

# A Study of the Origin and Uses of the Ordered Weighted Geometric Operator in Multicriteria Decision Making

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The ordered weighted geometric (OWG) operator is an aggregation operator that is based on the ordered weighted averaging (OWA) operator and the geometric mean. Its application in multicriteria decision making (MCDM) under multiplicative preference relations has been presented. Some families of OWG operators have been defined. In this article, we present the origin of the OWG operator and we review its relationship to the OWA operator in MCDM models. We show a study of its use in multiplicative decision-making models by providing the conditions under which reciprocity and consistency properties are maintained in the aggregation of multiplicative preference relations performed in the selection process. © 2003 Wiley Periodicals, Inc.

## 1. INTRODUCTION

In any multicriteria decision-making (MCDM) problem the final solution must be obtained from the synthesis of performance degrees of criteria.<sup>1,2</sup> To this end, the aggregation of information is fundamental.

The ordered weighted geometric (OWG) operator is an aggregation operator that we define and characterize in Ref. 3, to design multiplicative decision-making models,<sup>4,5</sup> i.e., MCDM processes using multiplicative preference relations<sup>6</sup> to express the preferences about alternatives. It is based on the ordered weighted averaging (OWA) operator<sup>7</sup> and on the geometric mean. Recently, some families of OWG operators were presented in Ref. 8.

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In this article, we study the basic ideas that justify the definition of the OWG operator and show its relationship to the OWA operator in the MCDM models. Additionally, we analyze two important aspects of its application in multiplicative decision-making processes:

- (1) The conditions under which the reciprocity property is maintained when aggregating multiplicative preference relations and, in particular, we provide a necessary and sufficient condition to this end
- (2) The conditions under which the consistency property is maintained when aggregating multiplicative preference relations and we show that this property generally is not maintained

To do this, our study is set out as follows. In Section 2 we present the OWG operator and study the foundations of its definition. In Section 3 we analyze its relationship to the OWA operator. In Section 4 we study the necessary and sufficient conditions under which the reciprocity property is maintained when aggregating reciprocal multiplicative preference relations using an OWG operator. In Section 5 we show that OWG operators generally do not maintain Saaty's consistency property<sup>6</sup> in the aggregation process but the geometric mean does. Finally, in Section 6 we draw our conclusions.

## 2. THE OWG OPERATOR AND ITS ORIGIN

In this section, we show why and how the OWG operator is defined in multiplicative decision-making models.

In Ref. 9 we consider MCDM problems where the information about the alternatives is represented using *fuzzy preference relations* and we design a *fuzzy majority* guided choice scheme that follows two steps to achieve a final decision from the synthesis of performance degrees of the majority of criteria: (i) aggregation and (ii) exploitation. This choice scheme is based on the quantifier-guided aggregation operator, the OWA operator,<sup>7</sup> which implements the concept of fuzzy majority in the aggregation phase by means of the *fuzzy quantifiers*<sup>10</sup> used to calculate its weighting vector.

**DEFINITION 1.** *In Ref. 7 an OWA operator of dimension  $n$  is a function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  that has associated to it a set of weights or weighting vector  $\mathbf{W} = (w_1, \dots, w_n)$  such that  $w_i \in [0, 1]$  and  $\sum_{i=1}^n w_i = 1$  and is defined to aggregate a list of values  $\{p_1, \dots, p_n\}$  according to the following expression:*

$$\phi(p_1, \dots, p_n) = \sum_{i=1}^n w_i \cdot q_i$$

being  $q_i$  the  $i$ th largest value in the set  $\{p_1, \dots, p_n\}$ .

Yager<sup>7</sup> suggests a way of calculating the weights of the OWA operator using fuzzy quantifiers representing the concept of fuzzy majority, which in the case of a nondecreasing relative quantifier  $Q$  is expressed as follows:

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \dots, n$$

When a fuzzy quantifier  $Q$  is used to compute the weights of the OWA operator  $\phi$ , then it is symbolized by  $\phi_Q$ .

The MCDM problem when the experts express their preferences using multiplicative preference relations has been solved by Saaty using the decision analytic hierarchical process (AHP), which obtains the set of solution alternatives by means of the eigenvector method.<sup>6</sup> However, this decision process is not guided by the concept of fuzzy majority. As shown in Refs. 11 and 12, the proper aggregation operator of ratio-scale measurements is not the *arithmetic mean* but the *geometric mean*. However, the geometric mean does not allow the concept of fuzzy majority to be incorporated in the decision processes. Therefore, if we want to design a decision scheme for multiplicative preference relations that allows decision makers to implement the concept of fuzzy majority to obtain the final solution, then it is necessary to introduce a new class of operator to aggregate ratio-scale measurements, allowing the implementation of the fuzzy majority concept.

In Ref. 4, we obtained the transformation function between multiplicative and fuzzy preference relations, which is given in the following result:

PROPOSITION 1. *From Ref. 4, suppose that we have a set of alternatives  $X = \{x_1, \dots, x_n\}$  and associated with it a multiplicative reciprocal preference relation  $A = (a_{ij})$ , with  $a_{ij} \in [1/9, 9]$  and  $a_{ij} \cdot a_{ji} = 1$ , for all  $i, j$ . Then the corresponding fuzzy reciprocal preference relation  $P = (p_{ij})$  associated to  $A$ , with  $p_{ij} \in [0, 1]$  and  $p_{ij} + p_{ji} = 1, \forall i, j$ , is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2} (1 + \log_9 a_{ij})$$

The above transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. In this way, e.g., if we want to aggregate a set of values  $\{a_1, \dots, a_n\}$  given on the basis of a positive ratio scale, we can use the OWA operator not on the set of values  $\{a_1, \dots, a_n\}$  but on the set of values  $\{p_1, \dots, p_n\}$  obtained using the foregoing transformation function  $f$ , i.e.,  $p_i = f(a_i) = 1/2 (1 + \log_9 a_i)$ . Thus, we obtain

$$p = \phi(p_1, \dots, p_n) = \sum_{i=1}^n w_i \cdot q_i$$

being  $q_i$  the  $i$ th largest value in  $\{p_1, \dots, p_n\}$ . If we denote  $b_i$  as the  $i$ th largest value in  $\{a_1, \dots, a_n\}$ , as  $f$  is an increasing function, then  $q_i = f(b_i) = 1/2(1 + \log_9 b_i)$ , and, therefore,

$$\begin{aligned}
 p &= \sum_{i=1}^n w_i \cdot \frac{1}{2} (1 + \log_9 b_i) = \frac{1}{2} \left( 1 + \sum_{i=1}^n w_i \cdot \log_9 b_i \right) \\
 &= \frac{1}{2} \left( 1 + \sum_{i=1}^n \log_9 b_i^{w_i} \right) = \frac{1}{2} \left( 1 + \log_9 \prod_{i=1}^n b_i^{w_i} \right)
 \end{aligned}$$

This last expression justifies the definition of the OWG operator as an aggregation operator to aggregate information given on a ratio scale.

**DEFINITION 2.** *From Ref. 3. An OWG operator of dimension  $n$  is a function  $\phi^G : \mathbb{R}^n \rightarrow \mathbb{R}$ , to which a set of weights or weighting vector is associated  $\mathbf{W} = (w_1, \dots, w_n)$ , such that  $w_i \in [0, 1]$  and  $\sum_i w_i = 1$ , and it is defined to aggregate a list of values  $\{a_1, \dots, a_n\}$  according to the following expression:*

$$\phi^G(a_1, \dots, a_n) = \prod_{i=1}^n b_i^{w_i},$$

where  $b_i$  is the  $i$ th largest value in the set  $\{a_1, \dots, a_n\}$ .

From Ref. 3 we show that the OWG operator satisfies the following properties:

- (1) It is an or-and operator, i.e., it remains between the minimum and the maximum of the arguments:  $\min(a_1, \dots, a_m) \leq \phi^G(a_1, \dots, a_m) \leq \max(a_1, \dots, a_m)$ .
- (2) It is commutative:  $\phi^G(a_1, \dots, a_m) = \phi^G(a_{\pi(1)}, \dots, a_{\pi(m)})$  for all  $\pi$ .
- (3) It is idempotent:  $\phi^G(a_1, \dots, a_m) = a$ , if  $a_i = a$  for all  $i$ .
- (4) It is increasing monotonous:  $\phi^G(a_1, \dots, a_m) \geq \phi^G(d_1, \dots, d_m)$ , if  $a_i \geq d_i$  for all  $i$ .
- (5) It leads to the geometric mean when  $w_i = 1/m$  for all  $i$ :  $\phi_{GM}^G(a_1, a_2, \dots, a_m) = \prod_{k=1}^m (a_k)^{1/m}$ .
- (6) It leads to the maximum when  $\mathbf{W} = [1, 0, \dots, 0]$ :  $\phi^G(a_1, a_2, \dots, a_m) = \max_{i=1}^m (a_i)$ .
- (7) It leads to the minimum when  $\mathbf{W} = [0, \dots, 0, 1]$ :  $\phi^G(a_1, a_2, \dots, a_m) = \min_{i=1}^m (a_i)$ .

Because the OWG operator is based on the OWA operator, it is clear that the weighting vector  $\mathbf{W}$  can be obtained by the same method used in the case of the OWA operator, i.e., the vector may be obtained using a fuzzy quantifier  $Q$ , representing the concept of fuzzy majority. When a fuzzy quantifier  $Q$  is used to compute the weights of the OWG operator  $\phi^G$ , then it is symbolized by  $\phi_Q^G$ .

Consequently, the OWG operator is defined to implement the concept of fuzzy majority in the multiplicative decision-making processes and its expression is obtained from the OWA operator using the transformation function  $f$  that relates the multiplicative to the fuzzy preference relations. We used it in Refs. 4 and 5 to design different multiplicative decision-making models and has been used by other authors to define different families of OWG operators.<sup>8</sup>

**3. THE RELATIONSHIPS BETWEEN THE OWG AND THE OWA OPERATORS IN DECISION-MAKING CONTEXTS**

In this section we analyze the relationships between the OWG and OWA operators when they are used as aggregation operators in the MCDM problems. Such relationships are established via the foregoing transformation function  $f$ .

**3.1. The Relationship Between the OWG and the OWA Operators to Define Choice Degrees of Alternatives**

In Ref. 13 we define two quantifier-guided choice degrees of alternatives using the ideas presented in Ref. 14: *quantifier-guided dominance degree* (QGDD) and *quantifier-guided nondominance degree* (QGNDD) to solve MCDM problems under fuzzy preference relations, which are based on the use of the OWA operator  $\phi_Q$ .

DEFINITION 3. In Ref. 3 if  $P = (p_{ij})$  is a fuzzy preference relation over the set of alternatives  $X = \{x_1, \dots, x_n\}$ , for the alternative  $x_i$  we define the QGDD<sub>*i*</sub>, used to quantify the dominance that  $x_i$  has over all the others in a fuzzy majority sense, as follows:

$$QGDD_i = \phi_Q(p_{i1}, \dots, p_{in})$$

DEFINITION 4. In Ref. 3 if  $P = (p_{ij})$  is a fuzzy preference relation over the set of alternatives  $X = \{x_1, \dots, x_n\}$ , for the alternative  $x_i$  we define the QGNDD<sub>*i*</sub>, used to quantify the degree to which  $x_i$  is not dominated by a fuzzy majority of the remaining alternatives, as follows:

$$QGNDD_i = \phi_Q(1 - p_{i1}^s, \dots, 1 - p_{in}^s)$$

where  $p_{ji}^s = \max\{p_{ji} - p_{ij}, 0\}$  represents the degree to which  $x_i$  is strictly dominated by  $x_j$ .

We note that if the fuzzy preference relation  $P = (p_{ij})$  is assumed reciprocal in the sense that  $p_{ij} + p_{ji} = 1$ , for all  $i, j$ , then we have

$$1 - p_{ji}^s = 1 - \max\{p_{ji} - p_{ij}, 0\} = \min\{1 - (p_{ji} - p_{ij}), 1\} = \min\{2p_{ij}, 1\}$$

and the expression of QGNDD<sub>*i*</sub> can be rewritten as

$$QGNDD_i = \phi_Q(p_{i1}^d, \dots, p_{in}^d) \quad \text{being} \quad p_{ij}^d = \min\{2p_{ij}, 1\}$$

The OWG operator gives us the opportunity to define the multiplicative versions of the foregoing QGDD and QGNDD for solving MCDM problems under multiplicative preference relations. Indeed, if we have a multiplicative reciprocal preference relation  $A = (a_{ij})$ ,  $a_{ij} \cdot a_{ji} = 1$ , then by applying function  $f$  we obtain the corresponding fuzzy reciprocal preference relation  $P = (p_{ij})$ ,  $p_{ij} = 1/2(1 + \log_\circ a_{ij})$ , where the QGDD and QGNDD are defined.

If we denote as  $b_{ij}$  the  $j$ th largest value of  $\{a_{i1}, \dots, a_{in}\}$ , we have  $q_{ij} = 1/2(1 + \log_\circ b_{ij})$  because  $f$  is an increasing function. Thus, we obtain

$$\begin{aligned}
 QGDD_i &= \phi_Q(p_{i1}, \dots, p_{in}) = \sum_{j=1}^n w_j \cdot q_{ij} = \sum_{j=1}^n w_j \cdot \left[ \frac{1}{2} \cdot (1 + \log_9 b_{ij}) \right] \\
 &= \frac{1}{2} \cdot \left( 1 + \sum_{j=1}^n w_j \cdot \log_9 b_{ij} \right) = \frac{1}{2} \cdot \left( 1 + \log_9 \prod_{j=1}^n b_{ij}^{w_j} \right) \\
 &= \frac{1}{2} \cdot [1 + \log_9 \phi_Q^G(a_{i1}, \dots, a_{in})] = f(\phi_Q^G(a_{i1}, \dots, a_{in}))
 \end{aligned}$$

This last expression gives us the justification for the following definition.

**DEFINITION 5.** [multiplicative QGDD (MQGDD)]. *If  $A = (a_{ij})$  is a multiplicative preference relation over the set of alternatives  $X = \{x_1, \dots, x_n\}$  for the alternative  $x_i$ , we define the MQGDD $_i$ , used to quantify the dominance that  $x_i$  has over all the others in a fuzzy majority sense, as follows:*

$$MQGDD_i = \phi_Q^G(a_{i1}, \dots, a_{in})$$

We should point out the following about this definition:

- (1) *It is obvious that with Definition 5 we have  $QGDD_i = f(MQGDD_i)$ .*
- (2) *To obtain a unique ordering among the set of alternatives by the application of a choice degree, it is obvious that we can use a particular expression of a choice degree or any one obtained by the application of an increasing function. In fact, if we use the expression  $MQGDD_i = 1/2 \cdot [1 + \log_9 \phi_Q^G(a_{i1}, \dots, a_{in})]$  as the definition of multiplicative quantifier-guided choice degree,<sup>6,7</sup> then it is clear that we would get the same ordering of alternatives as we would get by using the one given in Definition 5, because these two expressions are related by the bijective increasing function  $f$ .*
- (3) *Using the expression  $MQGDD_i = 1/2 \cdot [1 + \log_9 \phi_Q^G(a_{i1}, \dots, a_{in})]$  means defining the QGDD for a multiplicative preference relation  $A$  as the QGDD of the fuzzy preference relation obtained by applying the transformation function  $f$  to  $A$ , i.e.,  $P = f(A)$ .*

On the other hand, if we denote  $a_{ji}^s = \max\{a_{ji}/a_{ij}, 1\}$  and  $a_{ij}^d = 9/a_{ji}^s$ , then

$$\begin{aligned}
 p_{ij}^d &= 1 - p_{ji}^s = 1 - \max\{p_{ji} - p_{ij}, 0\} = 1 - \max\left\{\frac{1}{2} \log_9 \frac{a_{ji}}{a_{ij}}, 0\right\} \\
 &= 1 - \frac{1}{2} \log_9 \left( \max\left\{\frac{a_{ji}}{a_{ij}}, 1\right\} \right) \\
 &= 1 - \frac{1}{2} \log_9 a_{ji}^s = \frac{1}{2} (1 + \log_9 9) - \frac{1}{2} \log_9 a_{ji}^s \\
 &= \frac{1}{2} \left( 1 + \log_9 \frac{9}{a_{ji}^s} \right) = \frac{1}{2} (1 + \log_9 a_{ij}^d)
 \end{aligned}$$

therefore,

$$\begin{aligned} \text{QGNDD}_i &= \phi_Q(p_{i1}^d, \dots, p_{in}^d) = \sum_{j=1}^n w_j \cdot q_{ij}^d = \sum_{j=1}^n w_j \cdot \frac{1}{2} (1 + \log_9 b_{ij}^d) \\ &= \frac{1}{2} \cdot \left( 1 + \sum_{j=1}^n w_j \cdot \log_9 b_{ij}^d \right) = \frac{1}{2} \cdot \left[ 1 + \log_9 \prod_{j=1}^n (b_{ij}^d)^{w_j} \right] \\ &= \frac{1}{2} \cdot [1 + \log_9 \phi_Q^G(a_{i1}^d, \dots, a_{in}^d)] = f(\phi_Q^G(a_{i1}^d, \dots, a_{in}^d)) \end{aligned}$$

We note that for all  $a, b > 0$ ,  $1/(\max\{a, b\}) = \min\{1/a, 1/b\}$ , and thus  $a_{ij}^d = \min\{9 \cdot (a_{ij}/a_{ji}), 9\}$ . All this leads us to the following definition:

**DEFINITION 6.** [multiplicative QGNDD (MQGNDD)]. *If  $A = (a_{ij})$  is a multiplicative preference relation over the set of alternatives  $X$  for the alternative  $x_i$ , we define the MQGNDD<sub>*i*</sub> used to quantify the degree to which  $x_i$  is not dominated by a fuzzy majority of the remaining alternatives as follows:*

$$\text{MQGNDD}_i = \phi_Q^G(a_{i1}^d, \dots, a_{in}^d)$$

where  $a_{ij}^d = \min\{9 \cdot (a_{ij}/a_{ji}), 9\}$  represents the degree to which  $x_i$  is strictly dominated by  $x_j$  measured in  $[1/9, 9]$ .

In the two following propositions, we establish the consistency of the foregoing multiplicative choice degrees by comparing them with the priority vectors provided by Saaty’s eigenvector method applied in the AHP. In particular, we show that the ordering among the alternatives provided by Saaty’s eigenvector method from a multiplicative preference relation  $A = (a_{ij})$  and the one obtained by applying any of the two foregoing multiplicative quantifier-guided choice degrees are the same; therefore, we show both choice degrees do not change the informative content of the multiplicative preference relation.

**PROPOSITION 2.** *If  $x_i, x_j \in X$ , assuming that for a given consistent multiplicative preference relation  $A = (a_{ij})$ ,  $a_{ij} \cdot a_{jk} = a_{ik}$ , for all  $i, j, k$ ,<sup>11</sup> without loss of generality, the eigenvector method provides a priority vector  $\alpha = (\alpha_1, \dots, \alpha_n)$  verifying  $\alpha_i \leq \alpha_j$ , and then the MQGDD satisfies the equivalent relationship:*

$$\text{MQGDD}_i \leq \text{MQGDD}_j$$

*Proof.* We know that as  $A$  is consistent, the relation between the weights  $\alpha_i, \alpha_j$ , and the judgment  $a_{ij}$  is given by  $a_{ij} = s(\alpha_i)/s(\alpha_j)$  being  $s$  an increasing function,<sup>7</sup> and in particular  $a_{ij} = \alpha_i/\alpha_j$ . Then, the expression of the MQGDD<sub>*i*</sub> reduces to

$$\begin{aligned} \text{MQGDD}_i &= \prod_{t=1}^n a_{it}^{w_t} = \prod_{t=1}^n \left( \frac{s(\alpha_i)}{s(\alpha_t)} \right)^{w_t} = \frac{\prod_{t=1}^n s(\alpha_i)^{w_t}}{\prod_{t=1}^n s(\alpha_t)^{w_t}} \\ &= \frac{s(\alpha_i)^{\sum_{t=1}^n w_t}}{\prod_{t=1}^n s(\alpha_t)^{w_t}} = \frac{s(\alpha_i)}{C} \leq \frac{s(\alpha_j)}{C} = \text{MQGDD}_j, \end{aligned}$$

where  $C = \prod_{t=1}^n s(\alpha_t)^{w_t}$ . ■

PROPOSITION 3. *If  $x_i, x_j \in X$ , assuming that for a given consistent multiplicative preference relation  $A$ , without loss of generality, the eigenvector method provides a priority vector  $\alpha$  verifying  $\alpha_i \leq \alpha_j$ , and then the MQGNDD satisfies the equivalent relationship:*

$$\text{MQGNDD}_i \leq \text{MQGNDD}_j$$

*Proof.* The assumption that  $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$  implies

- (1)  $1/\alpha_1 \geq 1/\alpha_2 \geq \dots \geq 1/\alpha_n$  and because  $s$  is an increasing function,  $a_{i1} \geq a_{i2} \geq \dots \geq a_{in}$
- (2)  $a_{it} \leq a_{ii}$ , for all  $i \leq t$ , which implies that

$$a_{it}^d = \min\left\{9 \cdot \frac{a_{it}}{a_{ii}}, 9\right\} = \begin{cases} 9 \cdot a_{it}^2 & \text{if } t \geq i \\ 9 & \text{otherwise} \end{cases}$$

In the case of the  $\text{MQGNDD}_i$ , we have the following expression:

$$\text{MQGNDD}_i = \prod_{t=1}^n (a_{it}^d)^{w_t} = \prod_{t=1}^{i-1} (a_{it}^d)^{w_t} \cdot \prod_{t=i}^n (a_{it}^d)^{w_t} = \prod_{t=1}^{i-1} 9^{w_t} \cdot \prod_{t=i}^n (9a_{it}^2)^{w_t} = 9 \cdot \prod_{t=i}^n (a_{it}^2)^{w_t}$$

Because  $a_{it} \leq 1$ , for all  $t \geq i$ , then

$$\begin{aligned} \text{MQGNDD}_i &= 9 \cdot \prod_{t=i}^n (a_{it})^{2w_t} = 9 \cdot \prod_{t=i}^{j-1} (a_{it})^{2w_t} \cdot \prod_{t=j}^n (a_{it})^{2w_t} \\ &\leq 9 \cdot \prod_{t=i}^{j-1} 1^{2w_t} \cdot \prod_{t=j}^n (a_{it}^2)^{w_t} = 9 \cdot \prod_{t=j}^n (a_{it}^2)^{w_t} = \text{MQGNDD}_j \end{aligned} \quad \blacksquare$$

### 3.2. The Relationship Between the OWG and the OWA Operators to Design a Selection Process

At this point, we note that the foregoing multiplicative choice degrees of alternatives and the OWG operator can be used to build a selection process based on fuzzy majority to solve MCDM problems, where the experts express their preferences on the set of alternatives  $X$  by means of a set of multiplicative preference relations  $\{A^1, \dots, A^m\}$ . This selection process can be designed as a multiplicative version of the selection process based on the OWA operator proposed in Ref. 13 for MCDM problems with fuzzy preference relations.

Thus, the multiplicative selection process based on fuzzy majority is structured in two phases:

- (1) *The aggregation phase.* This phase defines a collective multiplicative preference relation,  $A^c = (a_{ij}^c)$ , which indicates the global preference according to the fuzzy



majority of the experts' opinions.  $A^c$  is obtained from  $\{A^1, \dots, A^m\}$  by means of the following expression:

$$a_{ij}^c = \phi_Q^G(a_{ij}^1, \dots, a_{ij}^m)$$

where  $\phi_Q^G$  is the OWG operator guided by the concept of fuzzy majority represented by the fuzzy linguistic quantifier  $Q$ .

- (2) *The exploitation phase.* Using the quantifier-guided choice degrees defined for multiplicative preference relations, this phase transforms the global information about the alternatives into a global ranking of them, supplying the set of solution alternatives. According to the exploitation scheme designed in Refs. 3 and 4, the choice degrees can be applied in three steps:

**Step 1.** Using  $\phi_Q^G$  we obtain the following two sets of choice degrees of alternatives from  $A^c$ :

$$[MQGDD_1^c, \dots, MQGDD_n^c] \text{ and } [MQGNDD_1^c, \dots, MQGNDD_n^c]$$

The application of each choice degree of alternatives over  $X$  allows us to obtain the following sets of alternatives:

$$X^{MQGDD} = \{x_i | x_i \in X, MDQGDD_i^c = \sup_j MQGDD_j^c\}$$

$$X^{MQGNDD} = \{x_i | x_i \in X, MDQGNDD_i^c = \sup_j MQGNDD_j^c\}$$

**Step 2.** The application of the conjunction selection policy, obtaining the following set of alternatives:  $X^{QGCP} = X^{MQGDD} \cap X^{MQGNDD}$ . If  $X^{QGCP} \neq \emptyset$ , then end. Otherwise, continue.

**Step 3.** The application of one of the two sequential selection policies, according to either a dominance or nondominance criterion, i.e.,

- *The dominance-based sequential selection process MQG-DD-NDD.* To apply the QGDD over  $X$  to obtain the set  $X^{MQGDD}$ . If  $card(X^{MQGDD}) = 1$  then end, and this is the solution set. Otherwise, continue obtaining

$$X^{MQG-DD-NDD} = \{x_i | x_i \in X^{MQGDD}, MDQGNDD_i^c = \sup_{x_j \in X^{MQGDD}} MQGNDD_j^c\}$$

This is the selection set of alternatives.

*The nondominance-based sequential selection process MQG-NDD-DD.* To apply the QGNDD over  $X$  to obtain the set  $X^{MQGNDD}$ . If  $card(X^{MQGNDD}) = 1$  then end, and this is the solution set. Otherwise, continue obtaining

$$X^{MQG-NDD-DD} = \{x_i | x_i \in X^{MQGNDD}, MDQGDD_i^c = \sup_{x_j \in X^{MQGNDD}} MQGDD_j^c\}$$

This is the selection set of alternatives.

We should point out that in Ref. 4 we have shown that the transformation function  $f$  also connects both OWA and OWG operators in the sense expressed in the following result.

**PROPOSITION 4.** *If  $\{A^1, \dots, A^m\}$  is a set of multiplicative preference relations and  $A^c$  the collective multiplicative preference relation obtained using the OWG operator  $\phi_Q^G$ , and if  $\{P^1, \dots, P^m\}$  is the set of additive fuzzy preference relations*

obtained via the transformation function  $f$ ,  $P^k = f(A^k)$ , and  $P^d$  is the collective fuzzy preference relation obtained using the OWA operator  $\phi_Q$  then  $P^d = f(A^c)$ .

**4. THE PRESERVATION OF THE RECIPROCITY PROPERTY IN THE MULTIPLICATIVE SELECTION PROCESS BASED ON THE OWG OPERATOR**

In multiplicative MCDM models we assume that the multiplicative preference relations to express the judgements are reciprocal. However, it is well known that reciprocity generally is not preserved after aggregation is performed in the selection process. In this section, we study the conditions under which the reciprocity property is maintained when aggregating multiplicative reciprocal preference relations using an OWG operator guided by a relative fuzzy quantifier.

Suppose that a group of experts  $E = \{e_1, \dots, e_m\}$  provide preferences about the alternatives  $X = \{x_1, \dots, x_n\}$  by means of the multiplicative preference relations  $\{A^1, \dots, A^m\}$ , which are reciprocal,  $a_{ij}^k \cdot a_{ji}^k = 1$ , for all  $i, j, k$ .<sup>6</sup>

Then, in the aggregation phase of the foregoing multiplicative selection process, we derive a collective preference relation  $A^c = (a_{ij}^c)$  by using an OWG operator  $\phi_Q^G$  guided by a linguistic quantifier  $Q$ . Each  $a_{ij}^c$  indicates the global preference between every pair of alternatives  $x_i$  and  $x_j$  according to the majority of expert opinions represented by  $Q$ :

$$a_{ij}^c = \phi_Q^G(a_{ij}^1, \dots, a_{ij}^m) = \prod_{k=1}^m (b_{ij}^k)^{w_k}$$

with  $b_{ij}^k$  the  $k$ th largest value in the set  $\{a_{ij}^1, \dots, a_{ij}^m\}$  and  $w_k = Q(k/m) - Q[(k-1)/m]$ , for all  $k$ .

Because we are assuming  $A^k$  is reciprocal and then  $a_{ji}^k = 1/a_{ij}^k$ ; therefore, if  $\{b_{ij}^1, \dots, b_{ij}^m\}$  are ordered from highest to lowest, then  $\{b_{ji}^1, \dots, b_{ji}^m\}$ , being  $b_{ji}^k = 1/b_{ij}^k$ , are ordered from lowest to highest, and consequently we have

$$\begin{aligned} a_{ij}^c \cdot a_{ji}^c &= \prod_{k=1}^m (b_{ij}^k)^{w_k} \prod_{k=1}^m (b_{ji}^k)^{w_{m-k+1}} = \prod_{k=1}^m (b_{ij}^k)^{w_k} \cdot \prod_{k=1}^m \left(\frac{1}{b_{ij}^k}\right)^{w_{m-k+1}} \\ &= \prod_{k=1}^m (b_{ij}^k)^{w_k - w_{m-k+1}} = \prod_{k=1}^m (b_{ij}^k)^{\bar{w}_k} \end{aligned}$$

where  $\bar{w}_k = \{Q(k/m) - Q[(k-1)/m]\} - \{Q[(m-k+1)/m] - Q[(m-k)/m]\}$ . If we denote  $A(k) = Q(k/m) + Q[1 - (k/m)]$  then  $\bar{w}_k = A(k) - A(k - 1)$ . The following result is obvious.

**PROPOSITION 5.** *If  $Q$  is a nondecreasing linguistic quantifier with membership function verifying*

$$Q(1 - x) = 1 - Q(x)$$

then the collective multiplicative preference relation  $A^c$ , obtained by aggregating the set of multiplicative preference relations,  $\{A^1, \dots, A^m\}$ , using the OWG operator  $\phi_Q^G$ , is reciprocal.

*Proof.* If  $Q(1 - x) = 1 - Q(x)$ , then  $A(k) = 1$ , for all  $k$  and, consequently,  $\bar{w}_k = A(k) - A(k - 1) = 0$ , for all  $k$ . This implies that

$$a_{ij}^c \cdot a_{ji}^c = \prod_{k=1}^m (b_{ij}^k)^{\bar{w}_k} = \prod_{k=1}^m (b_{ij}^k)^0 = \prod_{k=1}^m 1 = 1, \quad \forall i, j \quad \blacksquare$$

In the case that  $Q$  is a nondecreasing relative fuzzy quantifier with membership function,

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x - a}{b - a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

$a, b \in [0, 1]$ , the election of a suitable fuzzy quantifier representing the concept of fuzzy majority that we wish to implement in our MCDM problem is reduced to choosing adequate values for the parameters  $a$  and  $b$ .

#### 4.1. The Sufficient Condition for Parameters $a$ and $b$

The problem that we have to solve is

*What condition do the parameters  $a$  and  $b$  have to meet so that  $a_{ij}^c \cdot a_{ji}^c = 1, \quad \forall i, j$*

*Note 1.* If all the individual multiplicative reciprocal preference relations are the same, i.e.,  $A^1 = \dots = A^m = A$ , then we will have  $A^c = A$ , no matter which OWG operator  $\phi_Q^G$  we use.

We distinguish three possible cases, according to the value of  $a + b$ : (A)  $a + b = 1$ , (B)  $a + b < 1$ , and (C)  $a + b > 1$ .

##### 4.1.1. Case A: $a + b = 1$

In this case  $1 - a = b$  ( $1 - b = a$ ); therefore,

$$Q(1 - x) = \begin{cases} 0 & 0 \leq 1 - x < a \\ \frac{1 - x - a}{b - a} & a \leq 1 - x \leq b \\ 1 & b < 1 - x \leq 1 \end{cases} = \begin{cases} 1 - 0 & 0 \leq x < a \\ 1 - \frac{x - a}{b - a} & a \leq x \leq b \\ 1 - 1 & b < x \leq 1 \end{cases} = 1 - Q(x)$$

If we apply the Proposition 5, the reciprocity property is maintained after the aggregation phase is performed. This is summarized in the following proposition.

PROPOSITION 6. *If  $Q$  is a relative nondecreasing linguistic quantifier with parameters  $a$  and  $b$  verifying  $a + b = 1$ , then the OWG operator guided by  $Q$  maintains multiplicative reciprocity.*

EXAMPLE 1. *Let's assume the following three multiplicative reciprocal preference relations*

$$A^1 = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 1 & 5 \\ 1/5 & 1/5 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 3 & 9 \\ 1/3 & 1 & 3 \\ 1/9 & 1/3 & 1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 4 & 8 \\ 1/4 & 1 & 2 \\ 1/8 & 1/2 & 1 \end{pmatrix}$$

*If we take the parameter values  $a = 0.4$  and  $b = 0.6$ , then we have*

$$Q(0) = Q(1/3) = 0, \quad Q(2/3) = Q(1) = 1$$

*therefore, the associated weighing vector is  $\mathbf{W} = (w_1, w_2, w_3) = (0, 1, 0)$ . The collective multiplicative preference relation that we obtain is*

$$A^c = \begin{pmatrix} 1 & 3 & 8 \\ 1/3 & 1 & 3 \\ 1/8 & 1/3 & 1 \end{pmatrix}$$

*which obviously is reciprocal.*

4.1.2. Case B:  $a + b < 1$

In this case, we have  $1 - a > b$  ( $1 - b > a$ ) and as  $a \leq b$  we have  $a < 1/2$ . We will start by assuming that  $b \geq 1/2$ , which implies that  $1 - b \leq b$ , and, consequently,

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x-a}{b-a} & a \leq x < 1-b \\ \frac{x-a}{b-a} & 1-b \leq x < b \\ 1 & b \leq x < 1-a \\ 1 & 1-a \leq x < 1 \end{cases} \quad Q(1-x) = \begin{cases} 1 & 0 \leq x < a \\ 1 & a \leq x < 1-b \\ \frac{1-x-a}{b-a} & 1-b \leq x < b \\ \frac{1-x-a}{b-a} & b \leq x < 1-a \\ 0 & 1-a \leq x < 1 \end{cases}$$

$$A(x) = \begin{cases} 1 & 0 \leq x < ma \\ \frac{x+m(b-2a)}{m(b-a)} & ma \leq x < m(1-b) \\ \frac{1-2a}{b-a} & m(1-b) \leq x < mb \\ \frac{m-x-m(b-2a)}{m(b-a)} & mb \leq x < m(1-a) \\ 1 & m(1-a) \leq x < m. \end{cases}$$

It is clear that there exists  $h_1, h_2, h_3, h_4 \in \{1, \dots, m\}$  such that  $h_1 - 1 < m \cdot a \leq h_1, h_2 - 1 < m(1 - b) \leq h_2, h_3 - 1 < m \cdot b \leq h_3$ , and  $h_4 - 1 < m(1 - a) \leq h_4$ ; therefore,

$$\begin{aligned}
 A(0) &= \dots = A(h_1 - 1) = 1 \\
 A(k) &= \frac{k + m(b - 2a)}{m(b - a)}, \quad k = h_1, \dots, h_2 - 1 \\
 A(j) &= \frac{1 - 2a}{b - a}, \quad j = h_2, \dots, h_3 - 1 \\
 A(l) &= \frac{m - l - m(b - 2a)}{m(b - a)}, \quad l = h_3, \dots, h_4 - 1 \\
 A(h_4) &= \dots = A(m) = 1
 \end{aligned}$$

Moreover, we have  $m - h_4 = h_1 - 1, m - h_3 = h_2 - 1$ ; thus,

$$\begin{aligned}
 \bar{w}_1 &= \dots = \bar{w}_{h_1-1} = 0 = \bar{w}_{h_4+1} = \dots = \bar{w}_m \\
 \bar{w}_{h_1} &= \frac{h_1 - ma}{m(b - a)} = -\bar{w}_{h_4} \geq 0 \\
 \bar{w}_{h_1+1} &= \dots = \bar{w}_{h_2-1} = \frac{1}{m(b - a)} = -\bar{w}_{h_3+1} = \dots = -\bar{w}_{h_4-1} \geq 0 \\
 \bar{w}_{h_2} &= \frac{h_2 - mb}{m(b - a)} = -\bar{w}_{h_3} \geq 0 \\
 \bar{w}_{h_2+1} &= \dots = \bar{w}_{h_3-1} = 0
 \end{aligned}$$

The expression of  $a_{ij}^c \cdot a_{ji}^c$  reduces to

$$a_{ij}^c \cdot a_{ji}^c = \left(\frac{b_{ij}^{h_1}}{b_{ij}^{h_4}}\right)^{\bar{w}_{h_1}} \cdot \prod_{k=h_1+1}^{h_2-1} \left(\frac{b_{ij}^k}{b_{ij}^{m-k+1}}\right)^{1/[m(b-a)]} \cdot \left(\frac{b_{ij}^{h_2}}{b_{ij}^{h_3}}\right)^{\bar{w}_{h_2}}$$

As  $\{b_{ij}^1, \dots, b_{ij}^m\}$  is ordered from highest to lowest, every fraction in the foregoing expression is greater or equal to 1; therefore, the whole product is as well, i.e.  $a_{ij}^c \cdot a_{ji}^c \geq 1$ . In the case of  $b < 1/2$  following a similar reasoning, we reach the same conclusion. Summarizing, we have obtained the following result.

PROPOSITION 7. *If  $\{A^1, \dots, A^m\}$  is a finite set of individual multiplicative reciprocal preference relations and  $Q$  is a nondecreasing relative quantifier with membership function*

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x - a}{b - a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

with  $a + b < 1$ , then the collective multiplicative preference relation obtained using the OWG operator  $\phi_Q^G$ ,  $A^c = (a_{ij}^c)$ , and  $a_{ij}^c = \phi_Q^G(a_{ij}^1, \dots, a_{ij}^m)$  verifies  $a_{ij}^c \cdot a_{ji}^c \geq 1$ .

EXAMPLE 2. In the case of  $a = 0$  and  $b = 0.5$  (symbolizing the fuzzy quantifier of “at least one-half”), the collective multiplicative preference relation that we obtain is

$$A^c = \begin{pmatrix} 1 & 3.63 & 8.65 \\ 0.69 & 1 & 4.22 \\ 0.17 & 0.44 & 1 \end{pmatrix}$$

If  $a = 0.15$  and  $b = 0.35$  the collective multiplicative preference relation is

$$A^c = \begin{pmatrix} 1 & 3.90 & 8.91 \\ 0.91 & 1 & 4.79 \\ 0.19 & 0.48 & 1 \end{pmatrix}$$

In both cases it is clear that the condition  $a_{ij}^c \cdot a_{ji}^c \geq 1$  is verified.

#### 4.1.3. Case C: $a + b > 1$

As in the previous subsection, we have to distinguish two subcases: (i)  $a < 1/2$  and (ii)  $a \geq 1/2$ . We study just the first one because following a similar reasoning the same result is obtained in both subcases.

The expression of  $A(x)$  when  $a < 1/2$  is the following:

$$A(x) = \begin{cases} 1 & 0 \leq x < m(1 - b) \\ \frac{m - x - ma}{m(b - a)} & m(1 - b) \leq x < ma \\ \frac{1 - 2a}{b - a} & ma \leq x < m(1 - a) \\ \frac{x - ma}{m(b - a)} & m(1 - a) \leq x < mb \\ 1 & mb \leq x < m. \end{cases}$$

As in the previous case, there exists  $r_1, r_2, r_3, r_4 \in \{1, \dots, m\}$  such that  $r_1 - 1 < m(1 - b) \leq r_1, r_2 - 1 < ma \leq r_2, r_3 - 1 < m(1 - a) \leq r_3, r_4 - 1 < mb \leq r_4, m - r_4 = r_1 - 1$ , and  $m - r_3 = r_2 - 1$ ; therefore,

$$\bar{w}_1 = \dots = \bar{w}_{r_1-1} = 0 = \bar{w}_{r_4+1} = \dots = \bar{w}_m$$

$$\bar{w}_{r_1} = \frac{m - r_1 - mb}{m(b - a)} = -\bar{w}_{r_4} \leq 0$$

$$\bar{w}_{r_1+1} = \dots = \bar{w}_{r_2-1} = \frac{-1}{m(b - a)} = -\bar{w}_{r_3+1} = \dots = -\bar{w}_{r_4-1} \leq 0$$

$$\bar{w}_{r_2} = \frac{r_2 - 1 - ma}{m(b - a)} = -\bar{w}_{r_3} \leq 0$$

$$\bar{w}_{r_2+1} = \dots = \bar{w}_{r_3-1} = 0$$

The expression of  $a_{ij}^c \cdot a_{ji}^c$  reduces to

$$a_{ij}^c \cdot a_{ji}^c = \left(\frac{b_{ij}^{r_4}}{b_{ij}^{r_1}}\right)^{\bar{w}_{r_4}} \cdot \prod_{k=r_1+1}^{r_2-1} \left(\frac{b_{ij}^{m-k+1}}{b_{ij}^k}\right)^{1/[m(b-a)]} \cdot \left(\frac{b_{ij}^{r_3}}{b_{ij}^{r_2}}\right)^{\bar{w}_{r_3}}$$

As  $\{b_{ij}^1, \dots, b_{ij}^m\}$  is ordered from highest to lowest, every fraction in the foregoing expression is lower or equal to 1 and so the whole product is as well, i.e.,  $a_{ij}^c \cdot a_{ji}^c \leq 1$ . Summarizing, we have obtained the following result.

PROPOSITION 8. *If  $\{A^1, \dots, A^m\}$  is a finite set of individual multiplicative reciprocal preference relations and  $Q$  is a nondecreasing relative quantifier with membership function*

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x - a}{b - a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

with  $a + b > 1$ , then the collective multiplicative preference relation obtained using the OWG operator  $\phi_Q^G, A^c = (a_{ij}^c)$ , and  $a_{ij}^c = \phi_Q^G(a_{ij}^1, \dots, a_{ij}^m)$  verifies  $a_{ij}^c \cdot a_{ji}^c \leq 1$ .

EXAMPLE 3. *In the case of  $a = 0.3$  and  $b = 0.8$  (fuzzy quantifier “most of”), the collective multiplicative preference relation obtained is*

$$A^c = \begin{pmatrix} 1 & 2.28 & 7.11 \\ 0.33 & 1 & 2.79 \\ 0.12 & 0.3 & 1 \end{pmatrix}.$$

*If  $a = 0.6$  and  $b = 0.9$ , the collective multiplicative preference relation is*

$$A^c = \begin{pmatrix} 1 & 1.44 & 5.85 \\ 0.28 & 1 & 2.29 \\ 0.12 & 0.24 & 1 \end{pmatrix}.$$

*In both cases it is clear that the condition  $a_{ij}^c \cdot a_{ji}^c \leq 1$  is verified.*

#### 4.2. The Necessity of the Condition $a + b = 1$

In this subsection, we will show that the condition  $a + b = 1$  is also a necessary condition to maintain reciprocity in the aggregation process of multiplicative preference relations.

If we suppose that  $A^c = (a_{ij}^c)$  is reciprocal no matter which set of individual multiplicative reciprocal preference relations  $\{A^1, \dots, A^m\}$  are used,  $a_{ij}^c \cdot a_{ji}^c = 1$ , for all  $i, j$  *what can we say about the parameters  $a$  and  $b$ ? Is it compulsory that  $a + b = 1$ ?*

We will prove that in fact  $a + b = 1$  because we will show that  $A^c$  is reciprocal and that  $a + b \neq 1$  are not compatible.

If  $a + b \neq 1$ , four cases have to be studied,

$a + b < 1$	$b \geq 1/2$
	$b < 1/2$
$a + b > 1$	$a < 1/2$
	$a \geq 1/2$

As we have done in the previous subsection, we will prove the necessity of  $a + b = 1$  for cases  $a + b < 1 \wedge b \geq 1/2$  and  $a + b > 1 \wedge a < 1/2$ , because in the other two cases the same result is obtained by similar reasoning.

4.2.1. Case B1:  $a + b < 1$  and  $b \geq 1/2$

To ensure that  $A^c = (a_{ij}^c)$  is reciprocal for every set of multiplicative reciprocal preference relations, the following two conditions have to be met:

- (1)  $\bar{w}_{h_1} = 0$  and  $\bar{w}_{h_2} = 0$
- (2)  $\bar{w}_{h_1+1} = \dots = \bar{w}_{h_2-1} = 0$

or, equivalently,

- (1)  $h_1 = ma$  and  $h_3 = mb$
- (2)  $h_1$  and  $h_2$  have to be consecutive numbers because  $1/[m(b - a)] \neq 0$ , i.e.,  $h_2 = h_1 + 1$

These two conditions lead to

$$m(a + b) = ma + mb = h_1 + h_3 = (h_2 - 1) + [m - (h_2 - 1)] = m$$

i.e.,  $a + b = 1$ , which contradicts  $a + b < 1$ .

4.2.2. Case C1:  $a + b > 1$  and  $a < 1/2$

Again, to guarantee the reciprocity of  $A^c = (a_{ij}^c)$  for every set of multiplicative reciprocal preference relations, the following has to be verified:

- (1)  $\bar{w}_{r_1} = \bar{w}_{r_2} = 0 \Leftrightarrow r_1 = m(1 - b) \wedge r_2 - 1 = m \cdot a$
- (2)  $\bar{w}_{r_1+1} = \dots = \bar{w}_{r_2-1} = 0 \Leftrightarrow r_2 = r_1 + 1$

and, consequently,

$$m(a + b) = ma + mb = r_2 - 1 + m - r_1 = r_1 + 1 - 1 + m - r_1 = m$$



i.e.,  $a + b = 1$ , which contradicts  $a + b < 1$ . Summarizing, we have obtained the following result.

PROPOSITION 9. *If  $Q$  is a nondecreasing relative fuzzy quantifier with membership function*

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x - a}{b - a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

*then the collective multiplicative preference relation,  $A^c$ , obtained by aggregating a set of multiplicative preference relations  $\{A^1, \dots, A^m\}$ , using the OWG operator  $\phi_Q^G$ , is reciprocal if and only if  $a + b = 1$ .*

**5. THE CONSISTENCY PROPERTY IN THE MULTIPLICATIVE SELECTION PROCESS BASED ON THE OWG OPERATOR**

In the case that multiplicative preference relations are consistent,  $A = (a_{ij})$ , verifying  $a_{ij} \cdot a_{jk} = a_{ik}$ , for all  $i, j, k$ , we showed in Section 3 that the ordering among the alternatives provided by Saaty's eigenvector method and the multiplicative choice degrees are the same. Therefore, knowing how to maintain the consistency property in the aggregation process could be of great interest to a decision maker.

It is easy to verify that the particular case of OWG operator, the geometric mean, that has associated a set of weights or weighting vector  $W = (w_1, \dots, w_n)$  such that  $w_i = 1/m$  for all  $i$  not only preserves reciprocity but the consistency property as well.

$$a_{ij}^c \cdot a_{jk}^c = \left( \prod_{k=1}^m a_{ij}^l \right)^{1/m} \cdot \left( \prod_{k=1}^m a_{jk}^l \right)^{1/m} = \left( \prod_{k=1}^m a_{ij}^l \cdot a_{jk}^l \right)^{1/m} = \left( \prod_{k=1}^m a_{ik}^l \right)^{1/m} = a_{ik}^c, \quad \forall i, j, k$$

However, we note that the consistency property generally is not maintained after aggregation is performed when using an OWG operator guided by a relative nondecreasing quantifier  $Q$ .

Indeed, as is well known, a consistent multiplicative preference relation  $A$  has to be reciprocal, i.e.,

$$a_{ij} \cdot a_{jk} = a_{ik}, \quad \forall i, j, k \Rightarrow a_{ij} \cdot a_{ji} = 1, \quad \forall i, j$$

This is easy to prove. First, taking

$$k = j = i \Rightarrow a_{ii} \cdot a_{ii} = 1, \quad \forall i$$

and therefore

$$a_{ij} \cdot a_{ji} = a_{ii} = 1, \quad \forall i, j$$

This means that to maintain the consistency property in the aggregation process, the parameters  $a, b \in [0, 1]$  of the relative nondecreasing fuzzy quantifier  $Q$  have to verify  $a + b = 1$ , as we have proved in the previous section. In example 1 we had

$$A^c = \begin{pmatrix} 1 & 3 & 8 \\ 1/3 & 1 & 3 \\ 1/8 & 1/3 & 1 \end{pmatrix}$$

which obviously is reciprocal but not consistent because  $a_{12} \cdot a_{23} \neq a_{13}$ .

## 6. CONCLUDING REMARKS

In this study we have studied the foundations and presented justifications of the origins of the OWG operator. We have also shown the main relationships between the OWG and the OWA operator in an MCDM context, where preferences are modeled by multiplicative preference relations. To do that we have used the function that transforms multiplicative preference relations into fuzzy preference relations, and the corresponding concepts, the OWA operator, and the QGDD and QGNDD, in the case of fuzzy preference relations. We have also presented an alternative selection process for MCDM problems to Saaty's AHP, with the advantage of allowing the decision makers to implement the concept of fuzzy majority in the decision process, which was not possible in the case of Saaty's AHP. On the other hand, we have given a necessary and sufficient condition to maintain reciprocity when aggregating a finite set of multiplicative reciprocal preference relations using OWG operators guided by fuzzy quantifiers. In the case of a nondecreasing relative fuzzy quantifier with parameters  $(a, b)$ , reciprocity is maintained only when  $a + b = 1$ . Furthermore, the greater the value of  $|a + b - 1|$  the more distant the collective multiplicative preference relation is from being reciprocal. Finally, we have given an example that shows that  $a + b = 1$  is a necessary but not sufficient condition to maintain the consistency property in the aggregation process.

In the future, we intend to study the order-induced aggregation<sup>15,16</sup> when using OWG operators and show its usefulness in multiplicative decision models. Additionally, we want to study the use of OWG operators to define consensus models<sup>17</sup> in multiplicative decision making.

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