On Numerical and Linguistic Quantification in Linguistic Approximation

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ABSTRACT

Interpretation of outputs of fuzzy systems often involves the use of linguistic approximation that assigns a linguistic label to a fuzzy set based on the predefined primary terms, linguistic modifiers and linguistic connectives. More formally linguistic approximation can be considered in the terms of re-translation rules that correspond to the translation rules in explicitation in computing with words such as the simple, modifier, composite, quantification and qualification rules. However most existing methods of linguistic approximation use the simple, modifier and composite re-translation rules only. Although these methods can provide a sufficient approximation of simple fuzzy sets the approximation of more complex ones that are typical in many practical applications of fuzzy systems may be less satisfactory. In particular quantification may be desirable in situations where the conclusions interpreted as quantified linguistic propositions can be more informative and natural. Quantification in linguistic approximation can provide an additional information about the scope of the linguistic labels assigned to a given fuzzy set enhancing the interpretability of the conclusions. This paper presents some aspects of linguistic approximation in the context of the re-translation rules and proposes an approach to linguistic approximation with the use of the quantification rules. Two methods of the quantification in linguistic approximation are considered with the use of the numerical and linguistic quantifiers. These quantifiers are based on the concepts of the non-fuzzy and fuzzy cardinalities of fuzzy sets, respectively. A number of examples are provided to illustrate the proposed approach.

1. INTRODUCTION

Most fuzzy systems including fuzzy decision support and fuzzy control systems provide outputs in the form of fuzzy sets that represent the inferred conclusions. Linguistic interpretation of such outputs often involves the use of linguistic approximation that assigns a linguistic label to a fuzzy set. Many methods of linguistic approximation have been developed and used in both fuzzy decision making [1, 3, 5, 6] and fuzzy control [2, 7]. These methods are usually based on combination of predefined primary terms (e.g. small, medium, large), linguistic modifiers or hedges (e.g. not, much, very, more or less) and their connectives (e.g. and, or) that form a linguistic label assigned to a given fuzzy set. For example Bonissone [1] has developed a linguistic approximation method based on feature extraction and pattern recognition techniques and used it in some problems of decision analysis and natural language processing. A more general approach to linguistic approximation has been proposed in [3] that uses a combination of segments of the membership function with well defined characteristics. The segments are labeled with the use of linguistic modifiers of the generated primitive terms and the final approximation is a combination of these labels. This technique has been demonstrated for a decision making application [3]. Similar principles have been used in linguistic approximation presented in [2] that considers only linguistic terms entering the inference mechanism of a linguistic fuzzy control system [2, 7]. A linguistic approximation method based on the use of the principles of evolutionary computation where primary terms, modifiers and connectives are treated as elements of a genetic program has been proposed in [5].

In general linguistic approximation can be considered as a complementary task to explicitation in computing with words [11]. Explicitation translates linguistic propositions into possibility distributions that are further processed by approximate reasoning to infer a possibility distribution of a conclusion. Linguistic approximation re-translates the induced possibility distribution into a linguistic proposition. Therefore linguistic approximation can be formalized in the terms of re-translation rules that correspond to the translation rules in explicitation such as the simple, modifier, composite, quantification and qualification rules. However most existing methods of linguistic approximation use the simple, modifier and composite re-translation rules only. Although these methods can provide a sufficient approximation of simple fuzzy sets the approximation of more complex ones that are typical in many practical applications of fuzzy systems may be less satisfactory. Therefore the question arises why not use in linguistic approximation also other re-translation rules corresponding to the translation rules in explicitation to advantage. In particular quantification may be desirable in situations where the conclusions interpreted as quantified linguistic propositions can be more informative and natural. Quantification in linguistic approximation can provide an additional information about the scope of the linguistic labels

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assigned to a given fuzzy set enhancing the interpretability of the conclusions. This paper presents some aspects of linguistic approximation in the context of the re-translation rules and proposes an approach to linguistic approximation with the use of quantification rules. The principles of re-translation rules in linguistic approximation are presented in section 2. Section 3 proposes two methods of quantification in linguistic approximation with the use of numerical and linguistic quantifiers based on the concepts of the non-fuzzy and fuzzy cardinalities of fuzzy sets, respectively. A number of examples are provided to illustrate the proposed approach. The concluding remarks are presented in section 4.

2. RE-TRANSLATION RULES IN LINGUISTIC APPROXIMATION

The fundamental concept used in fuzzy systems and more generally in computing with words is a linguistic proposition [9, 10, 11, 12, 13]. A simple linguistic proposition takes the form "*X is A*" where X is a variable over the universe of discourse U and A is a linguistic value corresponding to a fuzzy subset of U defined by a membership function μ_A. The variable X has an associated possibility distribution. It is described by a possibility distribution function π_X: $U \to [0,1]$ that assigns a degree of possibility to every value of X. Translation of linguistic propositions into the corresponding possibility distributions (i.e. explicitation) can be performed according to well known translations rules in fuzzy set theory [9, 10, 11, 12, 13]. For example in a simple proposition the possibility distribution function of X is equal to the membership function of A, i.e.

$$X \text{ is } A \to \pi_X = \mu_A$$

Translation of more complex propositions (e.g. modified, composite, qualified and quantified propositions) involves the use of translation rules such as modifier rules, composition rules, qualification rules and quantification rules [9, 10, 11, 12, 13]. Examples of the simple, modified and composite linguistic propositions, and their corresponding possibility distributions are presented in figure 1.

A complementary task to explicitation of linguistic propositions in computing with words is re-translation of the induced conclusions in the form of possibility distributions into propositions expressed in a natural language, i.e. linguistic propositions. It involves the use of linguistic approximation that assigns a linguistic label to a given fuzzy set.

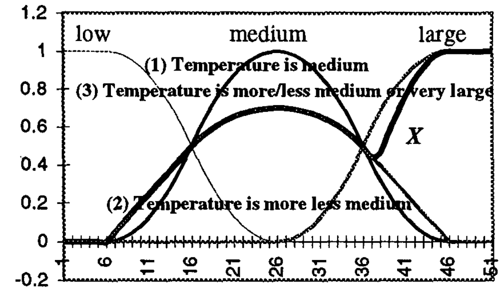


Fig. 1 Examples of (1) simple, (2) modified and (3) composite linguistic propositions

The problem of linguistic approximation can be defined as mapping from a set S of fuzzy subsets in a universe of discourse U, into a set of labels L, which are generated according to a grammar G and a vocabulary V [3]. Typically a solution of linguistic approximation is a linguistic description (label) LA composed of linguistic primary terms A, linguistic modifiers m and linguistic connectives c such that it is most suitable (meaningful) to describe a given fuzzy set (a possibility distribution of a linguistic variable). For example a given possibility distribution of a fuzzy set X in figure 1 describing temperature may be linguistically approximated to "*Temperature is more/less medium or very large*", i.e. $LA(X) \equiv X$ is $(m_1 A_1 \ c \ m_2 A_2)$ where $X \equiv$ Temperature, $m_1 \equiv$ more/less, $A_1 \equiv$ medium, $m_2 \equiv$ very, $A_2 \equiv$ large and $c \equiv$ or. It should be noted that the results of linguistic approximation are not unique and the quality of the provided solutions depends on the error of the approximation expressed typically as the degree of equality of fuzzy quantities [4, 8, 14], i.e. the original fuzzy set (and its segments) and a fuzzy set (and its segments) corresponding to the linguistic propositions in its linguistic approximation [6].

A common characteristic of the existing linguistic approximation methods is that, although not stated explicitly, they generate labels following the principles similar to the translation rules in explicitation. In the context of linguistic approximation these principles can be summarized as the following re-translation rules:

- Simple linguistic approximation

  Given the possibility distribution of a fuzzy set X, its linguistic approximation $LA(X)$ is a simple linguistic proposition as follows:

$$\pi_X \approx \mu_A \to LA(X) \equiv X \text{ is } A$$

  where π_X is a possibility distribution function of X, μ_A is a membership function of a linguistic term A, and $\approx$ stands for the equality of fuzzy quantities.

- Modified linguistic approximation

The modifier rule asserts that re-translation of the possibility distribution function is expressed in the the following form:

$$\pi_X \approx \mu_{mA} \rightarrow LA(X) \equiv X \ is \ mA$$

where μ_{mA} is a membership function of the modified linguistic term A induced by the linguistic modifier m. In other words m can be interpreted as an operator that transforms the fuzzy set A into the fuzzy set mA. For example if $m \equiv very$ then $\mu_{veryA}(x) = \mu_A^2(x)$.

- Composite linguistic approximation

The composite re-translation rules apply to linguistic approximation with composite linguistic propositions which are generated from linguistic terms through the use of binary connectives c such as the conjunction (and) and the disjunction (or) as follows

$$\pi_X \approx \mu_{AcB} \rightarrow LA(X) \equiv X \ is \ A \ c \ B$$

For example if c is the conjunction then the composite re-translation rule states that if the possibility distribution of X is equal to the intersection of A and B, i.e. $\mu_{A \wedge B}(x) = \min(\mu_A(x), \mu_B(x))$ then a linguistic approximation of X can be expressed by a composite proposition "*X is A and B*". It should be noted that the composite re-translation rule can also be applied to more general cases where A and B are defined on two different universes of discourse. More specifically, let U and V be two universes of discourse, and let A and B be fuzzy subsets of U and V, respectively. Then two propositions "*X is A*" and "*Y is B*" connected by the conjunction can be expressed by a composite proposition "*X is A and Y is B*" where the membership function is $\mu_{A \wedge B}(x, y) = \min(\mu_A(x), \mu_B(y))$.

To illustrate the above re-translation rules let us consider two simple problems of linguistic approximation illustrated in figure 2. The task is to assign linguistic labels to two different fuzzy sets X_1 and X_2 with the use of the simple, modifier and composite re-translation rules and the following elements:

- a set of primary terms T = {small, medium, large}
- a set of linguistic modifiers M = {not, very, more/less, indeed, above, below}
- a set of connectives C = {and, or}

It is easy to observe that both X_1 and X_2 can be linguistically approximated with the same label, i.e. $LA(X_1) = LA(X_2) \equiv$ *more/less medium or very large*. To distinguish these linguistic approximations one can also provide some additional information expressing the quality of the approximation such as the degree of equality of fuzzy quantities. However in some applications it may also be desirable to provide information about the scope of the generated linguistic labels. Following the observation of complementarity of linguistic approximation and explicitation in computing with words, it seems that the quantified re-translation rules can also be applied in linguistic approximation to advantage.

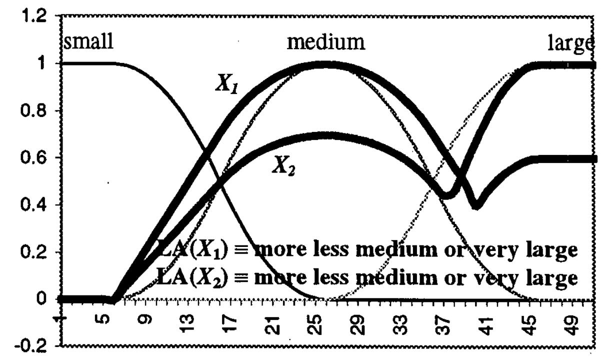


Fig. 2 Linguistic approximation of two different fuzzy sets

3. NUMERICAL AND LINGUISTIC QUANTIFICATION IN LINGUISITC APPROXIMATION

Quantification is a common means for expressing the scope of propositions. It plays a central role in commonsense knowledge representation and reasoning [12, 13]. The classical logic provides two types of quantification, i.e. universal and existential quantification that correspond to the quantifiers *all* and *some*, respectively. Fuzzy logic offers, in addition, a wide variety of fuzzy quantifiers such as few, several, usually, most [12, 13].

The linguistic proposition in the form of *X is A* (where *A* can be modified and/or composite) implicitly indicates that it is true for all values of X, i.e. *all X's are A* (or *100% of X's are A*). However when only a proportion of values of X satisfies the proposition then the scope specification of this proportion with other quantifiers can be desirable. The quantification rules in explicitation define the translation of quantified linguistic propositions into the canonical forms suitable for further processing such as assessing the truth of a given linguistic proposition. In linguistic approximation quantification can be used to provide the scope of the linguistic labels assigned to the approximated fuzzy set.

In general the quantification rules allow one to consider quantification in a linguistic proposition, i.e. "*QX is A*" where Q is a quantifier that can be interpreted in fuzzy logic as a fuzzy number. In general fuzzy numbers can represent linguistic quantifiers such as many, few, several, all, some, most (e.g. *many X's are large*). It should also be noted that in a specific case when a fuzzy number takes the form of a singleton fuzzyfier it can represent a numerical quantifier (e.g. 30% *of X's are large* or *0.3 of X's are large*). Examples of such fuzzy numbers with the corresponding numerical and linguistic quantifiers are illustrated in figure 3.

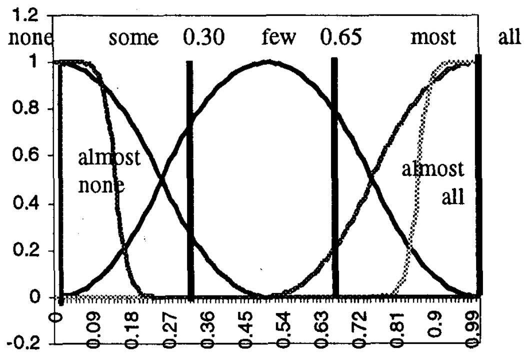


Fig. 3 An example of numerical and linguistic quantifiers

Referring to the principles of the quantification rules in explicitation [11], a quantified proposition can be generated in linguistic approximation of a given fuzzy set X with the following quantification re-translation rule:

$$\pi_{Card(A/X)} \approx \mu_Q \rightarrow LA(X) \equiv QX \text{ is } A$$

where μ_Q is a membership function of a fuzzy set (fuzzy number) corresponding to the quantifier Q. *Card(A/X)* denotes the number (or the proportion) of elements of X which are in A. In other words it can be considered as cardinality of a fuzzy set corresponding to the intersection of X and A. It is well known in fuzzy set theory that this cardinality can be expressed as a non-fuzzy (singleton fuzzyfier) or fuzzy number. More formally, the non-fuzzy cardinality of a fuzzy set F is typically defined as a sigma count as follows:

$$\Sigma Count(F) = \sum_{i=1}^{N} \mu_{F(i)}$$

where $\mu_{F(i)}$ is the grade of membership of the i^{th} value of U in the fuzzy set F. When Q relates to a proportion (e.g. most) then μ_Q is a mapping from [0,1] to [0,1], and so-called relative sigma count that expresses the proportion of elements of one fuzzy set X which are in another fuzzy set A can be defined as follows:

$$\Sigma Count(A/X) = \frac{\Sigma Count(A \cap X)}{\Sigma Count(X)}$$

It should be noted that the relative sigma count has been used in some linguistic approximation methods to measure the quality of approximation and to guide the matching process (e.g. [6]). In the presented approach its use is extended to quantification of the generated label with a numerical quantifier. For example let us consider two problems of linguistic approximation from the previous section (see figure 2) with the additional use of the quantified retranslation rule based on the non-fuzzy cardinality. Table 1 presents some results of the linguistic approximation with the calculated numerical quantifiers for a linguistic label with the highest relative sigma count, i.e. *more or less medium or very large* for both fuzzy sets. The components of such a label are further described in the terms of numerical quantifiers as follows:

$LA(X_1) \equiv 0.65$ of X_1 are more or less medium;
0.47 of X_1 are very large
$LA(X_2) \equiv 0.83$ of X_2 are more or less medium;
0.28 of X_2 are very large

Although the non-fuzzy cardinality has commonly been used in explicitation the fuzzy cardinality may be more

Table 1 Linguistic approximation with numerical quantification

Linguistic label A	$\Sigma Count(A/X_1)$	$\Sigma Count(A/X_2)$
small	0.15	0.17
medium	0.57	0.66
large	0.52	0.34
more or less medium	0.65	0.83
very large	0.47	0.28
medium or large	0.97	0.88
more or less medium or very large	1	1
	$LA(X_1) \equiv 0.65$ of X_1 are more or less medium; 0.47 of X_1 are very large	$LA(X_2) \equiv 0.83$ of X_2 are more or less medium; 0.28 of X_2 are very large

appropriate in linguistic approximation. In particular it can be used as a basis for linguistic quantification. In general the fuzzy cardinality of a fuzzy set A is expressed as a fuzzy number as it was proposed in [12, 13]. More specifically, let A_α be the α-level-set of A, i.e. non-fuzzy set defined by

$$A_\alpha = \{u_i | \mu_A(u_i) \geq \alpha\}, \quad 0 > \alpha \geq 1, u_i \in U, i = 1, \ldots, n$$

where $\mu_i = \mu_A(u_i), i = 1, \ldots, n$ is the grade of membership of u_i in A. Then the fuzzy cardinality *FECount(A)* can be represented by intersection of two fuzzy numbers corresponding to the fuzzy cardinalities *FGCount(A)* and *FLCount(A)* describing that at least n elements and at most n elements, respectively are in the fuzzy set A as follows:

$$FECount(A) = FGCount(A) \cap FLCount(A)$$

where

$$FGCount(A) = \sum_\alpha \alpha / Count(A_\alpha)$$

$$FLCount(A) = \sum_\alpha \alpha / Count(\overline{A}_\alpha)$$

where Σ stands for the union, *Count* (A_α) denotes the cardinality of the non-fuzzy set A_α and $\overline{A}$ is the complement of the fuzzy set A. Similarly the relative fuzzy cardinalities of two fuzzy sets can be defined as follows:

$$FECount(A/X) = FGCount(A/X) \cap FLCount(A/X)$$

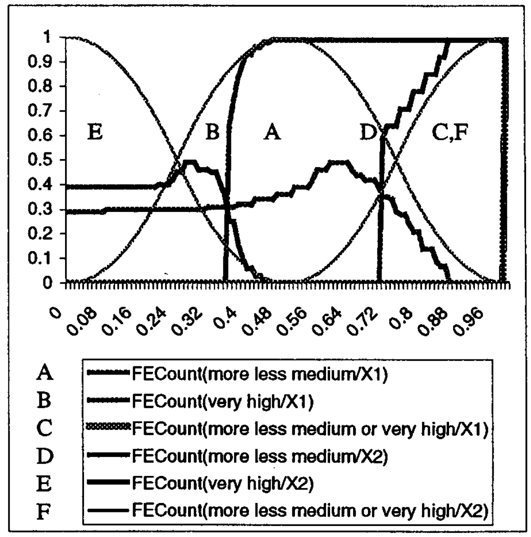
where

$$FGCount(A/X) = \sum_\alpha \alpha / \frac{Count(A_\alpha \cap X_\alpha)}{Count(X_\alpha)}$$

$$FLCount(A/X) = \sum_\alpha \alpha / \frac{Count(\overline{A}_\alpha \cap \overline{X}_\alpha)}{Count(\overline{X}_\alpha)}$$

These cardinalities can be used in linguistic quantification of linguistic propositions following the principles of the quantified re-translation rules as discussed before. The relative fuzzy cardinalities for the example of linguistic approximation of fuzzy sets X_1 and X_2 considered before are illustrated in figure 4. The final linguistic approximation of these sets is based on the assignment of linguistic quantifiers corresponding to the fuzzy cardinality *FECount*. The results confirm that *all* elements of the approximated fuzzy sets satisfy the composite label "*more or less medium or very large*". In addition the components of this label can be described in the terms of linguistic quantifiers for both fuzzy sets as follows:

$LA(X_1) \equiv$ few X_1 are more or less medium;
few X_1 are very large
$LA(X_2) \equiv$ most X_2 are more or less medium;
some/few X_2 are very large



$LA(X_1) \equiv$ few X_1's are more or less medium and few X_1's are very large
$LA(X_2) \equiv$ most X_2's are more or less medium and some/few X_2's are very large

Fig. 4 Relative fuzzy cardinality *FECount* and linguistic approximation with linguistic quantifiers

It seems that the linguistic quantifiers in linguistic approximation can be more meaningful and natural than the numerical quantifiers. However it should be noted that assignment of a linguistic quantifier to the relative sigma count can be considered as another linguistic approximation problem. However it seems that in this case a simple matching provides sufficient approximation of linguistic quantification.

4. CONCLUSIONS

The paper presented some aspects of linguistic approximation in the context of the re-translation rules and proposed an approach to linguistic approximation with the use of the quantification rules. Two methods of the quantification in linguistic approximation were considered with the use of the numerical and linguistic quantifiers based on the concepts of the non-fuzzy and fuzzy cardinalities of fuzzy sets, respectively. Based on the initial results it can be concluded that quantification and in particular linguistic quantification can be useful in linguistic approximation to enhance the interpretability of the generated linguistic labels. In particular it seems to be relevant in the problems where information about the scope of the linguistic labels assigned to the approximated fuzzy set is important. It includes commonsense knowledge representation and reasoning in many applications of fuzzy systems for decision support, decision making, optimization and control.

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