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A note on the reciprocity in the aggregation of fuzzy preference relations using OWA operators☆

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Abstract

In (Fuzzy Sets and Systems 97 (1998) 33), we presented a fuzzy multipurpose decision making model integrating different preference representations based on additive reciprocal fuzzy preference relations. The main aim of this paper is to complete the decision model studying conditions under which reciprocity property is maintained when aggregating preference relations using an OWA operator guided by a relative linguistic quantifier.

Keywords: Fuzzy preference relations; OWA operator; Linguistic quantifier; Reciprocity

1. Introduction

We assume multiperson decision making (MPDM) problems [3] being the experts' preferences about the alternatives represented by means of the fuzzy preference relations which are additive reciprocal [6].

Usually, the solution set of alternatives is achieved in two phases [5]: *aggregation phase* and *exploitation phase*. The aggregation phase leads us to the use of an aggregation operator for getting a collective preference relation. In [1], we use *the OWA operator* [7] guided by *fuzzy majority* like aggregation procedure to combine the preference relations. In the OWA operator, the concept of fuzzy majority can be incorporated by means of a *relative linguistic quantifier* [2,4,8,9] (e.g., such as "most of", "at least half", "as many as possible") used to compute the weighting vector [7].

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The problem that we can find is that the reciprocity property is not generally preserved when aggregation is carried out by means of the OWA operator guided by a relative linguistic quantifier. This paper is focused on the analysis of this problem.

In order to do that, this note is organized as follows. In Section 2, we present formally the decision making problem. In Section 3, we study reciprocity conditions and also give a few examples to illustrate everything. Finally, some conclusions are pointed out.

2. Presentation of the problem

We have a set of alternatives $X=\{x_1,\dots,x_n\}$, a set of experts $E=\{e_1,\dots,e_m\}$, and a set of fuzzy preference relations $\{P^1,\dots,P^m\}$, where $P^k=(p_{ij}^k)$, and p_{ij}^k represents the preference degree or intensity of alternative x_i over alternative x_j for expert e_k. We consider additive reciprocal fuzzy preferences relations to express the preferences, i.e., $p_{ij}^k+p_{ji}^k=1,\ \forall i,j,k$.

As we have said, using an OWA operator ϕ_Q guided by a linguistic quantifier Q, we derive a collective preference relation, $P^{\mathrm{c}}=(p_{ij}^{\mathrm{c}})$, that indicates the global preference between every pair of alternatives according to the majority of experts' opinions, which is represented by Q. In this case,

$$p_{ij}^{\mathrm{c}}=\phi_Q(p_{ij}^1,\dots,p_{ij}^m)=\sum_{k=1}^{m} w_k q_{ij}^k,$$

where q_{ij}^k is the kth largest value in the set $\{p_{ij}^1,\dots,p_{ij}^m\}$, Q is a relative non-decreasing quantifier with membership function

$$Q(x)=\begin{cases}0, & 0\leqslant x<a,\\ \dfrac{x-a}{b-a}, & a\leqslant x\leqslant b,\\ 1, & b<x\leqslant 1,\end{cases}$$

$a,b\in[0,1]$, and $w_k=Q(k/m)-Q((k-1)/m),\forall k$.

Note 1: We make note that the definition of Q implies that $a<b$.

Following this methodology, the first thing we have to do is to choose the suitable relative quantifier for representing the concept of fuzzy majority that we desire to implement in our MPDM problem, what reduces to choose adequate values for parameters a and b, computing afterwards the weights of the OWA operator using the above relation. Our objective in this paper is to give values of parameters a and b that maintain reciprocity property.

3. Reciprocity of collective preference relation

In the following two subsections we will demonstrate the following relation:

$$a+b=1 \iff p_{ij}^{\mathrm{c}}+p_{ji}^{\mathrm{c}}=1.$$

3.1. Sufficient condition

The problem to solve is: What condition do parameters a and b have to verify so that $p_{ij}^{\mathrm{c}} + p_{ji}^{\mathrm{c}} = 1,\ \forall i, j (i \neq j)$?

Note 2: We make note that if all the individual additive reciprocal fuzzy preference relations are the same, that is when $P^1 = \cdots = P^m = P$, then we will have $P^{\mathrm{c}} = P$, no matter what OWA operator ϕ_Q we do use.

As we are assuming P^k additive reciprocal then $p_{ji}^k = 1 - p_{ij}^k$, and therefore if $\{q_{ij}^k, \ldots, q_{ij}^m\}$ are ordered from largest to lowest, $\{q_{ji}^1, \ldots, q_{ji}^m\}$, being $q_{ji}^k = 1 - q_{ij}^k$, are ordered form lowest to largest, and in consequence we have

$$p_{ij}^{\mathrm{c}} + p_{ji}^{\mathrm{c}} = \sum_{k=1}^{m} w_k q_{ij}^k + \sum_{k=1}^{m} w_{m-k+1} q_{ji}^k = \sum_{k=1}^{m} w_k q_{ij}^k + \sum_{k=1}^{m} w_{m-k+1}(1 - q_{ij}^k)$$

$$= 1 + \sum_{k=1}^{m} (w_k - w_{m-k+1}) q_{ij}^k = 1 + \sum_{k=1}^{m} \bar{w}_k q_{ij}^k,$$

where

$$\bar{w}_k = \left[Q\left(\frac{k}{m}\right) - Q\left(\frac{k-1}{m}\right) \right] - \left[Q\left(\frac{m-k+1}{m}\right) - Q\left(\frac{m-k}{m}\right) \right].$$

If we denote $A(k) = Q(k/m) + Q(1 - (k/m))$ then $\bar{w}_k = A(k) - A(k-1)$.

We distinguish three possible cases, according to the values of $a + b$: (A) $a + b = 1$, (B) $a + b < 1$, (C) $a + b > 1$.

Case A: $a + b = 1$. In this case $1 - a = b$, $1 - b = a$ and we have

$$Q(1-x) = \left\{ \begin{array}{ll} 0, & 0 \leqslant 1 - x < a \\ \dfrac{1 - x - a}{b - a}, & a \leqslant 1 - x \leqslant b \\ 1, & b < 1 - x \leqslant 1 \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} 0, & b < x \leqslant 1 \\ \dfrac{b + a - x - a}{b - a}, & a \leqslant x \leqslant b \\ 1, & 0 \leqslant x < a \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} 1 - 0, & 0 \leqslant x < a \\ 1 - \dfrac{x - a}{b - a}, & a \leqslant x \leqslant b \\ 1 - 1, & b < x \leqslant 1 \end{array} \right\} = 1 - Q(x).$$

This implies that

$$A(k) = Q\left(\frac{k}{m}\right) + Q\left(1 - \frac{k}{m}\right) = Q\left(\frac{k}{m}\right) + 1 - Q\left(\frac{k}{m}\right) = 1, \quad \forall k,$$

and $\bar{w}_k = A(k) - A(k-1) = 0,\ \forall k$, and therefore $p_{ij}^{\mathrm{c}} + p_{ji}^{\mathrm{c}} = 1,\ \forall i, j$.

Summarising, we have stated the following results:

Proposition 1. *If Q is a linguistic quantifier with membership function verifying*

$$Q(1-x) = 1 - Q(x), \quad \forall x,$$

then the collective fuzzy preference relation, obtained by aggregating a set of additive reciprocal fuzzy preference relations, using an OWA operator guided by Q, is additive reciprocal.

Proposition 2. *If Q is a relative non-decreasing linguistic quantifier with parameters a and b verifying $a+b=1$, then the OWA operator guided by Q preserves additive reciprocity.*

Example 1. Suppose that we have a set of four alternatives and a set of six experts that provide their opinion using the following additive reciprocal fuzzy preference relations:

$$P^1 = \begin{pmatrix} 0.5 & 0.17 & 0.67 & 0.5 \\ 0.83 & 0.5 & 1 & 0.67 \\ 0.33 & 0 & 0.5 & 0.17 \\ 0.5 & 0.33 & 0.83 & 0.5 \end{pmatrix},$$

$$P^2 = \begin{pmatrix} 0.5 & 0.38 & 0.58 & 0.84 \\ 0.62 & 0.5 & 0.69 & 0.9 \\ 0.42 & 0.31 & 0.5 & 0.8 \\ 0.16 & 0.1 & 0.2 & 0.5 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0.5 & 0.1 & 0.6 & 0.7 \\ 0.9 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.9 \\ 0.3 & 0.6 & 0.2 & 0.5 \end{pmatrix},$$

$$P^4 = \begin{pmatrix} 0.5 & 0.33 & 0.17 & 0.67 \\ 0.67 & 0.5 & 0.33 & 0.17 \\ 0.83 & 0.67 & 0.5 & 1 \\ 0.33 & 0.83 & 0 & 0.5 \end{pmatrix},$$

$$P^5 = \begin{pmatrix} 0.5 & 0.34 & 0.2 & 0.96 \\ 0.66 & 0.5 & 0.33 & 0.98 \\ 0.8 & 0.67 & 0.5 & 0.99 \\ 0.04 & 0.02 & 0.01 & 0.5 \end{pmatrix},$$

$$P^6 = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 1 \\ 0.5 & 0.5 & 0.8 & 0.6 \\ 0.3 & 0.2 & 0.5 & 0.8 \\ 0 & 0.4 & 0.2 & 0.5 \end{pmatrix}.$$

Using the linguistic quantifier with the pair of values $(0.25, 0.75)$ and the corresponding OWA operator with weight vector $(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, 0)$, the collective preference relation is:

$$P^{\mathrm{c}} = \begin{pmatrix} 0.5 & 0.315 & 0.538 & 0.785 \\ 0.685 & 0.5 & 0.685 & 0.64 \\ 0.462 & 0.315 & 0.5 & 0.865 \\ 0.215 & 0.36 & 0.135 & 0.5 \end{pmatrix}.$$

Case B: $a+b<1$. In this case, we have that $1-a>b$, $1-b>a$ and as a consequence of being $a<b$ we have $a<\frac{1}{2}$. We can assume for now that $b \geqslant \frac{1}{2}$, what implies that $1-b \leqslant b$, letting for later the other case $b<\frac{1}{2}$.

Case B1: $b \geqslant \frac{1}{2}$. Now we have that $0 \leqslant a < 1-b \leqslant b < 1-a \leqslant 1$, and consequently

$$Q(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ \dfrac{x-a}{b-a}, & a \leqslant x < 1-b, \\ \dfrac{x-a}{b-a}, & 1-b \leqslant x < b, \\ 1, & b \leqslant x < 1-a, \\ 1, & 1-a \leqslant x \leqslant 1, \end{cases}$$

$$Q(1-x) = \begin{cases} 1, & 0 \leqslant x < a, \\ 1, & a \leqslant x < 1-b, \\ \dfrac{1-x-a}{b-a}, & 1-b \leqslant x < b, \\ \dfrac{1-x-a}{b-a}, & b \leqslant x < 1-a, \\ 0, & 1-a \leqslant x \leqslant 1 \end{cases}$$

with $x \in [0,1]$ and

$$A(y) = \begin{cases} 1, & 0 \leqslant y < ma, \\ \dfrac{y+m(b-2a)}{m(b-a)}, & ma \leqslant y < m(1-b), \\ \dfrac{1-2a}{b-a}, & m(1-b) \leqslant y < mb, \\ \dfrac{m-y-m(b-2a)}{m(b-a)}, & mb \leqslant y < m(1-a), \\ 1, & m(1-a) \leqslant y \leqslant m \end{cases}$$

with $y\in[0,m]$. It is clear that there exist $h_1,h_2,h_3,h_4\in\{1,\dots,m\}$ such that

$$h_1-1<ma\leqslant h_1,\quad h_2-1<m(1-b)\leqslant h_2,$$

$$h_3-1<mb\leqslant h_3,\quad h_4-1<m(1-a)\leqslant h_4,$$

and in consequence:

$$A(0)=\cdots=A(h_1-1)=1,$$

$$A(k)=\frac{k+m(b-2a)}{m(b-a)},\quad k=h_1,\dots,h_2-1,$$

$$A(j)=\frac{1-2a}{b-a},\quad j=h_2,\dots,h_3-1,$$

$$A(l)=\frac{m-l-m(b-2a)}{m(b-a)},\quad l=h_3,\dots,h_4-1,$$

$$A(h_4)=\cdots=A(m)=1.$$

Moreover, it is clear that $m-h_4=h_1-1$, $m-h_3=h_2-1$, so

$$\bar{w}_1=\cdots=\bar{w}_{h_1-1}=0,\quad \bar{w}_{h_1}=\frac{h_1-ma}{m(b-a)},$$

$$\bar{w}_{h_1+1}=\cdots=\bar{w}_{h_2-1}=\frac{1}{m(b-a)},\quad \bar{w}_{h_2}=\frac{h_3-mb}{m(b-a)},$$

$$\bar{w}_{h_2+1}=\cdots=\bar{w}_{h_3-1}=0,\quad \bar{w}_{h_3}=\frac{mb-h_3}{m(b-a)},$$

$$\bar{w}_{h_3+1}=\cdots=\bar{w}_{h_4-1}=\frac{-1}{m(b-a)},\quad \bar{w}_{h_4}=\frac{ma-h_1}{m(b-a)},$$

$$\bar{w}_{h_4+1}=\cdots=\bar{w}_m=0.$$

The expression for $p_{ij}^{\mathrm{c}}+p_{ji}^{\mathrm{c}}$ reduces to

$$p_{ij}^{\mathrm{c}}+p_{ji}^{\mathrm{c}}=1+\bar{w}_{h_1}(q_{ij}^{h_1}-q_{ij}^{h_4})+\sum_{k=h_1+1}^{h_2-1}\frac{1}{m(b-a)}(q_{ij}^{k}-q_{ij}^{m-k+1})+\bar{w}_{h_2}(q_{ij}^{h_2}-q_{ij}^{h_3}),\quad \forall i,j.$$

As we have that $\{q_{ij}^1,\dots,q_{ij}^m\}$ are ordered from largest to lowest, then it is clear that $p_{ij}^{\mathrm{c}}+p_{ji}^{\mathrm{c}}\geqslant 1$, $\forall i,j$.

Example 2. Suppose again the same set of additive reciprocal preference relations as in Example 1. Using the linguistic quantifier "*at least half*" with the pair of values $(0,0.5)$ and the corresponding

OWA operator with weight vector $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)$, then the collective preference relation is

$$P^c = \begin{pmatrix} 0.5 & 0.4 & 0.66 & 0.94 \\ 0.8 & 0.5 & 0.87 & 0.85 \\ 0.69 & 0.55 & 0.5 & 0.96 \\ 0.38 & 0.61 & 0.41 & 0.5 \end{pmatrix}.$$

Case B2: $b < \frac{1}{2}$. In this case we have that $0 \leqslant a < b < 1 - b \leqslant 1 - a \leqslant 1$, and therefore

$$Q(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ \dfrac{x-a}{b-a}, & a \leqslant x < b, \\ 1, & b \leqslant x < 1-b, \\ 1, & 1-b \leqslant x < 1-a, \\ 1, & 1-a \leqslant x \leqslant 1, \end{cases}$$

$$Q(1-x) = \begin{cases} 1, & 0 \leqslant x < a, \\ 1, & a \leqslant x < b, \\ 1, & b \leqslant x < 1-b, \\ \dfrac{1-x-a}{b-a}, & 1-b \leqslant x < 1-a, \\ 1, & 1-a \leqslant x \leqslant 1, \end{cases}$$

$$A(y) = \begin{cases} 1, & 0 \leqslant y < ma, \\ \dfrac{y+m(b-2a)}{m(b-a)}, & ma \leqslant y < mb, \\ 2, & mb \leqslant y < m(1-b), \\ \dfrac{m-y+m(b-2a)}{m(b-a)}, & m(1-b) \leqslant y < m(1-a), \\ 1, & m(1-a) \leqslant y \leqslant m. \end{cases}$$

There exist $l_1, l_2, l_3, l_4 \in \{1, \ldots, m\}$ such that

$$l_1 - 1 < ma \leqslant l_1, \quad l_2 - 1 < mb \leqslant l_2,$$

$$l_3 - 1 < m(1-b) \leqslant l_3, \quad l_4 - 1 < m(1-a) \leqslant l_4,$$

$$m - l_4 = l_1 - 1, \quad m - l_3 = l_2 - 1.$$

Thus,

$$\bar{w}_1 = \cdots = \bar{w}_{l_1 - 1} = 0, \quad \bar{w}_{l_1} = \frac{l_1 - ma}{m(b-a)},$$

$$\bar{w}_{l_1+1} = \cdots = \bar{w}_{l_2-1} = \frac{1}{m(b-a)}, \quad \bar{w}_{l_2} = \frac{mb - l_2 + 1}{m(b-a)},$$

$$\bar{w}_{l_2+1} = \cdots = \bar{w}_{l_3-1} = 0, \quad \bar{w}_{l_3} = \frac{l_2 - 1 - mb}{m(b-a)},$$

$$\bar{w}_{l_3+1} = \cdots = \bar{w}_{l_4-1} = \frac{-1}{m(b-a)}, \quad \bar{w}_{l_4} = \frac{ma - l_1}{m(b-a)},$$

$$\bar{w}_{l_4+1} = \cdots = \bar{w}_m = 0.$$

The expression for $p^{\mathrm{c}}_{ij} + p^{\mathrm{c}}_{ji}$ reduces to

$$p^{\mathrm{c}}_{ij} + p^{\mathrm{c}}_{ji} = 1 + \bar{w}_{l_1}(q^{l_1}_{ij} - q^{l_4}_{ij}) + \sum_{k=l_1+1}^{l_2-1} \frac{1}{m(b-a)}(q^{k}_{ij} - q^{m-k+1}_{ij}) + \bar{w}_{l_2}(q^{l_2}_{ij} - q^{l_3}_{ij}), \quad \forall i,j$$

Example 3. Suppose again the same set of additive reciprocal preference relations as in Example 1. Using the linguistic quantifier with the pair of values $(0.15, 0.35)$ and the corresponding OWA operator with weight vector $(\frac{1}{3}, \frac{7}{12}, \frac{1}{12}, 0, 0, 0)$, then the collective preference relation is

$$P^{\mathrm{c}} = \begin{pmatrix} 0.5 & 0.42 & 0.53 & 0.96 \\ 0.84 & 0.5 & 0.87 & 0.91 \\ 0.78 & 0.64 & 0.5 & 0.99 \\ 0.38 & 0.66 & 0.41 & 0.5 \end{pmatrix}.$$

Summarising, we have obtained the following result:

Proposition 3. *Let $\{P^1, \ldots, P^m\}$ be a finite set of individual additive reciprocal preference relations, and Q a relative non-decreasing quantifier with membership function*

$$Q(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ \dfrac{x-a}{b-a}, & a \leqslant x \leqslant b, \\ 1, & b < x \leqslant 1 \end{cases}$$

with $a + b < 1$. Then, the collective preference relation $P^{\mathrm{c}} = (p^{\mathrm{c}}_{ij})$, $p^{\mathrm{c}}_{ij} = \phi_Q(p^1_{ij}, \ldots, p^m_{ij})$, obtained using the OWA operator ϕ_Q, verifies $p^{\mathrm{c}}_{ij} + p^{\mathrm{c}}_{ji} \geqslant 1$, $\forall i,j$.

Case C: $a + b > 1$. As in the previous case, we have to distinguished two sub-cases: $a < \frac{1}{2}$ and $a \geqslant \frac{1}{2}$.

Case C1: $a<\frac{1}{2}$. The expressions for $Q(x)$, $Q(1-x)$ and $A(x)$ are, respectively,

$$Q(x)=\begin{cases} 0, & 0\leqslant x<1-b,\\ 0, & 1-b\leqslant x<a,\\ \dfrac{x-a}{b-a}, & a\leqslant x<1-a,\\ \dfrac{x-a}{b-a}, & 1-a\leqslant x<b,\\ 1, & b\leqslant x\leqslant 1,\end{cases}$$

$$Q(1-x)=\begin{cases} 1, & 0\leqslant x<1-b,\\ \dfrac{1-x-a}{b-a}, & 1-b\leqslant x<a,\\ \dfrac{1-x-a}{b-a}, & a\leqslant x<1-a,\\ 0, & 1-a\leqslant x<b,\\ 0, & b\leqslant x\leqslant 1,\end{cases}$$

$$A(y)=\begin{cases} 1, & 0\leqslant y<m(1-b),\\ \dfrac{m-y-ma}{m(b-a)}, & m(1-b)\leqslant y<ma,\\ \dfrac{1-2a}{b-a}, & ma\leqslant y<m(1-a),\\ \dfrac{y-ma}{m(b-a)}, & m(1-a)\leqslant y<mb,\\ 1, & mb\leqslant y\leqslant m.\end{cases}$$

There exist $r_1,r_2,r_3,r_4\in\{1,\dots,m\}$ such that

$$r_1-1<m(1-b)\leqslant r_1,\quad r_2-1<ma\leqslant r_2,$$

$$r_3-1<m(1-a)\leqslant r_3,\quad r_4-1<mb\leqslant r_4,$$

$$m-r_4=r_1-1,\quad m-r_3=r_2-1$$

and therefore

$$\bar{w}_1=\cdots=\bar{w}_{r_1-1}=0,\quad \bar{w}_{r_1}=\frac{m-r_1-mb}{m(b-a)}\leqslant 0,$$

$$\bar{w}_{r_1+1}=\cdots=\bar{w}_{r_2-1}=\frac{-1}{m(b-a)},\quad \bar{w}_{r_2}=\frac{r_2-1-ma}{m(b-a)}\leqslant 0,$$

$$\bar{w}_{r_2+1}=\cdots=\bar{w}_{r_3-1}=0,\quad \bar{w}_{r_3}=-\bar{w}_{r_2},\quad \bar{w}_{r_3+1}=\cdots=\bar{w}_{r_4-1}=\frac{1}{m(b-a)},$$

$$\bar{w}_{r_4}=-\bar{w}_{r_1},\quad \bar{w}_{r_4+1}=\cdots=\bar{w}_m=0.$$

The expression for $p^c_{ij} + p^c_{ji}$ reduces to

$$p^c_{ij} + p^c_{ji} = 1 + \bar{w}_{r_1}(q^{r_1}_{ij} - q^{r_4}_{ij}) + \sum_{k=r_1+1}^{r_2-1} \frac{-1}{m(b-a)}(q^k_{ij} - q^{m-k+1}_{ij}) + \bar{w}_{r_2}(q^{r_2}_{ij} - q^{r_3}_{ij}) \leqslant 1, \quad \forall i,j.$$

Example 4. Using the linguistic quantifier "*most of*" with the pair of values $(0.3, 0.8)$ and the corresponding OWA operator with weight vector $(0, \frac{1}{15}, \frac{1}{3}, \frac{1}{3}, \frac{4}{15}, 0)$, then the collective preference relation is

$$P^c = \begin{pmatrix} 0.5 & 0.25 & 0.49 & 0.76 \\ 0.66 & 0.5 & 0.64 & 0.59 \\ 0.42 & 0.27 & 0.5 & 0.85 \\ 0.19 & 0.31 & 0.12 & 0.5 \end{pmatrix}.$$

Case C2: $a \geqslant \frac{1}{2}$. In this case, following a similar reasoning as in case b2, we have that

$$p^c_{ij} + p^c_{ji} = 1 + \bar{w}_{s_1}(q^{s_1}_{ij} - q^{s_4}_{ij}) + \sum_{k=s_1+1}^{s_2-1} \frac{-1}{m(b-a)}(q^k_{ij} - q^{m-k+1}_{ij}) + \bar{w}_{s_2}(q^{s_2}_{ij} - q^{s_3}_{ij}) \leqslant 1, \quad \forall i,j.$$

being $s_1, s_2, s_3, s_4 \in \{1, \ldots, m\}$ such that

$$s_1 - 1 < m(1-b) \leqslant s_1, \quad s_2 - 1 < m(1-a) \leqslant s_2,$$

$$s_3 - 1 < ma \leqslant s_3, \quad s_4 - 1 < mb \leqslant s_4,$$

$$m - s_4 = s_1 - 1, \quad m - s_3 = s_2 - 1,$$

and

$$\bar{w}_1 = \cdots = \bar{w}_{s_1-1} = 0, \quad \bar{w}_{s_1} = \frac{m(1-b) - s_1}{m(b-a)} \leqslant 0,$$

$$\bar{w}_{s_1+1} = \cdots = \bar{w}_{s_2-1} = \frac{-1}{m(b-a)},$$

$$\bar{w}_{s_2} = \frac{(s_2 - 1) - m(1-a)}{m(b-a)} \leqslant 0,$$

$$\bar{w}_{s_2+1} = \cdots = \bar{w}_{s_3-1} = 0, \quad \bar{w}_{s_3} = -\bar{w}_{s_2},$$

$$\bar{w}_{s_3+1} = \cdots = \bar{w}_{s_4-1} = \frac{1}{m(b-a)},$$

$$\bar{w}_{s_4} = -\bar{w}_{s_1}, \quad \bar{w}_{s_4+1} = \cdots = \bar{w}_m = 0.$$

Consequently $p^c_{ij} + p^c_{ji} \leqslant 1, \ \forall i,j$.

Example 5. Using, in this case, the linguistic quantifier "*as many as possible*" with the pair of values $(0.5, 1)$ and the corresponding OWA operator with weight vector $(0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then the collective preference relation is

$$P^{\mathrm{c}} = \begin{pmatrix} 0.5 & 0.2 & 0.31 & 0.62 \\ 0.6 & 0.5 & 0.45 & 0.39 \\ 0.34 & 0.13 & 0.5 & 0.59 \\ 0.06 & 0.15 & 0.04 & 0.5 \end{pmatrix}.$$

If $(a, b) = (0.7, 0.9)$, the weighting vector is $(0, 0, 0, 0, \frac{2}{3}, \frac{1}{3})$ and the collective preference relation is

$$P^{\mathrm{c}} = \begin{pmatrix} 0.5 & 0.15 & 0.19 & 0.5 \\ 0.58 & 0.5 & 0.33 & 0.32 \\ 0.32 & 0.13 & 0.5 & 0.59 \\ 0.03 & 0.07 & 0.01 & 0.5 \end{pmatrix}.$$

Summarising, we have obtained the following result:

Proposition 4. *Let $\{P^1, \ldots, P^m\}$ be a finite set of individual additive reciprocal preference relations, and Q a relative non-decreasing quantifier with membership function*

$$Q(x) = \begin{cases} 0, & 0 \leqslant x < a, \\ \dfrac{x-a}{b-a}, & a \leqslant x \leqslant b, \\ 1, & b < x \leqslant 1 \end{cases}$$

with $a + b > 1$. Then, the collective preference relation $P^{\mathrm{c}} = (p_{ij}^{\mathrm{c}})$, $p_{ij}^{\mathrm{c}} = \phi_Q(p_{ij}^1, \ldots, p_{ij}^m)$, obtained using the OWA operator ϕ_Q, verifies $p_{ij}^{\mathrm{c}} + p_{ji}^{\mathrm{c}} \leqslant 1$, $\forall i, j$.

3.2. Necessity of condition $a + b = 1$

We have given a sufficient condition on the parameters a and b ($a+b=1$) to ensure that $P^{\mathrm{c}} = (p_{ij}^{\mathrm{c}})$ is reciprocal for every set of reciprocal fuzzy preference relations. In what follows, we will show that the above condition is a necessary condition as well.

Therefore, if we impose that $P^{\mathrm{c}} = (p_{ij}^{\mathrm{c}})$ is reciprocal no matter which set of individual reciprocal fuzzy preference relations $\{P^1, \ldots, P^m\}$ we do start with, that is $p_{ij}^{\mathrm{c}} + p_{ji}^{\mathrm{c}} = 1$, $\forall i, j$, what can we say about parameters a and b?, is it compulsory that $a + b = 1$? We will prove that indeed $a + b = 1$ as we will show that being $P^{\mathrm{c}} = (p_{ij}^{\mathrm{c}})$ reciprocal and $a + b \neq 1$ lead to a contradiction.

In the case of being $a+b\neq 1$, four cases have to be studied,

$a+b<1$	$b\geqslant\frac{1}{2}$
	$b<\frac{1}{2}$
$a+b>1$	$b<\frac{1}{2}$
	$a\geqslant\frac{1}{2}$

Case B1: $a+b<1$ and $b\geqslant\frac{1}{2}$. To ensure that $P^c=(p_{ij}^c)$ is reciprocal for every set of reciprocal fuzzy preference relations, the following two conditions have to be verified:

1. $\bar{w}_{h_1}=0$ and $\bar{w}_{h_2}=0$,
2. $\bar{w}_{h_1+1}=\cdots=\bar{w}_{h_2-1}=0$.

Or equivalently

1. $h_1=ma$ and $h_3=mb$,
2. h_1 and h_2 have to be consecutive numbers because $1/m(b-a)\neq 0$, that is $h_2=h_1+1$.

All this leads to

$$m(a+b)=ma+mb=h_1+h_3=(h_2-1)+[m-(h_2-1)]=m$$

that is $a+b=1$, which contradicts being $a+b<1$.

Case B2: $a+b<1$ and $b<\frac{1}{2}$. Again, to guarantee the reciprocity of $P^c=(p_{ij}^c)$ for every set of reciprocal fuzzy preference relations, it has to be:

1. $\bar{w}_{l_1}=\bar{w}_{l_2}=0\Leftrightarrow l_1=ma\wedge l_2-1=mb$,
2. $\bar{w}_{l_1+1}=\cdots=\bar{w}_{l_2-1}=0\Leftrightarrow l_2=l_1+1$

and therefore

$$mb=l_2-1=l_1=ma\Leftrightarrow a=b,$$

which contradicts being $a<b$.

Case C1: $a+b>1$ and $a<\frac{1}{2}$. $P^c=(p_{ij}^c)$ is reciprocal when

1. $\bar{w}_{r_1}=\bar{w}_{r_2}=0\Leftrightarrow r_1=m(1-b)\wedge r_2-1=ma$,
2. $\bar{w}_{r_1+1}=\cdots=\bar{w}_{r_2-1}=0\Leftrightarrow r_2=r_1+1$

and consequently

$$m(a+b)=ma+mb=r_2-1+m-r_1=(r_1+1)-1+m-r_1=m$$

that is $a+b=1$, which contradicts being $a+b<1$.

Case C2: $a+b>1$ and $a\geqslant\frac{1}{2}$. The imposition $p_{ij}^c+p_{ji}^c=1$, $\forall i,j$, for every set of reciprocal fuzzy preference relations $\{P^1,\ldots,P^m\}$ implies that

1. $\bar{w}_{s_1}=\bar{w}_{s_2}=0\Leftrightarrow s_1=m(1-b)\wedge s_3=ma$,
2. $\bar{w}_{s_1+1}=\cdots=\bar{w}_{s_2-1}=0\Leftrightarrow s_2=s_1+1$

and therefore

$$m - mb = m(1 - b) = s_1 = s_2 - 1 = m - s_3 = m - ma$$

that is $a = b$ which contradicts being $a < b$.

4. Conclusions

We have obtained a necessary and sufficient conditions to ensure the additive reciprocity of the collective preference relation obtained when aggregating any finite set of additive reciprocal fuzzy relations using OWA operators guided by a relative non-decreasing linguistic quantifier with parameters (a, b). We have shown that additive reciprocity is maintained when $a + b = 1$ and not when $a + b \neq 1$. Moreover, as we can see from the examples given, the bigger the value of $|a + b - 1|$ the more distant the collective preference relation is from being additive reciprocal, in the sense that the bigger is $|p_{ij}^{c} + p_{ji}^{c} - 1|$.

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