents characteristics belonging to the maximum aggregation operator, and the I-LOWA operator presents characteristics belonging to the minimum aggregation operator. This peculiarity will be used later in the definition of one of our weighted aggregation operators.

Clearly, an I-LOWA operator also verifies the previously mentioned properties and axioms of the LOWA operator.

1) The LOWA Operator Guided by Fuzzy Majority: How to calculate the weighting vector of the LOWA operator W is a basic question. Yager proposed in [28] and [30] two ways to do so. The first approach is to use some kind of learning mechanism using sample data; the second approach is to try to give some semantics or meaning to the weights. We consider the latter approach, because our idea is to show the concept of *fuzzy majority* by means of the weighting vector in the LOWA operator aggregations.

Traditionally, the majority is defined as a threshold number of individuals. Fuzzy majority is a soft majority concept which is manipulated via a fuzzy logic based calculus of linguistically quantified propositions. In [17], Kacprzyk specified fuzzy majority rule by means of a *linguistic quantifier* to derive various solutions concepts for group decision making problems in a numerical setting. Here, we shall work in a similar way, but in the field of quantifier-guided aggregations. Before showing how do so, we will introduce the concept of fuzzy linguistic quantifier.

2) Fuzzy Linguistic Quantifier: Human discourse is very rich and diverse in its quantifiers, e.g., *about 5, almost all, a few, many, most, as many as possible, nearly half, at least half.* Zadeh, using Fuzzy logic, introduced the concept of *linguistic quantifier* to represent the large number of possible quantifiers [36]. Zadeh suggested that the semantic of a linguistic quantifier can be captured by using fuzzy subsets for its representation. He distinguished between two types of linguistic quantifiers: *absolute* and *proportional.* Absolute quantifiers are used to represent amounts that are absolute in nature such as *about 2* or *more than 5*. These absolute linguistic quantifiers are closely related to the concept of the counting or number of elements. Proportional quantifiers are used to represent amounts that are relative in nature such as such as *most, at least half.* A proportional quantifier can be represented by a fuzzy subset Q in the unit interval, [0,1], such that for any $r \in [0, 1], Q(r)$ indicates the degree to which the proportion r is compatible with the meaning of the quantifier it represents.

A proportional quantifier, Q: $[0, 1] \rightarrow [0, 1]$, satisfies

$$Q(0) = 0, \quad \text{and} \quad \exists r \in [0, 1] \text{ such that } Q(r) = 1.$$

A nondecreasing quantifier satisfies

$$\forall a, b \text{ if } a > b \text{ then } Q(a) \geq Q(b).$$

The membership function of a nondecreasing proportional quantifier can be represented as

$$Q(r) = \begin{cases} 0 & \text{if } r < a \\ \dfrac{r - a}{b - a} & \text{if } a \leq r < b \\ 1 & \text{if } r \geq b \end{cases}$$

with $a, b, r \in [0, 1]$.

Some examples of proportional quantifiers are shown in Fig. 2, where the parameters, (a, b) are (0.3, 0.8), (0, 0.5), and (0.5, 1), respectively.

In [28] and [30], Yager suggested an interesting way to compute the weights of the OWA aggregation operator using

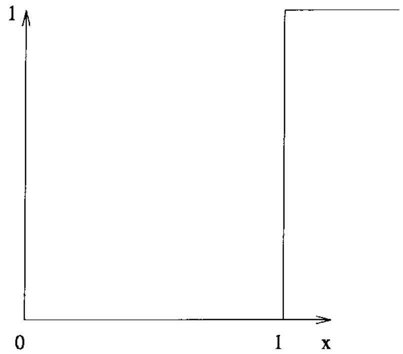

Fig. 3. Fuzzy linguistic quantifier "All."

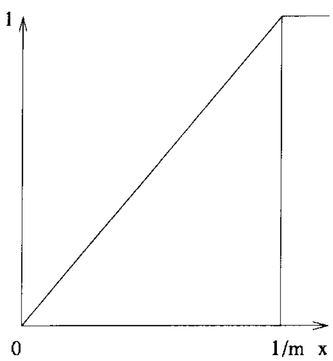

Fig. 4. Fuzzy linguistic quantifiers "At least m."

linguistic quantifiers, which, in the case of a nondecreasing proportional quantifier Q, is given by the expression

$$w_i = Q(i/n) - Q((i-1)/n), \qquad i = 1, \cdots, n.$$

When a fuzzy linguistic quantifier Q is used to compute the weights of LOWA operator ϕ, it is symbolized by ϕ_Q. Therefore, when a fuzzy linguistic quantifier Q is used to compute the weights of the I-LOWA operator ϕ^I, it is symbolized by ϕ^I_Q.

Clearly, depending on the fuzzy linguistic quantifier that is chosen to calculate the weights, it is possible to observe the following properties:

1) if the fuzzy linguistic quantifier is *"All"*, as is shown in Fig. 3, whose membership function is

$$Q(r) = \begin{cases} 0 & \text{if } 0 \le r < 1 \\ 1 & \text{otherwise} \end{cases}$$

then $\phi_Q(a_1, \cdots, a_m) = \text{MIN}(a_1, \cdots, a_m)$ and $\phi^I_Q(a_1, \cdots, a_m) = \text{MAX}(a_1, \cdots, a_m)$;

2) if the fuzzy linguistic quantifier is *"At least m"* ($m \in N$), as is shown in Fig. 4, whose membership function is

$$Q_m(r) = \begin{cases} \dfrac{r}{(1/m)} & \text{if } r < (1/m), r \in [0,1] \\ 1 & \text{if } r \ge (1/m) \end{cases}$$

then $\phi_{Qm}(a_1, \cdots, a_m) = \text{MAX}(a_1, \cdots, a_m)$ and $\phi^I_{Qm}(a_1, \cdots, a_m) = \text{MIN}(a_1, \cdots, a_m)$.

Where "MAX" stands for *maximum operator* and "MIN" stands for *minimum operator*.

III. Aggregation Operators for Linguistic Weighted Information

In this section, the following three aggregation operators for linguistic weighted information based on the direct computation on labels are presented:

1) *Linguistic Weighted Disjunction* (LWD);
2) *Linguistic Weighted Conjunction* (LWC); and
3) *Linguistic Weighted Averaging* (LWA).

Following Cholewa's studies [2] and Montero's aggregation model [20], if we want to aggregate weighted information we have to define two aggregations as follows:

- the aggregation of importance degrees (weights) of information; and
- the aggregation of weighted information (information combined with weights).

The first aspect consists of obtaining a collective importance degree from individual importance degrees that characterizes the final result of aggregation operator. In the three operators, as the importance degrees are linguistic values, this is solved using the LOWA operator guided by the concept of fuzzy majority.

The aggregation of weighted information involves the transformation of the weighted information under the importance degrees. The transformation form depends upon the type of aggregation of weighted information being performed [32]. In [26] and [27], Yager discussed the effect of the importance degrees in the types of aggregation "MAX" and "MIN" and suggested a class of functions for importance transformation in both types of aggregation. For MIN type aggregation he suggested a family of t-conorms acting on the weighted information and the negation of the weights, which presents the nonincreasing monotonic property in the weights. For MAX type aggregation, he suggested a family of t-norms acting on weighted information and the weight, which presents the nondecreasing monotonic property in the weights. In [32], Yager proposed a general specification of the requirements that any *importance transformation function* g must satisfy for any type of the aggregation operator. The function g must have the following properties:

1) if $a > b$ then $g(w, a) \ge g(w, b)$;
2) $g(w, a)$ is monotone in w;
3) $g(0, a) = \text{ID}$; and
4) $g(1, a) = a$;

with $a, b \in [0, 1]$ expressing the satisfaction with regards to a criterion $w \in [0, 1]$ the weight associated to the criterion, and "ID" an identity element, which is such that if we add it to our aggregations it does not change the aggregated value. Condition one means that the function g is monotonically nondecreasing in the second argument, that is, if the satisfaction with regards to the criteria is increased the overall satisfaction should not decrease. The second condition may be viewed as a requirement that the effect of the importance be consistent. It does not specify whether g is monotonically nonincreasing or nondecreasing in the first argument, but must be one of these. It should be noted that conditions three and four actually determine the type of monotonicity obtained from two. If $a >$ ID, the $g(w,a)$ is monotonically nondecreasing in w, while if $a <$ ID, then it is monotonically nonincreasing. The third condition is a manifestation of the imperative that zero importance items do not affect the aggregation process. The final condition is essentially a boundary condition which states that the assumption of all importances equal to one effectively is like not including importances at all [32].

Considering the aforementioned ideas and assuming a linguistic framework, that is a label set, S, to express the information and a label set, L, to express the weights, we propose using the following aggregations of weighted information for the three aggregation operators, with their respective aggregation operators and transformation functions.

- Linguistic weighted disjunction
  a) aggregation operator: MAX linguistic aggregation;
  b) transformation function: $g = \text{MIN}(w,a)$.
- Linguistic weighted conjunction
  a) aggregation operator: MIN linguistic aggregation;
  b) Transformation function: $g = \text{MAX}(\text{Neg}(w), a)$.
- Linguistic weighted averaging
  a) aggregation operator: LOWA or I - LOWA;
  b) Transformation function: $g_{(\text{LOWA})} = LC^{\rightarrow}(w,a)$ or $g_{(\text{I-LOWA})} = LI^{\rightarrow}(w,a)$.

The first two aggregations are based on *canonical generalizations of weighted disjunction and conjunction of fuzzy goals,* defined by Dubois and Prade in a possibility theory setting [7], [8]. The latter is based on the combination of the LOWA and I-LOWA operator with several *linguistic conjunction functions* $(LC^{\rightarrow})$ and several *linguistic implication functions* $(LI^{\rightarrow})$, respectively. Therefore, the LWA operator is a type of fuzzy majority guided weighted aggregation operator.

In the next subsections, we present each aggregation operator of linguistic weighted information in detail. In order to complete the presentation, in the final subsection we provide some evidence of the rationality of their aggregation, checking some of the axioms that they verify. We shall demonstrate that all the operators proposed combine appropriately the weighted information in such a way that the final aggregation is the "best" representation of the overall individual information.

A. Linguistic Weighted Disjunction and Conjunction

Let $\{(c_1,a_1),\cdots,(c_m,a_m)\}$ be a set of weighted opinions expressed by a set of experts, $E = \{e_1,\cdots,e_m\}$, to evaluate an alternative, x_j, where a_i shows the opinion of expert e_i, assessed linguistically on the label set, $S, a_i \in S$, and c_i the relevance degree of expert e_i, assessed linguistically on the label set $L, c_i \in L$.

Definition 1: The aggregation of the set of weighted individual opinions, $\{(c_1,a_1),\cdots,(c_m,a_m,)\}$, according to the Linguistic Weighted Disjunction (LWD) operator is defined as

$$(c_E, a_E) = LWD[(c_1,a_1),\cdots,(c_m,a_m)]$$

where the opinion of the group, a_E, is obtained as

$$a_E = \text{MAX}_{i=1,\cdots,m} \text{MIN}(c_i, a_i)$$

and the importance degree of the opinion of the group, c_E, is obtained as

$$c_E = \phi_Q(c_1,\cdots,c_m).$$

Definition 2: The aggregation of the set of weighted individual opinions, $\{(a_1,c_1),\cdots,(a_m,c_m)$ according to the Linguistic Weighted Conjunction (LWC) operator is defined as

$$(c_E, a_E) = LWC[(c_1,a_1),\cdots,(c_m,a_m)]$$

where the opinion of the group, a_E, is obtained as

$$a_E = \text{MIN}_{i=1,\cdots,m} \text{MAX}(\text{Neg}(c_i), a_i),$$

and the importance degree of the opinion of the group, c_E, is obtained as

$$c_E = \phi_Q(c_1,\cdots,c_m).$$

Remark: It is clear that both definitions always require the condition $S = L$.

In the definition of the LWD operator, the transformation function is the "MIN" function, that is, one of the t-norms proposed by Yager in [26] and [27] for the "MAX" type aggregation operator, but defined linguistically, and satisfies the properties proposed for any g [32]. Something similar happens in the definition of the LWC operator. In both operators it should be possible to choose any other function of the families proposed by Yager in [26] and [27], but always defined linguistically. In any case, both operators try to reduce the effect of elements with low importance. To do so, in the first operator, the elements with low importance are transformed into small values and in the second one into large values.

Since c_i expresses the degree of importance of the opinion of expert e_i in the overall opinion, then

- when $c_i = s_T$, the opinion of e_i has a direct influence on the acceptance (rejection) of alternative x_j;
- when $c_i = s_0$, the opinion of e_i has no influence on the acceptance (rejection) of alternative x_j.

As the LOWA operator, ϕ, is an *"orand"* operator, the importance degree of opinion of the group, c_E, verifies the following expression:

$$\text{MIN}(c_1,c_2,\cdots,c_m) \le c_E \le \text{MAX}(c_1,c_2,\cdots,c_m).$$

B. Linguistic Weighted Averaging

Before defining the linguistic weighted aggregation (LWA) operator, and assuming $S = L$, consider the following two families of connectives:

1) Linguistic conjunction functions $(LC^{\rightarrow})$.

The linguistic conjunction functions that we shall use are the following t-norms, which are monotonically nondecreasing in the weights and satisfy the properties required for any transformation function, g, [11]:

a) *the classical MIN operator*:

$$LC_1^{\rightarrow}(w,a) = \text{MIN}(w,a);$$

b) *the nilpotent MIN operator*:

$$LC_2^{\rightarrow}(w,a) = \begin{cases} \text{MIN}(w,a) & \text{if } w > \text{Neg}(a) \\ s_0 & \text{otherwise;} \end{cases}$$

c) *the weakest conjunction*:

$$LC_3^{\rightarrow}(w,a) = \begin{cases} \text{MIN}(w,a) & \text{if MAX}(w,a) = s_T \\ s_0 & \text{otherwise.} \end{cases}$$

2) Linguistic implication functions ($LI^{\rightarrow}$). The linguistic implication functions that we shall use are monotonically nonincreasing in the weights and satisfy the properties required for any transformation function g [11]:

a) *Kleene–Dienes's implication function*:

$$LI_1^{\rightarrow}(w,a) = \text{MAX}(\text{Neg}(w), a);$$

b) *Gödel's implication function*:

$$LI_2^{\rightarrow}(w,a) = \begin{cases} s_T & \text{if } w \le a \\ a & \text{otherwise;} \end{cases}$$

c) *Fodor's implication function*:

$$LI_3^{\rightarrow}(w,a) = \begin{cases} s_T & \text{if } w \le a \\ \text{MAX}(\text{Neg}(w), a) & \text{otherwise.} \end{cases}$$

Definition 3: The aggregation of the set of weighted individual opinions, $\{(c_1, a_1), \cdots, (c_m, a_m)\}$, according to the Linguistic Weighted Averaging (LWA) operator is defined as

$$(c_E, a_E) = LWA[(c_1, a_1), \cdots, (c_m, a_m)],$$

where the importance degree of the group opinion, c_E, is obtained as

$$c_E = \phi_Q(c_1, \cdots, c_m).$$

and, the opinion of the group, a_E, is obtained as

$$a_E = f[g(c_1, a_1), \cdots, g(c_m, a_m)],$$

where $f \in \{\phi_Q, \phi_Q^I\}, g \in \{LC_1^{\rightarrow}, LC_2^{\rightarrow}, LC_3^{\rightarrow}\}$ if $f = \phi_Q$, and $g \in \{LI_1^{\rightarrow}, LI_2^{\rightarrow}, LI_3^{\rightarrow}\}$ if $f = \phi_Q^I$.

It should be observed that according to the class of transformation functions proposed by Yager in [26] and [27] for MIN type aggregation, when the aggregation operator, f, is the I-LOWA operator, ϕ_Q^I, and given that ϕ_Q^I is an aggregation operator with characteristics of a MIN type aggregation operator (as was seen in the presentation of the LOWA operator), then we have decided to use the linguistic implications functions, $LI^{\rightarrow}$, as the transformation function type. Something similar happens when f is the LOWA operator ϕ_Q.

Lemma 1: The linguistic weighted disjunction operator, LWD, is a particular weighted aggregation operator of the LWA operator type.

Proof: Suppose that we have a group of m experts. If a linguistic nondecreasing relative quantifier, Q_m, "*At least* m", is chosen, as is shown in Fig. 3, the LOWA operator, ϕ_{Qm}, as an aggregation operator, and the following linguistic conjunction function as a transformation function,

$$LC_1^{\rightarrow}(w,a) = \text{MIN}(w,a),$$

then since the weighting vector is $W = [w_1 = 1, w_2 = 0, \cdots, w_m = 0]$,

$$\begin{aligned} a_E &= \phi_{Qm}[\text{MIN}(c_1, a_1), \cdots, \text{MIN}(c_m, a_m)] \\ &= \text{MAX}_{i=1,\cdots,m}\text{MIN}(c_i, a_i), \end{aligned}$$

is verified, and therefore,

$$LWA[(c_1, a_1), \cdots, (c_m, a_m)] = LWD[(c_1, a_1), \cdots, (c_m, a_m)].$$

Lemma 2: The linguistic weighted conjunction operator, LWC, is a particular weighted aggregation operator of the LWA operator type.

Proof: Assuming the above linguistic quantifier, if it is chosen the I-LOWA operator, ϕ_{Qm}^I, as aggregation operator, and the following linguistic implication operator as transformation function,

$$LI^{\rightarrow}(w,a) = \text{MAX}(\text{Neg}(w), a),$$

then as the weighting vector is

$$W = [w_1 = 1, w_2 = 0, \cdots, w_m = 0],$$

it is verified

$$\begin{aligned} a_E &= \phi_{Qm}^I[\text{MAX}(\text{Neg}(c_1), a_1), \cdots, \text{MAX}(\text{Neg}(c_m), a_m)] \\ &= \text{MIN}_{i=1,\cdots,m}\text{MAX}(\text{Neg}(c_i), a_i), \end{aligned}$$

and therefore,

$$\begin{aligned} &LWA[(c_1, a_1), \cdots, (c_m, a_m)] \\ &\quad = LWC[(c_1, a_1), \cdots, (c_m, a_m)]. \end{aligned}$$

C. Axiomatic of the Aggregation Operators for Linguistic Weighted Information

Previous works on the aggregation of fuzzy weighted opinions, developed in a numerical setting, are those by Cholewa, Montero and Dubois and Koning. Cholewa [2] offers a collection of axioms that weighted aggregations should follow, and proposes the weighted arithmetic mean as a typical aggregation operator that satisfies these axioms, Montero [20] characterizes the fuzzy majority rule and studies the existence of absolutely decisive groups, and Dubois and Koning [10] analyze briefly the different axiomatic approaches existing for weighted aggregation.

As was mentioned earlier, in [2] a complete set of axioms in the fuzzy set setting for heterogeneous groups is given. Some of these axioms are *independence of alternatives*, *commutativity*, etc. Obviously a particular weighted aggregation operator does not have to satisfy all the axioms together, it must satisfy those that its special application circumstances require. Bellows, we are going to postulate an axiomatic approach with ten axioms, and we shall check which axioms our weighted aggregation operators verify. Specifically, Axioms I–VI are obtained directly from those proposed by Cholewa in [2], but defined in linguistic setting, and others are proposed by ourselves.

As has been shown in the subsection above, if we choose the linguistic quantifier, Q, "*At least* m", and an appropriate transformation function, the LWA operator is a generalization of the LWD and LWC operators. Therefore, here, we shall only study the axiomatic of the LWA operator, and in those cases where the axiom not be verified, then, we shall study what happens with the LWD and LWC operators.

Assume the following framework:

Let $X = \{x_1, \cdots, x_n\} (n \geq 2)$ be a finite non-empty set of alternatives to be evaluated.

Let $E = \{e_1, \cdots, e_m\} (m \geq 2)$ be a group of the experts to analyze X.

Let $S = \{s_i: i = 0, \cdots, T\}$ be a label set to voice experts' opinions and their respective importance degrees.

Axiom I: Independence of alternatives $x_i \in X$. The collective opinion for $x_i, (c_E, a_E)$, only depends on the individual opinions for $x_i, [(c_1, a_1), \cdots, (c_m, a_m)]$. This means that linguistic functions v and w exist for the aggregation of linguistic weighted opinions

$$v: (S \times S)^m \rightarrow S$$

and for the aggregation of the powers of aggregated opinions

$$w: S^m \rightarrow S$$

such that

$$(c_E, a_E) = (w[c_1, \cdots, c_m], v[(c_1, a_1), \cdots, (c_m, a_m)]).$$

It is basically technical, and is satisfied by the definition of the LWA operator, with the LOWA operator being the function w and the composition of the I-LOWA operator with a linguistic implication operator $(LI^{\rightarrow})$ or the composition of the LOWA operator with a linguistic conjunction function $(LC^{\rightarrow})$ the function v.

Axiom II: Commutativity. Having fixed an alternative, $x_i \in X$, then

$$LWA[(c_1, a_1), \cdots, (c_m, a_m)] = LWA[\pi(c_1, a_1), \cdots, \pi(c_m, a_m)]$$

where π is a permutation over the set of weighted opinions. Clearly it is satisfied, because the I-LOWA operator as well as the LOWA operator use "ordered" weighted average of the arguments.

Axiom III: Associativity. Having fixed an alternative $x_i \in X$, then

$$LWA[(c_1, a_1), \cdots, (c_m, a_m)] = LWA[LWA[(c_1, a_1), \cdots, (c_{m-1}, a_{m-1})], (c_m, a_m)].$$

This axiom is not verified by the LWA operator, because neither the I-LOWA operator nor the LOWA operator satisfy the associativity property as was demonstrated in [15]. So, it is not verified by the LWD and LWC operators because the aggregated weights are obtained by means of the LOWA operator.

Axiom IV: Quasi-equivalence and increasingness of power. If everyone agrees on an opinion, "a", about an alternative, $x_i \in X$, and thus, $a_1 = a_2 = \cdots = a_m = a$, then if

$$(c_E, a_E) = LWA[(c_1, a), \cdots, (c_m, a)]$$

the collective opinion, (c_E, a_E), must satisfy the following conditions:

1) $a_E = a$ (quasi-equivalence); and
2) $c_E > \text{MAX}(c_1, \cdots, c_m)$ (increasingness of power).

This axiom is not verified for any of its conditions.

- The LWA operator is not quasi-equivalent. For example, suppose a set of nine labels. Then, if we want to aggregate the linguistic weighted opinions $[(s_0, s_6), (s_1, s_6)]$ of two experts, having fixed the linguistic quantifier, Q_2, "*At least 2*", then the two weights are $w_1 = 1, w_2 = 0$. Thus,

  Case 1: If $f = \phi^I_{Q_2}$ and $g = LI^{\rightarrow}(w, a) = \text{MAX}(\text{Neg}(w), a)$, then

$$\phi^I_{Q_2}(\text{MAX}(\text{Neg}(s_0), s_6), \text{MAX}(\text{Neg}(s_1), s_6)) = s_7 \neq s_6.$$

  Case 2: $f = \phi_{Q_2}$ and $g = LC^{\rightarrow}(w, a) = \text{MIN}(w, a)$,

$$\phi_{Q_2}(\text{MIN}(s_0, s_6), \text{MIN}(s_1, s_6)) = s_1 \neq s_6.$$

  Since this example of the LWA operator is the case in which it works as the LWD and LWC operators then these do not verify this property either.
- The LWA operator does not verify the increasingness of power. Clearly it is a consequence of the property of the LOWA operator of being an "orand" operator, and therefore

$$\text{MIN}(c_1, \cdots, c_m) \leq c_E \leq \text{MAX}(c_1, \cdots, c_m).$$

Axiom V: Positive sensitivity in its strongest form. A weighted collective opinion is increased if and only if any weighted individual opinion is increased. This means that if (a_E, c_E) is the weighted collective opinion obtained for x_i as

$$(c_E, a_E) = LWA[(c_1, a_1), \cdots, (c_m, a_m)]$$

and (c_E, b_E) is the weighted collective opinion obtained for x_i as

$$(c_E, b_E) = LWA[(c_1, b_1), \cdots, (c_m, b_m)]$$

with $a_j \geq b_j$, then

$$a_E > b_E \text{ if and only if } \exists e_k \in E \text{ such that } a_k > b_k.$$

Clearly this axiom is not verified by the LWA operator. Example: suppose a label set with eight elements. Let $[(s_1, s_4), (s_2, s_5)]$ be two weighted opinions to be aggregated. Considering the linguistic quantifier, Q_2, "*At least 2*", then the two weights are $w_1 = 1, w_2 = 0$ and then, *Case 1:* if $f = \phi^I_{Q_2}$ and $g = \text{MAX}(\text{Neg}(w), a)$, then

$$(c_E, a_E) = LWA[(s_1, s_4), (s_2, s_5)] = (s_2, s_5), \quad \text{and}$$

Case 2: if $f = \phi_{Q_2}$ and $g = \text{MIN}(w, a)$, then

$$(c_E, a_E) = LWA[(s_1, s_4), (s_2, s_5)] = (s_2, s_2).$$

If the first expert changes his opinion by (s_1, s_5) then

Case 1: $(c_E, b_E) = LWA[(s_1, s_5), (s_2, s_5)] = (s_2, s_5)$,

Case 2: $(c_E, b_E) = LWA[(s_1, s_5), (s_2, s_5)] = (s_2, s_2)$, and therefore, although the opinion of an expert has been increased, however, the collective opinion has not been increased, independently of the aggregation operator of the weighted opinions considered in the LWA operator. The LWD and LWC operators do not verify this axiom as also happens in the axiom above.

Axiom VI: Neutrality of complement. If $(c_j, a_j)^c$ is the complement of weighted opinion (c_j, a_j), such that $(c_j, a_j)^c = (c_j, \text{Neg}(a_j))$, then, having fixed an alternative, $x_i \in X$,

$$LWA[(c_1, a), \cdots, (c_m, a)]^c = LWA[(c_1, a_1)^c, \cdots, (c_m, a_m)^c].$$

This axiom is not verified by the LWA operator. Example: consider a label set with eight elements. Let $[(s_4, s_6), (s_1, s_5)]$ be the weighted opinions to be aggregated and its complement weighted opinions, $[(s_4, s_1), (s_1, s_2)]$, then having fixed the linguistic quantifier, Q_2, "*At least 2*", then the two weights are $w_1 = 1, w_2 = 0$, and then

Case 1: if $f = \phi^I_{Q_2}$ and $g = LI^{\rightarrow}(w, a) = \text{MAX}(\text{Neg}(w), a)$, then

$$(c_E, a_E) = LWA[(s_4, s_6), (s_1, s_5)]^c = (s_4, s_6) \neq (s_4, s_3) = LWA[(s_4, s_1), (s_1, s_2)].$$

Case 2: And if $f = \phi_{Q_2}$ and $g = LC^{\rightarrow}(w, a) = \text{MIN}(w, a)$, then

$$(c_E, a_E) = LWA[(s_4, s_6), (s_1, s_5)]^c = (s_4, s_4) \neq (s_4, s_1) = LWA[(s_4, s_1), (s_1, s_2)].$$

Thus, the LWD and LWC operators do not verify this axiom either.

Axiom VII: Positive sensitivity in its weaker form. If an expert increases his weighted opinion for x_i then the collective weighted opinion for x_i cannot decrease. This means that if (a_j, c_j) and (b_j, c_j) are such that, $a_j \leq b_j$, then if

$$(c_E, a_E) = LWA[(c_1, a_1), \cdots, (c_j, a_j), \cdots, (c_m, a_m)]$$

and

$$(c_E, b_E) = LWA[(c_1, a_1), \cdots, (c_j, b_j), \cdots, (c_m, a_m)]$$

then

$$a_E \leq b_E.$$

Obviously, this axiom is satisfied, it is a consequence of the monotonic property of the LOWA and I-LOWA operators.

Axiom VIII: Neutrality with respect to alternatives. If we have two alternatives, x_i and x_k, and the weighted opinions known about both are (c_j, a_j) and (c_j, a'_j), such that, $a_j = a'_j, \forall j$, then

$$LWA[(c_1, a_1), \cdots, (c_m, a_m)] = LWA[(c_1, a'_1), \cdots, (c_m, a'_m)].$$

Clearly this axiom is satisfied too.

Axiom IX: Unrestricted domain. Having fixed any alternative, $x_i \in X$, for any set of weighted individual opinions, $\{(c_1, a_1), \cdots, (c_m, a_m)\}$, there is a weighted collective opinion, (c_E, a_E), which may be constructed, i.e.,

$$\forall [(c_1, a_1), \cdots, (c_m, a_m)] \in (S \times S)^m, \exists (c_E, a_E) \in (S \times S) \text{ such that } (c_E, a_E) = LWA[(c_1, a_1), \cdots, (c_m, a_m)].$$

It is satisfied in accordance with the LWA operator definition.

Axiom X: The LWA operator is an "orand" operator. This is a property of the LWA operator presented here in the form of an axiom. This property is postulated in the following sense: having fixed an alternative, $x_i \in X$, for any set of weighted individual opinions, $\{(c_1, a_1), \cdots, (c_m, a_m)\}$, if (c_E, a_E) is such that

$$(c_E, a_E) = LWA[(c_1, a_1), \cdots, (c_m, a_m)]$$

then

1) $\text{MIN}(a_1, \cdots, a_m, c_1, \cdots, c_m) \leq a_E \leq \text{MAX}(a_1, \cdots, a_m, c_1, \cdots, c_m)$; and
2) $\text{MIN}(c_1, \cdots, c_m) \leq c_E \leq \text{MAX}(c_1, \cdots, c_m)$.

This property is a consequence of the property of the LOWA and I-LOWA operators of being "orand" operators.

In conclusion, the LWD operator, the LWC operator, and the LWA operator verify the following axioms: *independence of alternatives*, *commutativity*, *positive sensitivity in its weaker form*, *neutrality with respect to alternatives*, *unrestricted domain*, and *being an "orand" operator*. The fulfillment of those axioms provides evidence of rational aggregation of these operators in particular frameworks. In the following sections we shall show an application of the use of these aggregation operators for linguistic weighted information in the choice processes for alternatives in heterogeneous groups.

IV. Example of Application

Assuming the set of seven labels presented in Section II, that is,

$$S = \{s_6 = P, s_5 = VH, s_4 = H, s_3 = M \\ s_2 = L, s_1 = VL, s_0 = N\}$$

suppose an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible options where to invest the money:

- x_1 car company;
- x_2 food company;
- x_3 computer company; and
- x_4 arms company.

The investment company has a group of four consultancy departments:

d_1 risk analysis department;
d_2 growth analysis department;
d_3 social-political impact analysis department; and
d_4 environmental impact analysis department.

In each department there is one expert with different importance degrees (c_i for the expert of the department d_i)

$$\{c_1 = s_4, c_2 = s_5, c_3 = s_2, c_4 = s_6\}.$$

The assessments of the option set by the experts from each department are the following (a_{ij} is the assessment assigned to the option x_i by the expert from department d_j):

1) For e_1: $\{a_{11} = s_1, a_{21} = s_5, a_{31} = s_4, a_{41} = s_6\}$
2) For e_2: $\{a_{12} = s_1, a_{22} = s_3, a_{32} = s_3, a_{42} = s_1\}$
3) For e_3: $\{a_{13} = s_4, a_{23} = s_5, a_{33} = s_3, a_{43} = s_2\}$
4) For e_4: $\{a_{14} = s_1, a_{24} = s_6, a_{34} = s_1, a_{44} = s_0\}$.

Thus, using the linguistic weighted conjunction (LWD) operator the issues are the following.

1) The collective assessments on alternatives are the following (a_i for collective assessment of the alternative x_i):

$$\begin{aligned} a_1 &= \mathrm{MAX}_{j=1,\dots,4}\{\mathrm{MIN}(c_j, a_1 j)\} \\ &= \mathrm{MAX}\{\mathrm{MIN}(s_4, s_1), \mathrm{MIN}(s_5, s_1), \mathrm{MIN}(s_5, s_4), \\ &\qquad \mathrm{MIN}(s_6, s_1)\} = s_2 \\ a_2 &= \mathrm{MAX}_{j=1,\dots,4}\{\mathrm{MIN}(c_j, a_2 j)\} \\ &= \mathrm{MAX}\{\mathrm{MIN}(s_4, s_5), \mathrm{MIN}(s_5, s_3), \mathrm{MIN}(s_5, s_5), \\ &\qquad \mathrm{MIN}(s_6, s_6)\} = s_6 \\ a_3 &= \mathrm{MAX}_{j=1,\dots,4}\{\mathrm{MIN}(c_j, a_3 j)\} \\ &= \mathrm{MAX}\{\mathrm{MIN}(s_4, s_4), \mathrm{MIN}(s_5, s_3), \mathrm{MIN}(s_5, s_3), \\ &\qquad \mathrm{MIN}(s_6, s_1)\} = s_4 \\ a_4 &= \mathrm{MAX}_{j=1,\dots,4}\{\mathrm{MIN}(c_j, a_4 j)\} \\ &= \mathrm{MAX}\{\mathrm{MIN}(s_4, s_6), \mathrm{MIN}(s_5, s_1), \mathrm{MIN}(s_5, s_2), \\ &\qquad \mathrm{MIN}(s_6, s_0)\} = s_4. \end{aligned}$$

2) The collective importance degree, c_E, meaning the credibility degree of the solution, with the linguistic quantifier, Q, "*As many as possible*" with the par (0.5, 1) and $W = [0, 0, 0.5, 0.5]$ is

$$c_E = \phi_Q(c_1, c_2, c_3, c_4) = \phi_Q(s_4, s_5, s_5, s_6) = s_4.$$

Clearly alternative x_2 is the best assessed one.

V. Conclusions

In this paper, various aggregation operators for the linguistic weighted information are presented. These operators are very useful for modeling those processes in which there are various information sources and the information is linguistic in nature and is not equally relevant. Their aggregation has been checked examining some of the axioms that an acceptable weighted aggregation operator must verify.

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