Construction of Ignorance Functions from Overlap Functions.

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Abstract

In this work we present some general results on ignorance functions. In particular we focus on the relation between overlap functions and ignorance functions, with a special stress on how the latter can be constructed from the former.

Keywords: Overlap Function, Ignorance Function.

1 Introduction

In real life there are many situations on which the lack of knowledge of an expert plays an important role ([4, 5]). For instance, if an expert is asked to classify the pixels of a given image in those belonging to an object and those belonging to the background, for some of them he or she will not know clearly how to make this classification. The bigger this lack of certainty is, the worse the results provided by the expert will be. So it would be useful to have a way of measuring this lack of knowledge. If we require the expert to give us a fuzzy membership function representing an image, for pixels with very high membership value, the expert is quite sure that they belong either to the background or to the object in that image (depending on how the function is supposed to represent the image). So the membership value can be understood as a number to measure how certain the expert is that a given pixel belongs to the object, and how certain he or she is that the same pixel belongs to the background, then we can define a binary function, that we are going to call *ignorance function* measuring his or her "uncertainty". Or, in other words, ignorance functions give us a value for the lack of knowledge the expert suffers when dealing with a particular problem.

The structure of this work is the following. In Section 2 we start recalling some very well-known concepts. In Section 3 we introduce the concept of ignorance function. We relate ignorance functions to overlap functions in Section 4. Section 5 is devoted to a method to construct interval-valued fuzzy sets using ignorance functions in such a way that the information these functions contain is reflected in the interval-valued membership. Finally, we end with some conclusions and comments.

2 Preliminary results

An automorphism of the unit interval is a bijective and continuous mapping $\varphi : [0,1] \to [0,1]$ such that $\varphi(0) = 0$ and $\varphi(1) = 1$. A strict negation is a continuous bijection $n : [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$. If n is a strict negation such that $n(n(x)) = x$ for all $x \in [0,1]$, we will call n a strong negation. Let us also remind that a binary aggregation function ([1]) is an increasing mapping $M : [0,1]^2 \to [0,1]$ such that $M(0,0) = 0$ and $M(1,1) = 1$.

A fuzzy set Q over a finite, non-empty referential set X will be determined by a membership function $\mu_Q : X \to [0,1]$. That is, for us, a fuzzy set will be an expression of the type

$$Q = \{(x, \mu_Q(x) \mid x \in [0,1]\}$$

Let's define $L([0,1])$ as the set of all closed subintervals of the closed unit interval $[0,1]$. An interval-valued fuzzy set (IVFS) over the referential X is an expression of the type:

$$A = \{(x, \mathbf{A}(x) \mid x \in X\}$$

where $\mathbf{A} : X \to L([0,1])$. From now on, we will use bold letters to denote intervals. Moreover, if $\mathbf{x} \in L([0,1])$, then we will write $\mathbf{x} = [x_L, x_U]$, where $x_L \in [0,1]$ denotes the lower bound of the interval $\mathbf{x}$ and $x_U \in [0,1]$, the upper bound. Clearly, we will always have $x_L \leq x_U$.

A concept that will very important for us in the subsequent developments is that of overlap function ([2]).

Definition 1 An overlap function is a mapping $G_O : [0,1]^2 \to [0,1]$ such that:

(G_O1) $G_O(x,y) = G_O(y,x)$ for all $x, y \in [0,1]$;

(G_O2) $G_O(x,y) = 0$ if and only if $x = 0$ or $y = 0$;

(G_O3) $G_O(x,y) = 1$ if and only if $x = y = 1$;

(G_O4) G_O is non decreasing.

(G_O5) G_O is continuous.

3 Definition and main properties of ignorance functions

We start by presenting the definition of ignorance function, which can be found in [3]

Definition 2 An ignorance function is a continuous, commutative mapping $G_i : [0,1]^2 \to [0,1]$ such that

(G_i1) $G_i(x,y) = 0$ if and only if $x = 1$ or $y = 1$;

(G_i2) $G_i(x,y) = 1$ if $x = y = 0.5$;

(G_i3) G_i is decreasing in $[0.5,1] \times [0.5,1]$;

(G_i4) G_i is increasing in $[0,0.5] \times [0,0.5]$

Observe that our definition implies that we have assume that a value of 0.5 corresponds to complete lack of knowledge of the expert.

Example The mapping

$$G_i(x,y) = \begin{cases} 2\min(1-x, 1-y) \text{ if } x, y \geq \frac{1}{2} \\ \frac{1}{2\min(1-x,1-y)} \text{ in other case.} \end{cases}$$

provides an example of ignorance function.

We will denote by $\mathcal{I}g$ the set of ignorance functions. As first remark, notice that, if we take $G_1, G_2 \in \mathcal{I}g$, then the mappings

$$G_{1,2}(x,y) = \min(G_1(x,y), G_2(x,y))$$

and

$$G^{1,2}(x,y) = \max(G_1(x,y), G_2(x,y))$$

are also ignorance functions. Moreover, if, given $G_1, G_2 \in \mathcal{I}g$ we denote by $\leq_i$ the ordering defined by $G_1 \leq_i G_2$ if and only if $G_1(x,y) \leq G_2(x,y)$ for all $x, y \in [0,1]$, the supremum of $\mathcal{I}g$ is given by

$$G_{\sup}(x,y) = \begin{cases} 0 \text{ if } \max(x,y) = 1; \\ 1 \text{ in other case.} \end{cases}$$

whereas the infimum is given by

$$G_{\inf}(x,y) = \begin{cases} 1 \text{ if } x = y = 0.5; \\ 0 \text{ in other case.} \end{cases}$$

Observe that none of both mappings are overlap functions. All these considerations together lead us to the following structural result.

Proposition 1 The space $(\mathcal{I}g, \leq_i)$ is a partial ordered, non-complete lattice which has not either supremum or infimum.

Ignorance functions can be characterized in a first approach as follows.

Theorem 1 A mapping $G_i : [0,1] \times [0,1] \to [0,1]$ with $G_i(0,0) < 1$ is an ignorance function with $G_i(0,0) < G_i(x,y)$ for all $x, y \in (0,0.5]$ if and only if, for all $x, y \in [0,1]$, the mappings

$$M_1(x,y) = G_i(1-\frac{x}{2}, 1-\frac{y}{2}) \text{ with } x, y \in [0,1] \ ;$$

and

$$M_2(x,y) = \frac{1}{1-G_i(0,0)}(G_i(\frac{x}{2},\frac{y}{2}) - G_i(0,0))$$

are continuous and symmetric aggregation functions such that $M_1(x,0) = M_1(0,x) = 0$ for all $x \in [0,1]$ and $M_2(x,y) \neq 0$ if $\max(x,y) > 0$.

Proof. If G_i is an ignorance function in the conditions of the theorem, then M_1 and M_2 are clearly aggregation functions as stated. Conversely, suppose that M_1 and M_2 are aggregation functions in the conditions stated in the theorem. Then we have that

$$G_i(x,y) = M_1(2(1-x), 2(1-y))$$

for all $x, y \geq 0.5$, whereas

$$G_i(x,y) = (1-G_i(0,0))M_2(2x,2y) + G_i(0,0)$$

for all $x, y \leq 0.5$. This two mappings can easily be extended to symmetric and continuous mappings in the whole unit square. Hence the result is proved. □

4 Relation between ignorance functions and overlap functions

There is a very close relation between ignorance functions and overlap functions, as the next results shows.

Theorem 2 Let $G_0 : [0,1]^2 \to [0,1]$ be an overlap function. Then, the mapping $G_i : [0,1]^2 \to [0,1]$ given by

$$G_i(x,y) = \frac{G_0(1-x,1-y)}{G_0(0.5,0.5)}$$

if $G_0(1-x,1-y) \leq G_0(0.5,0.5)$, and

$$G_i(x,y) = \frac{G_0(0.5,0.5)}{G_0(1-x,1-y)}$$

otherwise, is an ignorance function.

Proof. Symmetry of G_i is clear. If $G_i(x,y) = 0$, then there are two possibilities:

a) If $G_0(1-x,1-y) \leq G_0(0.5,0.5)$, then $G_0(1-x,1-y) = 0$, which means by definition of overlap function that $(1-x)(1-y) = 0$, that is, $x = 1$ or $y = 1$.

b) If $G_0(1-x,1-y) > G_0(0.5,0.5)$, then $G_0(0.5,0.5) = 0$, which contradicts the definition of overlap function

On the other hand, if $x = 1$ or $y = 1$, then $(1-x)(1-y) = 0$, so $G_0(1-x,1-y) = 0 \leq G_0(0.5,0.5)$ and $G_i(x,y) = 0$.

The fact that $G_i(0.5,0.5)$ is evident from the definition.

Let $x_1, x_2, y \in [0.5,1]$ such that $0.5 \leq x_1 \leq x_2$ and $0.5 \leq y$. Therefore $0.5 \geq 1-x_1 \geq 1-x_2$ and $0.5 \geq 1-y$. Then $G_0(1-x_1,1-y) \leq G_0(0.5,0.5)$ and $G_0(1-x_2,1-y) \leq G_0(0.5,0.5)$. Therefore $G_i(x_1,y) = \frac{G_0(1-x_1,1-y)}{G_0(0.5,0.5)} \geq \frac{G_0(1-x_2,1-y)}{G_0(0.5,0.5)} = G_i(x_2,y)$.

In the same way, let $x_1, x_2, y \in [0,0.5]$ such that $x_1 \leq x_2 \leq 0.5$ and $y \leq 0.5$. Then $1-x_1 \geq 1-x_2 \geq 0.5$ and $1-y \geq 0.5$, therefore $G_0(1-x_1,1-y) \geq G_0(0.5,0.5)$ and $G_0(1-x_2,1-y) \geq G_0(0.5,0.5)$. Then $G_i(x_1,y) = \frac{G_0(0.5,0.5)}{G_0(1-x_1,1-y)} \leq \frac{G_0(0.5,0.5)}{G_0(1-x_2,1-y)} = G_i(x_2,y)$.

Finally, as by definition every overlap function is continuous, so is G_i □

We can get stronger result.

Theorem 3 Let n be a strong negation and G_i an ignorance function such that $G_i(x,y) = 1$ if and only if $x = y = 0.5$. Then the mapping

$$G_O(x,y) = G_i(n(n(0.5)x), n(n(0.5)y))$$

is an overlap function. And reciprocally, let G_O be an overlap function and $n : [0,1] \to [0,1]$ a strong negation. Then the mapping

$$G_i(x,y) = \begin{cases} \frac{G_O(n(x),n(y))}{G_O(n(0.5),n(0.5)} & \text{if } \frac{G_O(n(x),n(y))}{G_O(n(0.5),n(0.5)} \leq 1 \\ \frac{G_O(n(0.5),n(0.5))}{G_O(n(x),n(y)} & \text{in other case,} \end{cases}$$

is an ignorance function.

Proof. It is just an easy comprobation, bearing in mind the definitions of ignorance functions, overlap functions and strong negations □

Moreover, the connection between overlap functions and ignorance functions allows us to go a bit further.

Proposition 2 Let G_O be an overlap functions such that for all $x, y, \alpha \in [0,1]$ the identity $G_0(\alpha \cdot x, y) = G_0(x, \alpha \cdot y)$. Let G_i be the ignorance function given by Theorem 3, with $N(x) = 1 - x$. Then

$$G_i(1-\alpha \cdot x, 1-y) = G_i(1-x, 1-\alpha \cdot y)$$

for all $x, y, \alpha \in [0,1]$.

Proof. It is a straightforward calculation $\square$

Proposition 3 Let $g : [0,1] \to [0,1]$ be a continuous, increasing mapping such that $g(x) = 0$ if and only if $x = 0$ and $g(x) = 1$ if and only if $x = 1$. Then, the mapping $G_i(x, y)$ defined as

$$\frac{g((1-x)\cdot(1-y))}{g(0.5\cdot 0.5)}$$

if $g((1-x)\cdot(1-y)) \leq g(0.5 \cdot 0.5)$, and

$$\frac{g(0.5\cdot 0.5)}{g((1-x)\cdot(1-y))}$$

otherwise is an ignorance function such that $G_i(1-\alpha \cdot x, 1-y) = G_i(1-x, 1-\alpha \cdot y)$ for all $x, y, \alpha \in [0,1]$ and $G_i(0,0) = g(0.5 \cdot 0.5)$.

Proof. (G_i1) As the product is symmetric, so is G_i. (G_i2) $G_i(x,y) = 0$ if and only if $g((1-x)\cdot(1-y)) = 0$ if and only if $(1-x)\cdot(1-y) = 0$ if and only if $x = 1$ or $y = 1$. (G_i3) Direct. (G_i4) If $x_1, x_2, y \geq 0.5$ and $0.5 \leq x_1 \leq x_2$, then $0.5 \geq 1 - x_1 \geq 1 - x_2$. Under these conditions $G_i(x_1, y) = \frac{g((1-x_1)\cdot(1-y))}{g(0.5\cdot 0.5)} \geq \frac{g((1-x_2)\cdot(1-y))}{g(0.5\cdot 0.5)} = G_i(x_2, y)$. (G_i5) Similar to that for (G_i4). G_i is continuous from the continuity of g. As the mapping $G_0(x,y) = g(x \cdot y)$ is an overlap function ([]), we are in the conditions of Proposition 2, and we have $G_i(1-\alpha \cdot x, 1-y) = G_i(1-x, 1-\alpha \cdot y)$. Finally $G_i(0,0) = \frac{g(0.5\cdot 0.5)}{g(1)} = g(0.5 \cdot 0.5)$. $\square$

Example If we take the following function

$$g(x) = \begin{cases} x \text{ if } x \leq 0.2 \\ 0.2 \text{ if } 0.2 \leq x \leq 0.8 \\ 4x - 3 \text{ if } x \geq 0.8 \end{cases}$$

we have that

$$G_i(x,y) = \begin{cases} \frac{(1-x)\cdot(1-y)}{0.20} \text{ if } (1-x)(1-y) \leq 0.20 \\ 1 \text{ if } 0.20 \leq (1-x)\cdot(1-y) \leq 0.8 \\ \frac{0.20}{4((1-x)\cdot(1-y))-3} \text{ otherwise,} \end{cases}$$

is an ignorance function.

Proposition 4 Let φ be an automorphism of the unit interval. Then

$$G_i(x,y) = \begin{cases} \frac{\varphi((1-x)\cdot(1-y))}{\varphi(0.25)} \text{ if } (1-x)\cdot(1-y) \leq 0.25 \\ \frac{\varphi(0.25)}{\varphi((1-x)\cdot(1-y))} \text{ otherwise,} \end{cases}$$

is an ignorance function such that $G_i(1-\alpha \cdot x, 1-y) = G_i(1-x, 1-\alpha \cdot y)$ for all $\alpha \in [0,1]$ and $G_i(0,0) = \varphi(0.25)$.

Proof. Similar to Proposition 3 $\square$

Example 1) If we take $\varphi(x) = x$ for all $x \in [0,1]$ we have mapping $G_I(x,y)$ defined as

$$G_i(x,y) = 4(1-x)\cdot(1-y)$$

if $(1-x)\cdot(1-y) \leq 0.25$, and

$$G_i(x,y) = \frac{1}{4((1-x)\cdot(1-y))}$$

otherwise, is an ignorance function.

2) If $\varphi(x) = \sqrt{x}$, then the mapping

$$G_i(x,y) = 2\sqrt{(1-x)\cdot(1-y)}$$

if $(1-x)\cdot(1-y) \leq 0.25$, and

$$G_i(x,y) = \frac{1}{2\sqrt{(1-x)\cdot(1-y)}}$$

otherwise, is an ignorance function.

5 Ignorance functions and interval valued fuzzy sets

Given a fuzzy set, ignorance functions can be used as a mean to get an interval-valued fuzzy set such that the amplitude of the membership interval for a given point represents the lack of knowledge that the expert suffers when determining the fuzzy set. so, the greater the amplitude of the interval-valued membership, the greater the lack of knowledge of the expert. This process is further developed in the following result.

Theorem 3 Let μ_Q be the membership function of a given fuzzy set Q, and let G_i be an ignorance function, as given by Definition 2. Then the interval-valued fuzzy set defined by the interval-valued fuzzy membership function $\mathbf{A}$ given by

$$\mathbf{A}(x) = [\mu_Q(x)(1 - G_i(\mu_Q(x), 1-\mu_Q(x))), \\ \mu_Q(x) + G_i(\mu_Q(x), 1-\mu_Q(x))(1-\mu_Q(x))]$$

for all $x \in X$, is such that, for any $x \in X$

$$W(\mathbf{A}(x)) = G_i(\mu_Q(x), 1 - \mu_Q(x)),$$

where, for an interval $\mathbf{x} = [x_L, x_U] \in L([0,1])$, we have that $W(\mathbf{x}) = x_U - x_L$.

Proof First of all, observe that $\mathbf{A}$ defines a valid interval-valued membership, as it is clear just by comparing both bounds of the interval. The assertion on the amplitude of the membership interval follows from a straight calculation □

We have now the following result.

Theorem 4 In the setting of the previous theorem:

(i) The membership interval $\mathbf{A}(x)$ reduces to single point for a given $x \in X$ if and only if $G_i(\mu_Q(x), 1 - \mu_Q(x)) = 0$;

(ii) The membership interval $\mathbf{A}(x)$ is equal to the whole unit interval $[0,1]$ if and only if $G_i(\mu_Q(x), 1 - \mu_Q(x)) = 1$.

Proof Both items are just straightforward calculations.

Corollary 1 In the setting of the previous theorem

(i) The membership interval $\mathbf{A}(x)$ reduces to single point for a given $x \in X$ if and only if $\mu_Q(x) = 0$ or $\mu_Q(x) = 1$;

(ii) The membership interval $\mathbf{A}(x)$ is equal to the whole unit interval $[0,1]$ if $mu_Q(x) = 0.5$.

Proof It follows from the definition of ignorance functions and Theorem 3 □

So the interval-valued fuzzy representation we have constructed provides a mean to take into account the lack of knowledge of the expert when determining the fuzzy set membership we are considering. In particular, that measure is given by the amplitude of the different interval-valued memberships we obtain. Moreover, this measure is a local one, as it depends only on the membership value assigned to a point, regardless what are the memberships for the other points in the set we are considering.

6 Concluding remarks

In this work we have presented the concept of ignorance function as a tool to handle the lack of knowledge of an expert when dealing with a particular problem. We have related these ignorance functions with the concept of overlap function, showing how we can get ones from the others. Finally we have introduced a method to construct interval-valued fuzzy sets such that the corresponding membership functions contain significant information about the lack of knowledge of the expert.

We think that the concept of ignorance function and the related subjects considered in this paper open a wide range of possibilities. In particular, we hope to use them in a nearby future to improve some well-known image processing methods. We also expect to find applications for them in fields such as decision making, where lack of knowledge can play an important role. Furthermore, the theoretical development of the concept of ignorance function is still quite short, and we intend to push it further to develop possible connections with other widely used concepts of the fuzzy sets theory, starting from aggregation functions.

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References

[1] G. Beliakov, A. Pradera, T. Calvo(Eds.). *Aggregation Functions: A Guide for Practitioners* ,Springer, Berlin, 2007.

[2] H. Bustince, J. Fernandez, R. Mesiar, J. Montero, Overlap functions, submitted.

[3] H. Bustince, M. Pagola, E. Barrenechea, J. Fernandez, P. Melo-Pinto, P. Couto, H.R. Tizhoosh, J. Montero. An application to the calculation of the threshold in prostate ultrasound images, in *Fuzzy Sets and Systems*,In Press, Available online 5 April 2009.

[4] D. Dubois. On degrees of truth, partial ignorance and contradiction, in *Proceedings of Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU*, L. Magdalena, M. Ojeda-Aciago, J.L. Verdegay (eds), Málaga (Spain), 31–38, 2008.

[5] F. Herrera, L. Martinez, P. J. Sanchez, Managing non-homogeneous information in group decision making, in *European Journal of Operational Research*, 166 (1), 115–132, 2005.