Rule-Based Modeling: Precision and Transparency

M. Setnes, R. Babuška, and H. B. Verbruggen

Abstract— This article is a reaction to recent publications on rule-based modeling using fuzzy set theory and fuzzy logic. The interest in fuzzy systems has recently shifted from the seminal ideas about complexity reduction toward data-driven construction of fuzzy systems. Many algorithms have been introduced that aim at numerical approximation of functions by rules, but pay little attention to the interpretability of the resulting rule base. We show that fuzzy rule-based models acquired from measurements can be both accurate and transparent by using a low number of rules. The rules are generated by product-space clustering and describe the system in terms of the characteristic local behavior of the system in regions identified by the clustering algorithm. The fuzzy transition between rules makes it possible to achieve precision along with a good qualitative description in linguistic terms. The latter is useful for expert evaluation, rule-base maintenance, operator training, control systems design, user interfacing, etc. We demonstrate the approach on a modeling problem from a recently published article.

Index Terms— Accuracy, fuzzy clustering, interpretation, rule-based modeling, transparency.

I. INTRODUCTION

Fuzzy models describe systems by establishing relations between the relevant variables in the form of *if–then* rules. One of the aspects that distinguish fuzzy modeling from black-box techniques like neural nets is that fuzzy models are to a certain degree transparent to interpretation and analysis. Traditionally, a fuzzy model is built by using expert knowledge in the form of linguistic rules. Recently, there is an increasing interest in obtaining fuzzy models from measured data. Different approaches have been proposed for this purpose, like fuzzy relational modeling [1], neural-network training techniques [2], and product-space clustering [3], [4]. However, most of these approaches emphasize the global quantitative accuracy of the resulting model, and little attention is paid to linguistic and qualitative aspects (see [3] and [5] for examples). Solutions to this problem have been sought for fuzzy neural networks [6] and for fuzzy rule-based models in general [7].

The increasing computational possibilities seem to have caused a shift in fuzzy systems away from the seminal ideas about complexity reduction and linguistic interpretation that lead to the introduction of fuzzy systems [8]. In the current literature, fuzzy systems are often labeled as transparent and physically interpretable, while they are actually used as black-box techniques. The aim of this article is to show that automated modeling techniques can be used to obtain not only accurate, but also transparent rule-based models from system measurements. In the next section, we present a method to identify a Takagi–Sugeno (TS) rule-based model [9] by means of product-space clustering. Section III shows how a linguistic model can be obtained from the TS rule-based model. In Section IV, these approaches are applied to a modeling problem from [3], and we show that transparent and simple rule bases can be obtained with high accuracy as well as good semantic properties. Finally, some concluding remarks are given in Section V.

II. THE TS FUZZY MODEL AND IDENTIFICATION BY CLUSTERING

A rule-based model of the TS type [9] is considered. It consists of a set of fuzzy rules, each describing a local input–output relation in a linear form

$$R_i: \textbf{If } x_1 \textbf{ is } A_{i1} \textbf{ and } \cdots \textbf{ and } x_n \textbf{ is } A_{in}$$
$$\textbf{then } \hat{y}_i = \boldsymbol{a}_i \boldsymbol{x} + b_i, \qquad i = 1, 2, \cdots, K. \quad (1)$$

Here R_i is the ith rule, $\boldsymbol{x} = [x_1, \cdots, x_n]^T \in \mathcal{X}$ is the vector of input (antecedent) variables, $A_{i1}, \cdots, A_{in}$ are fuzzy sets defined in the antecedent space, and y_i is the rule output. K denotes the number of rules in the rule base, and the aggregated output of the model $\hat{y} \in \mathcal{Y}$ is calculated by taking the weighted average of the rule consequents

$$\hat{y} = \frac{\sum_{i=1}^{K} \beta_i(\boldsymbol{x}) \hat{y}_i}{\sum_{i=1}^{K} \beta_i(\boldsymbol{x})} \quad (2)$$

where $\beta_i(\boldsymbol{x})$ is the degree of activation of the ith rule

$$\beta_i(\boldsymbol{x}) = \prod_{j=1}^{n} \mu_{A_{ij}}(x_j), \qquad i = 1, 2, \cdots, K \quad (3)$$

and $\mu_{A_{ij}}(x_j)\colon \mathbb{R} \to [0, 1]$ is the membership function of the fuzzy set A_{ij} in the antecedent of R_i.

The construction of a TS fuzzy model from measured data is solved in two steps: 1) structure identification and 2) parameter estimation. In the structure identification step, the antecedent and consequent variables of the model are determined. From the available data sequences, a regression matrix X and an output vector $\boldsymbol{y}$ are constructed

$$X = [\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N]^T, \boldsymbol{y} = [y_1, \cdots, y_N]^T. \quad (4)$$

Here $N \gg n$ is the number of samples used for identification.

In the parameter estimation step, the number of rules K, the antecedent fuzzy sets A_{ij}, and the parameters of the rule consequents $\boldsymbol{a}_i$, b_i for $i = 1, 2, \cdots, K$ are determined. Fuzzy clustering in the Cartesian product-space $\mathcal{X} \times \mathcal{Y}$ is applied to partition the training data into characteristic regions where the systems behavior is approximated by local linear models [10]. The data set Z to be clustered is formed by combining X and $\boldsymbol{y}$

$$Z = [X; \boldsymbol{y}]^T. \quad (5)$$

Given the training data Z and the number of clusters K, the Gustafson–Kessel (GK) clustering algorithm [11] is applied, which computes the fuzzy partition matrix U. Note that the problem definition is the same as in [3], where product-space clustering is also applied. However, the GK clustering algorithm involves an adaptive distance measure, making it more suitable for the identification of characteristic regions in the data than the Learning Vector Quantization clustering approach in [3]. The GK clustering algorithm is given in the Appendix. Each cluster represents a certain operation region of the system, and the number of clusters K equals the number of rules. Methods like cluster validity measures [12] or compatible cluster merging [13] can be applied to find a suitable number of clusters.

The fuzzy sets in the antecedent of the rules are obtained from the partition matrix U, whose ikth element $\mu_{ik} \in [0, 1]$ is the
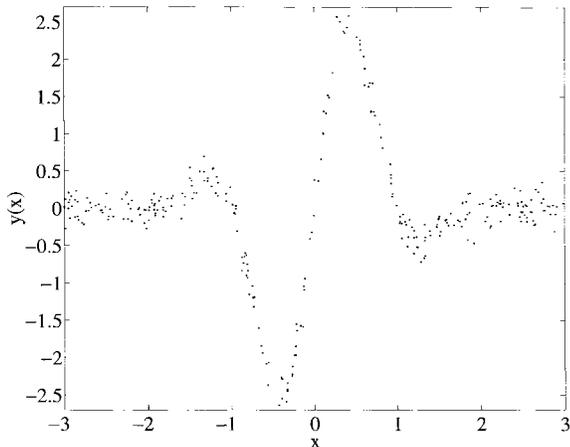
Fig. 1. Identification data contaminated by noise.

(a)

(b)

Fig. 2. (a) Rule antecedents and (b) rule consequents.

membership degree of the data object z_k in cluster i. The ith row of U contains a pointwise definition of a multidimensional fuzzy set. One-dimensional (1-D) fuzzy sets A_{ij} are obtained from the multidimensional fuzzy sets by projections onto the space of the input variables x_j

$$\mu_{A_{ij}}(x_{jk}) = \mathrm{proj}_j(\mu_{ik}) \qquad (6)$$

where proj is the pointwise projection operator [14]. The pointwise defined fuzzy sets A_{ij} are then approximated by suitable parametric functions in order to compute $\mu_{A_{ij}}(x_j)$ for any value of x_j.

The consequent parameters for each rule are obtained as a least-square estimate. Let X_e denote the matrix $[X; \mathbf{1}]$; Γ_i is a diagonal matrix in $\mathbb{R}^{N \times N}$ having the normalized membership degree $\gamma_i(\boldsymbol{x}_k) = \beta_i(\boldsymbol{x}_k) / \sum_{j=1}^{K} \beta_j(\boldsymbol{x}_k)$ as its kth diagonal element. Further, denote X', the matrix in $\mathbb{R}^{N \times KN}$ composed of matrices Γ_i and X_e

$$X' = [(\Gamma_1 X_e); (\Gamma_2 X_e); \cdots; (\Gamma_K X_e)]. \qquad (7)$$

Denote $\boldsymbol{\theta}'$, the vector in $\mathbb{R}^{K(n+1)}$ given by

$$\boldsymbol{\theta}' = \left[\boldsymbol{\theta}_1^T; \boldsymbol{\theta}_2^T; \cdots; \boldsymbol{\theta}_K^T \right]^T \qquad (8)$$

where $\boldsymbol{\theta}_i^T = [\boldsymbol{a}_i^T; b_i]$ for $1 \leq i \leq K$. The resulting least-squares problem $\boldsymbol{y} = X\boldsymbol{\theta}' + \epsilon$, where ϵ is the approximation error, has the solution

$$\boldsymbol{\theta}' = \left[(X')^T X' \right]^{-1} (X')^T \boldsymbol{y}. \qquad (9)$$

From (8), the parameters $\boldsymbol{a}_i$ and b_i are obtained by

$$\boldsymbol{a}_i = [\theta'_{q+1}, \theta'_{q+2}, \cdots, \theta'_{q+n}]^T, \qquad b_i = [\theta_{q+n+1}] \qquad (10)$$

where $q = (i-1)(n+1)$.

III. CONSTRUCTION OF A LINGUISTIC FUZZY MODEL

As we explained in the previous section, a TS model can be derived by fuzzy clustering, such that it approximates piecewise a nonlinear hypersurface by hyperplanes. A piecewise linear model can also be obtained by using a singleton model, a special case of the linguistic fuzzy model of the form

$$R_i: \textbf{If } x_1 \textbf{ is } A_{i1} \textbf{ and } \cdots \textbf{ and } x_n \textbf{ is } A_{in}$$
$$\textbf{then } \hat{y}_i = b_i, \qquad i = 1, 2, \cdots, K. \qquad (11)$$

It is easy to show that to obtain linear interpolation between the constant consequents b_i, the antecedent fuzzy sets must be defined by triangular membership functions that form a partition and the
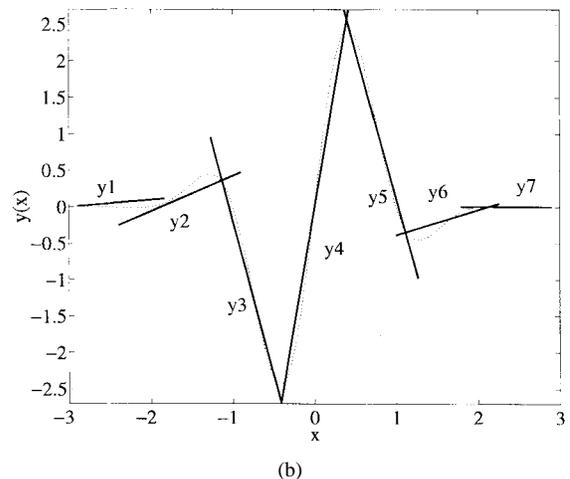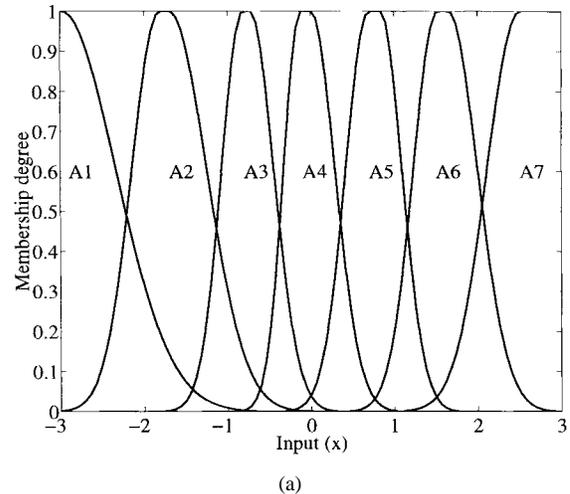
product intersection operator must be used [5]. In order to represent the piecewise linear mapping of the TS model, the cores of the membership functions in the linguistic model are chosen, such that they coincide with the intersection points of the adjacent membership functions in the affine TS model (1). (A core of a fuzzy set is a crisp set, $\mathrm{core}(A) = \{x | \mu_A(x) = 1\}$.) This is because each TS rule by itself results in a locally linear input–output mapping, while in the linguistic model, the linear relation is a consequence of the interpolation between the neighboring rules. Additional sets must be placed at the extreme points of the domain. Consider first a TS fuzzy model (1) with a scalar input x and scalar output $\hat{y}$. Let the fuzzy sets A_i, $i = 1, 2, \cdots, K$ be ordered, such that

$$\sup \mathrm{core}(A_i) < \inf \mathrm{core}(A_{i+1}), \qquad i = 1, 2, \cdots, K - 1. \qquad (12)$$

This condition ensures that the cores of the fuzzy sets A_i are disjunct. Let $\alpha = \{a'_i \,|\, i = 1, \cdots, K + 1\}$ denote a set of intersection points of the adjacent fuzzy sets A_i

$$\alpha = \{\inf \mathcal{X}, \{\mathrm{core}[\mathrm{norm}(A_i \cap A_{i+1})]$$
$$|\, i = 1, \cdots, K - 1\}, \sup \mathcal{X}\} \qquad (13)$$

where normalization of a fuzzy set is defined as $\mu_{\mathrm{norm}(A_i)} = \mu_{A_i}(x) / \sup_x \mu_{A_i}(x)$. Now, triangular membership functions $\mu_{A'_i}$ of the linguistic model can be constructed so that they form a partition,
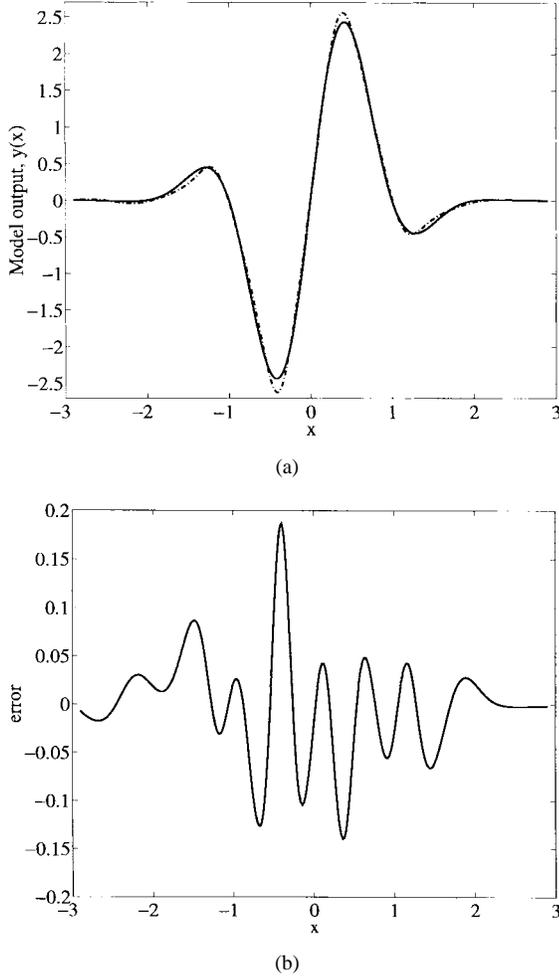
(a)

(b)

Fig. 3.   (a) Function approximation and (b) approximation error.

and their cores are the points a_i'

$$\mu_{A_1'}(x) = \max\left[0, \min\left(1, \frac{a_2' - x}{a_2' - a_1'}\right)\right] \quad (14)$$

$$\mu_{A_i'}(x) = \max\left[0, \min\left(\frac{x - a_{i-1}'}{a_i' - a_{i-1}'}, \frac{a_{i+1}' - x}{a_{i+1}' - a_i'}\right)\right],$$
$$i = 2, \cdots, K \quad (15)$$

$$\mu_{A_{K+1}'}(x) = \max\left[0, \min\left(\frac{x - a_K'}{a_{K+1}' - a_K'}, 1\right)\right]. \quad (16)$$

For the general model (11), the membership functions are derived per antecedent variable x_j, in the same way as above. In order to obtain a complete singleton model, identifying the rule consequents for all combinations of the antecedent fuzzy sets remains. The optimal consequent parameters b_i can be estimated by least-squares techniques. Let the degree of activation $\beta_i(\boldsymbol{x})$ of the ith rule be given by (3), and let the output $\hat{y}_k$ of the model corresponding to the input $\boldsymbol{x}_k$ be computed by

$$y_k = \frac{\sum_{i=1}^{K} \beta_i(\boldsymbol{x}_k)b_i}{\sum_{i=1}^{K} \beta_i(\boldsymbol{x}_k)}. \quad (17)$$

Let Γ denote the matrix in $\mathbb{R}^{N \times K}$ having the normalized degree of fulfillment $\gamma_{ki} = \beta_i(\boldsymbol{x}_k)/\sum_{j=1}^{K} \beta_i(\boldsymbol{x}_k)$ as its kith element; let

TABLE I
RULE-BASED MODEL IDENTIFIED FROM NOISY DATA

Rule	Antecedent	Consequent
$R_1:$	If x is A_1	Then y_1 is $0.0970x + 0.2907$
$R_2:$	If x is A_2	Then y_2 is $0.4854x + 0.9104$
$R_3:$	If x is A_3	Then y_3 is $-4.2325x - 4.4236$
$R_4:$	If x is A_4	Then y_4 is $6.4704x - 0.0268$
$R_5:$	If x is A_5	Then y_5 is $-4.0738x + 4.1996$
$R_6:$	If x is A_6	Then y_6 is $0.3450x - 0.7316$
$R_7:$	If x is A_7	Then y_7 is $-0.0029x + 0.0100$

$\boldsymbol{y}$ denote the vector in $\mathbb{R}^N$ having y_k as its kth component; and let $\boldsymbol{b} = [b_1, b_2, \cdots, b_K]^T$ denote the vector containing the consequent parameters. The least-squares problem given by (17), written in a matrix form $\boldsymbol{y} = \Gamma\boldsymbol{b} + \epsilon$, where ϵ is the approximation error and has the solution

$$\boldsymbol{b} = \left[\Gamma^T\Gamma\right]^{-1}\Gamma^T\boldsymbol{y}. \quad (18)$$

IV. EXAMPLE

A. TS Fuzzy Model with Linear Consequents

Using the TS-Fuzzy model structure, we apply the approach described in Section II to construct a fuzzy rule-based model of a system presented in [3]. The reader is encouraged to compare the results in this section with those in [3]. Consider a univariate function

$$y(x) = 3e^{-x^2}\sin(\pi x) + \eta \quad (19)$$

where η is Gaussian noise with zero mean and $\sigma^2 = 0.15$. By using random inputs x uniformly distributed in $[-3, 3]$, 300 samples of $y(x)$ were obtained from (19) (see Fig. 1). This gives the identification data $Z = \{(x_k, y_k)|k = 1, 2, \cdots, 300\}$. The data Z are clustered by the GK clustering algorithm, and $K = 7$ clusters are selected by means of the average within cluster distance validity measure [15]. The resulting TS fuzzy model consists of seven rules with linear consequent parts. The rules are given in Table I, and the fuzzy sets in the antecedent of the rules and the local linear models in the consequents are shown in Fig. 2. We now compare this model obtained from noisy data with the noise-free function (19), i.e., $\eta = 0$. Fig. 3 shows the function (19) and the approximation by the model of Table I. Considering 300 points x equally spaced in $[-3, 3]$, the model gives a mean-squared error of 0.0028 with a maximum error of 0.1868. The rule-based model of Table I performs better than all the models derived in [3], has fewer rules (seven compared to 30 in [3]), and is identified from noise contaminated data, while the identification data used in [3] were noise free. Our identification approach is based on product-space clustering, as was also the case in [3]. Also, the reasoning is identical with the mentioned article. The main difference between our approach and the one in [3] is that we use rules in which the consequents are linear functions. Further, the adaptive distance clustering algorithm can recognize clusters of various shapes and hence, can approximate functions more effectively.

B. Linguistic Model

The linguistic fuzzy model consists of eight rules with singleton consequents. The rules are given in Table II. The antecedent fuzzy sets obtained by (14)–(16) and the approximation of the noise-free function are shown in Fig. 4. Considering 300 equally spaced points
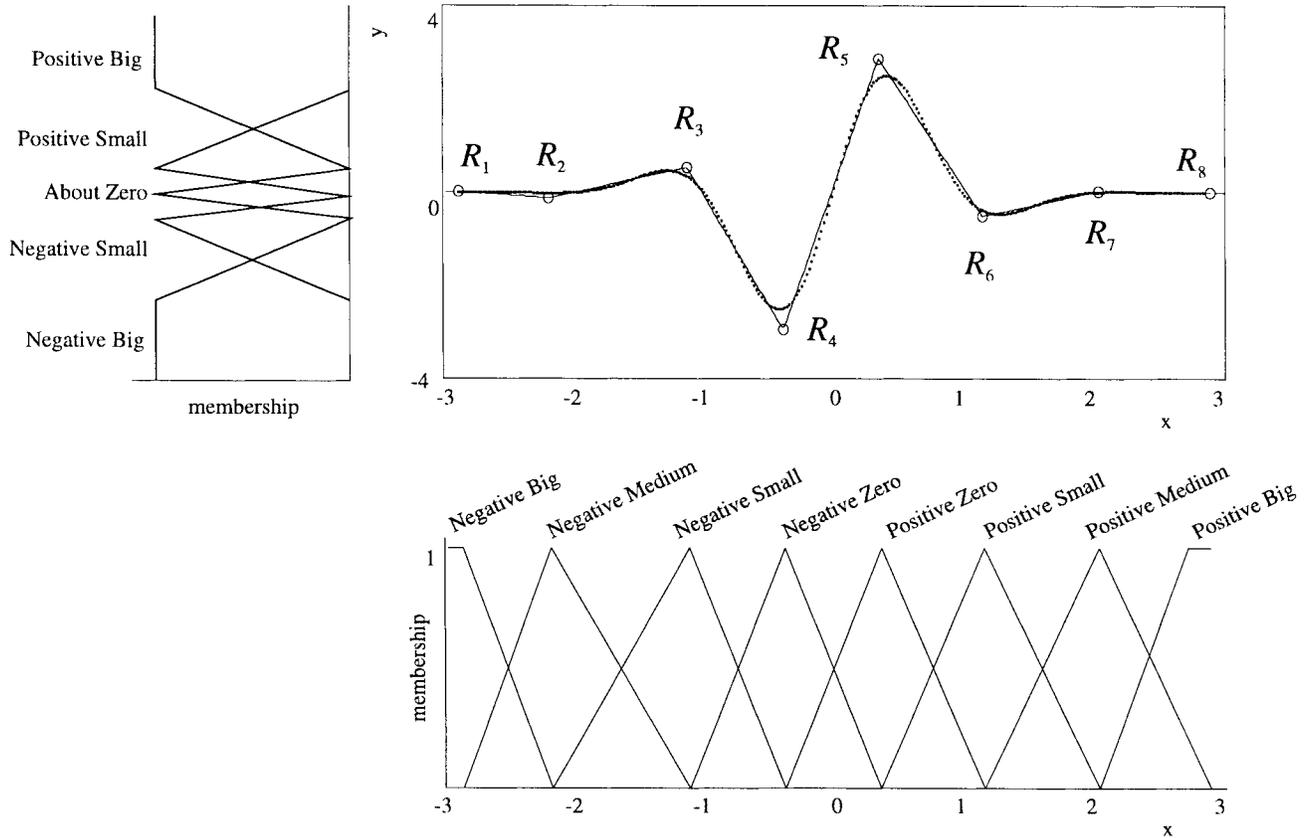
Fig. 4.   Antecedent membership functions and model output compared with the noise-free function. The circles denote the consequent singletons.

TABLE II
SINGLETON MODEL IDENTIFIED FROM NOISY DATA

Rule	Antecedent	Consequent singleton	Consequent label
R_1 :	If x is Negative Big	Then y_1 is 0.0328	About Zero
R_2 :	If x is Negative Medium	Then y_2 is -0.1202	About Zero
R_3 :	If x is Negative Small	Then y_3 is 0.5179	Positive Small
R_4 :	If x is Negative Zero	Then y_4 is -2.8572	Negative Big
R_5 :	If x is Positive Zero	Then y_5 is 2.7919	Positive Big
R_6 :	If x is Positive Small	Then y_6 is -0.4918	Negative Small
R_7 :	If x is Positive Medium	Then y_7 is 0.0300	About Zero
R_8 :	If x is Positive Big	Then y_7 is -0.0055	About Zero

$x \in [-3, 3]$, the singleton model gives a mean-squared error of 0.0096 with a maximum error of 0.3980. The accuracy of this model is lower than that of the TS model, but it is still comparable with the results presented in [3]. Notice, however, that our model consists of eight rules compared to 30 in the article mentioned. Moreover, our model was identified from data contaminated by noise, while the identification data used in [3] were noise free.

The singleton model presented in Table II can easily be interpreted linguistically. The numerical singletons can be grouped around some characteristic values, and they can be assigned linguistic terms. In our example, we obtain the terms Negative Big, Negative Small, About Zero, Positive Small, and Positive Big. One can see that the linguistic model describes the underlying function very well, giving an idea about the oscillations between small and large negative and positive output. Fuzzy sets defining the linguistic terms in the rule consequent are shown in Fig. 4.

V. CONCLUDING REMARKS

We have presented a method for constructing fuzzy rule-based models from system's measurements, which provides high accuracy as well as transparency and low complexity of the resulting rule base. The approach has been demonstrated on a modeling problem from the literature to give the reader a possibility to compare the results with those of fuzzy black-box modeling. It was our intention to show that construction of rule-based models from data can result in transparent fuzzy models suitable for linguistic interpretation. Such models are more in line with the paradigms of fuzzy systems. They enable an easy validation by experts and the possibility to insert additional rules based on the experience of experts, typically in regions that have not been covered by the measurements.

The modeling methodology described in this article has been successfully applied to many real-world problems in diverse fields, like ecology [16], biotechnology [17], finance [18], and process control [19]. It is our experience that when dealing with practical applications, the transparency of the models is of high importance. Fuzzy models have proven to be very suitable in providing such transparency for interpretation and analysis.

APPENDIX

The GK clustering algorithm [11]:

Given Z, choose $1 < K < N$, $m > 1$ and $\epsilon > 0$. Initialize $U^{(0)}$ (e.g., at random).

Repeat for $l = 1, 2, \cdots$.

Step 1) **Compute Cluster Means**

$$\boldsymbol{v}_i^{(l)} = \frac{\sum\limits_{k=1}^{N}[\mu_{ik}^{(l-1)}]^m \boldsymbol{z}_k}{\sum\limits_{k=1}^{N}[\mu_{ik}^{(l-1)}]^m}, \qquad i = 1, 2, \cdots, K.$$

Step 2) **Compute Covariance Matrices**

$$F_i = \frac{\sum\limits_{k=1}^{N}[\mu_{ik}^{(l-1)}]^m [\boldsymbol{z}_k - \boldsymbol{v}_i^{(l)}][\boldsymbol{z}_k - \boldsymbol{v}_i^{(l)}]^T}{\sum\limits_{k=1}^{N}[\mu_{ik}^{(l-1)}]^m},$$

$$i = 1, 2, \cdots, K.$$

Step 3) **Compute Distances**

$$D_{ik}^2 = \left[\boldsymbol{z}_k - \boldsymbol{v}_i^{(l)}\right]^T \left\{\left[\det(F_i)^{1/(n+1)} F_i^{-1}\right]\right\} \left[\boldsymbol{z}_k - \boldsymbol{v}_i^{(l)}\right],$$

$$i = 1, 2, \cdots, K, \quad k = 1, 2, \cdots, N.$$

Step 4) **Update Partition Matrix**

If $D_{ik} > 0$ for $1 \le i \le K, 1 \le k \le N$

$$\mu_{ik}^{(l)} = \frac{1}{\sum\limits_{j=1}^{K}(D_{ik}/D_{jk})^{2/(m-1)}}$$

otherwise

$$\mu_{ik}^{(l)} = 0 \text{ if } D_{ik} > 0, \text{ and } \mu_{ik}^{(l)} \in [0, 1] \text{ with } \sum_{i=1}^{K} \mu_{ik}^{(l)} = 1$$

until $\|U^{(l)} - U^{(l-1)}\| < \epsilon$.

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