ELSEVIER
Fuzzy Sets and Systems 131 (2002) 75–78
FUZZY sets and systems
www.elsevier.com/locate/fss

A note on the internal consistency of various preference representations

F. Chiclana, F. Herrera *, E. Herrera-Viedma

Department of Computer Science and Artificial Intelligence, ETS de Ingeniera Informatica, University of Granada,
Avda. Andalucia 38, 18071 - Granada, Spain

Abstract

In Chiclana et al. (Fuzzy Sets and Systems 97 (1998) 33) we presented a fuzzy multipurpose decision making model integrating different preference representations: preference orderings, utility functions and fuzzy preference relations. We complete the decision model studying its internal consistency. © 2001 Published by Elsevier Science B.V.

Keywords: Multipurpose decision making; Preference orderings; Utility functions; Fuzzy preference relations; Consistency

---

1. Introduction

The objective of a decision making process is to classify the alternatives $X = \{x_1, x_2, \ldots, x_n\}$ $(n \geqslant 2)$ from best to worst, using the information about them according to a set of general purposes (experts or criteria) $E = \{e_1, e_2, \ldots, e_m\}$ $(m \geqslant 2)$. In [2] we presented a multipurpose decision making (MPDM) model assuming that the experts' preferences can be provided in any of the following three preference representations:

(1) *Preference ordering of the alternatives*: $O^k = (o^k(1), \ldots, o^k(n))$, where $o^k(\cdot)$ is a permutation function over the index set, $\{1, \ldots, n\}$, for the expert, e_k.

(2) *Fuzzy preference relation*: $P^k \subset X \times X$, with membership function, $\mu_{P^k} : X \times X \rightarrow [0,1]$, where $\mu_{P^k}(x_i, x_j) = p_{ij}^k$ denotes the preference degree or intensity of the alternative x_i over x_j, and being P^k assumed additive reciprocal, i.e., $p_{ij}^k + p_{ji}^k = 1$.

(3) *Utility function*: $U^k = \{u_i^k, i = 1, \ldots, n\}$, $u_i^k \in [0,1]$, where u_i^k represents the utility evaluation given by the expert e_k to the alternative x_i.

This decision model obtains the set of solution alternatives in the following steps:

(1) Make the information uniform by means of the transformation function defined in Proposition 1, which assumes the fuzzy preference relations as the base element to make the preferences uniform.

Proposition 1. *Let X be a set of alternatives and λ_i^k represents an evaluation of alternative x_i indicating the performance of x_i according to a purpose e_k. Then, the intensity of preference of alternative x_i over alternative x_j, p_{ij}^k for e_k is given by the following transformation function* $p_{ij}^k = \varphi(\lambda_i^k, \lambda_j^k) = \frac{1}{2}[1 + \psi(\lambda_i^k, \lambda_j^k) -$

---

* Corresponding author. Tel.: +34-58-24-40-19; fax: +34-58-24-33-17.

E-mail address: herrera@decsai.ugr.es (F. Herrera).

0165-0114/01/$ - see front matter © 2001 Published by Elsevier Science B.V.
PII: S0165-0114(01)00256-1

$\psi(\lambda_j^k,\lambda_i^k)]$, *where* ψ *is a function verifying* (i) $\psi(z,z)=\frac{1}{2}, \forall z\in\mathscr{R}$, *and* (ii) ψ *is nondecreasing in the first argument and nonincreasing in the second argument.*

Corollary 1.1. *If* $\lambda_i^k=o^k(i)$, *and* $\psi(\lambda_i^k,\lambda_j^k)=F(\lambda_j^k-\lambda_i^k)$, *where F is any nondecreasing function, then* φ *transforms preference orderings into fuzzy preference relations.*

Corollary 1.2. *If* $\lambda_i^k=u_i^k$ *and*

$$\psi(z,y)=\begin{cases}\dfrac{s(z)}{s(z)+s(y)} & \text{if } (z,y)\neq(0,0),\\ \frac{1}{2} & \text{if } (z,y)=(0,0),\end{cases}$$

where $s\colon [0,1]\colon\to\mathscr{R}^+$ *is a nondecreasing and continuous function, verifying* $s(0)=0$, *then* φ *transforms utility values given on the basis of a ratio scale into fuzzy preference relations.*

(2) Aggregate the individual fuzzy preference relations $\{P^1,\ldots,P^m\}$ in a collective fuzzy preference relation P^{c} by means of a linguistic quantifier guided OWA operator ϕ_{Q} [11].

(3) Exploit P^{c} by means of following choice degrees of alternatives [3]:

  (a) *Quantifier guided dominance degree*, which is defined for the alternative, x_i, as $\mathrm{QGDD}_i=\phi_{\mathrm{Q}}(p_{ij}^{\mathrm{c}},\ j=1,\ldots,n, j\neq i)$.

  (b) *Quantifier guided nondominance degree*, which is defined for the alternative, x_i, as $\mathrm{QGNDD}_i=\phi_Q(1-p_{ji}^s, j=1,\ldots,n, j\neq i)$, $p_{ji}^s=\max\{p_{ji}^{\mathrm{c}}-p_{ij}^{\mathrm{c}},0\}$.

In Section 2 we study the internal consistency of this decision model by analyzing the consistency of the transformation function per each expert. Finally, we point out some concluding remarks.

2. The consistency of transformation function φ

In this section, we demonstrate that the transformation function proposed in Proposition 1 acts coherently according to both choice degrees of alternatives $(\mathrm{QGDD}_i, \mathrm{QGNDD}_i)$, because the ranking among the alternatives that we can obtain from any of the considered preference representations (preference ordering and utility values) is not disturbed if we apply any of the two choice degrees on the respective fuzzy preference relation obtained via a transformation function. Previously, we are going to present some interesting consequences followed from Proposition 1, and necessary to prove the consistency of φ.

Corollary 1.3. *Suppose a set of evaluations,* $\{\lambda_1^k,\ldots,\lambda_n^k\}$, *provided on X by an expert* e_k. *Without loss of generality, assume the following order,* $0<\lambda_1^k\leqslant\cdots\leqslant\lambda_n^k\leqslant 1$. *Then, the following restriction is verified*: $p_{i1}^k\geqslant\cdots\geqslant p_{ii-1}^k\geqslant p_{ii}^k=\frac{1}{2}\geqslant p_{ii+1}^k\geqslant\cdots\geqslant p_{in}^k$.

Proof. Let $s,i,j\in\{1,\ldots,n\}$ be such that $s<i<j$. Then, we have $0<\lambda_s^k\leqslant\lambda_i^k\leqslant\lambda_j^k\leqslant 1$, and therefore from Proposition 1

$$\left.\begin{array}{l}\psi(\lambda_i^k,\lambda_j^k)\leqslant\psi(\lambda_i^k,\lambda_s^k)\\ \psi(\lambda_s^k,\lambda_i^k)\leqslant\psi(\lambda_j^k,\lambda_i^k)\end{array}\right\}$$

$$\begin{aligned}&\Rightarrow\frac{1}{2}\{[\psi(\lambda_i^k,\lambda_j^k)-\psi(\lambda_i^k,\lambda_s^k)]\\ &\quad+[\psi(\lambda_s^k,\lambda_i^k)-\psi(\lambda_j^k,\lambda_i^k)]\}\\ &=\frac{1}{2}\{[1+\psi(\lambda_i^k,\lambda_j^k)-\psi(\lambda_j^k,\lambda_i^k)]\\ &\quad-[1+\psi(\lambda_i^k,\lambda_s^k)-\psi(\lambda_s^k,\lambda_i^k)]\}\\ &=p_{ij}^k-p_{is}^k\leqslant 0,\end{aligned}$$

which implies: $p_{i1}^k\geqslant\cdots\geqslant p_{ii-1}^k\geqslant p_{ii}^k=\frac{1}{2}\geqslant p_{ii+1}^k\geqslant\cdots\geqslant p_{in}^k$. □

Corollary 1.4. *Suppose a set of evaluations* $\{\lambda_1^k,\ldots,\lambda_n^k\}$ *provided on X by an expert* e_k. *Without loss of generality, assume the following order,* $0<\lambda_1^k\leqslant\cdots\leqslant\lambda_n^k\leqslant 1$. $\forall i,j,s\in\{1,\ldots,n\}$ *such that* $i<j$, *then* $p_{js}^k\geqslant p_{is}^k$.

Proof. From Proposition 1

$$\psi(\lambda_i^k,\lambda_s^k)\leqslant\psi(\lambda_j^k,\lambda_s^k),$$

$$\psi(\lambda_s^k,\lambda_j^k)\leqslant\psi(\lambda_s^k,\lambda_i^k),$$

and therefore

$$\begin{aligned} p_{is}^k - p_{js}^k &= \tfrac{1}{2}\{[1 + \psi(\lambda_i^k, \lambda_s^k) - \psi(\lambda_s^k, \lambda_i^k)] \\ &\quad - [1 + \psi(\lambda_i^k, \lambda_s^k) - \psi(\lambda_s^k, \lambda_j^k)]\} \\ &= \tfrac{1}{2}\{[\psi(\lambda_i^k, \lambda_s^k) - \psi(\lambda_j^k, \lambda_s^k)] \\ &\quad + [\psi(\lambda_s^k, \lambda_j^k) - \psi(\lambda_s^k, \lambda_i^k)]\} \leqslant 0 \end{aligned}$$

which implies $p_{js}^k \geqslant p_{is}^k, \forall i, j, s$, such that $i < j$. □

Proposition 2. *Let $i, j \in \{1, \ldots, n\}$ be such that $i < j$, assuming the evaluations given by an expert e_k verify $\lambda_i^k \leqslant \lambda_j^k$, then the dominance and nondominance choice degrees obtained from the fuzzy preference relation P^k satisfy the following relationships*: (i) $\mathrm{QGDD}_j^k \geqslant \mathrm{QGDD}_i^k$, *and* (ii) $\mathrm{QGNDD}_j^k \geqslant \mathrm{QGNDD}_i^k$.

Proof. Firstly, we prove that

$$\mathrm{QGDD}_j^k \geqslant \mathrm{QGDD}_i^k \quad \forall i, j.$$

Using Corollary 1.4 we know that if $i < j$, then $p_{jt}^k \geqslant p_{it}^k$, $\forall t$, and particularly, $\forall t \in \{1, \ldots, i-1\} \cup \{j+1, \ldots, n\}$. On the other hand, using Corollary 1.3 we have that $p_{it}^k \geqslant p_{it+1}^k$, $\forall i, t$, and therefore $\forall i < j$ we have that

$$p_{jt}^k \geqslant p_{it}^k \geqslant p_{it+1}^k, \quad \forall t$$

and particularly, $\forall t \in \{i, \ldots, j-1\}$. Therefore, concluding, the following relationship is satisfied,

$$\mathrm{QGDD}_j^k = \sum_{t=1_{t \neq j}}^{n} w_t \cdot p_{jt}^k \geqslant \sum_{t=1_{t \neq i}}^{n} w_t \cdot p_{it}^k = \mathrm{QGDD}_i^k$$

with w_t being the weights used in the OWA operator applied to obtain the degrees QGDD_j^k and QGDD_i^k. Secondly, we prove that

$$\mathrm{QGNDD}_j^k \geqslant \mathrm{QGNDD}_i^k \quad \forall i, j.$$

Using Corollary 1.4 we know that if $i < j$, then $p_{jt}^k \geqslant p_{it}^k$, $\forall t$. Using Corollary 1.3 we know that if $i < j$, then $p_{ti}^k \geqslant p_{tj}^k$, $\forall t$. Therefore, the following expression is satisfied:

$$\begin{aligned} p_{ti}^{k,s} &= \max\{p_{ti}^k - p_{it}^k, 0\} \geqslant \max\{p_{tj}^k - p_{jt}^k, 0\} \\ &= p_{tj}^{k,s}, \quad \forall t. \end{aligned}$$

Then, particularly we have

$$\begin{aligned} &1 - p_{ti}^{k,s} \leqslant 1 - p_{tj}^{k,s} \\ &\quad \forall t \in \{1, \ldots, i-1\} \cup \{j+1, \ldots, n\}. \end{aligned}$$

On the other hand, using Corollary 1.4 we know that $p_{t+1i}^k \geqslant p_{ti}^k$ $\forall t$. Using Corollary 1.3 we know that $p_{it}^k \geqslant p_{it+1}^k$ $\forall t$. Therefore, the following expression is satisfied:

$$\begin{aligned} p_{t+1i}^{k,s} &= \max\{p_{t+1i}^k - p_{it+1}^k, 0\} \geqslant \max\{p_{ti}^k - p_{it}^k, 0\} \\ &= p_{ti}^{k,s}, \quad \forall t. \end{aligned}$$

Then, particularly we have $1 - p_{ti}^{k,s} \geqslant 1 - p_{t+1i}^{k,s}$, $\forall t \in \{i, \ldots, j-1\}$, and therefore $\forall i < j$ we have that $1 - p_{tj}^{k,s} \geqslant 1 - p_{ti}^{k,s} \geqslant 1 - p_{t+1i}^{k,s}$, $t \in \{i, \ldots, j-1\}$. Concluding, the following relationship is satisfied:

$$\begin{aligned} \mathrm{QGNDD}_j^k &= \sum_{t=1_{t \neq j}}^{n} w_t(1 - p_{tj}^{k,s}) \geqslant \sum_{t=1_{t \neq i}}^{n} w_t(1 - p_{ti}^{k,s}) \\ &= \mathrm{QGNDD}_i^k. \end{aligned}$$

□

3. Concluding remarks

Proposition 2 guarantees the internal consistency of the decision process presented in [2]. We must remark that the issue is valid even if the elements of the main diagonal of the fuzzy preference relations are used in the calculus of the dominance and nondominance degrees and independent of the linguistic quantifiers used.

Finally, we should point out that the decision model presented in [2] is mainly used in *Social Theory* [4] and *outranking methods* [8,7]. Therefore, there exist

(i) other proposals on transformation functions, e.g. those based on the concept of *preference structures* (strict preference, indifference and incomparability) [5,6,7], and also

(ii) other proposals on choice degrees, e.g. those based on *scoring functions* (the leaving flow, entering flow the net flow, etc.) [1,6,7,10].

In consequence, a comparative study of our decision model with these ones similar to that done in [9] may be an interesting future research.

References

[1] D. Bouyssou, P. Perny, Ranking methods for valued preference relations: a characterization of a method based on leaving and entering flows, Eur. J. Oper. Res. 61 (1992) 186–194.

[2] F. Chiclana, F. Herrera, E. Herrera-Viedma, Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations, Fuzzy Sets and Systems 97 (1998) 33–48.

[3] F. Chiclana, F. Herrera, E. Herrera-Viedma, M.C. Poyatos, A classification method of alternatives for multiple preference ordering criteria based on fuzzy majority, J. Fuzzy Math. 4 (1996) 801–813.

[4] J.S. Kelly, Social choice bibliography, Soc. Choice Welfare 8 (1991) 97–169.

[5] P. Perny, M. Roubens, Fuzzy preference modeling, in: R. Slowinski (Ed.), Decision Analysis, Operations Research and Statistics, Kluwer Academic Publishers, Dordrecht, 1998, pp. 3–30.

[6] M. Pirlot, Ph. Vincke, Semiorders: Properties, Representations, Applications, Kluwer Academic Publishers, Dordrecht, 1997.

[7] B. Roy, Multicriteria Methodology for Decision Aiding, Kluwer Academic Publishers, Dordrecht, 1996.

[8] B. Roy, D. Bouyssou, Aide Multicritére á la Decision: Méthods et Cas, Economica, 1993.

[9] E. Triantaphyllow, C.T. Lin, Development and evaluation of five fuzzy multiattribute decision-making methods, Internat. J. Approx. Reasoning 18 (1996) 281–310.

[10] Ph. Vincke, Multicriteria Decision-Aid, Wiley, New York, 1992.

[11] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1988) 183–190.