The behavioral meaning of the median

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Summary. We generalize the notion of statistical preference to the theory of imprecise probabilities, by proposing an alternative notion of desirability of a gamble. As a natural consequence, we derive a general definition of median, providing it with a behavioral meaning. Furthermore, we show that, when we restrict to absolutely continuous probability distributions, a random variable is statistically preferred to another one if and only if the the median of their difference is positive.

Key words: Stochastic orderings, Statistical preference, Imprecise Probabilities, Desirable Gambles, Median

1 Introduction

Several preference relations between random variables have been proposed in the literature. One of them, called *statistical preference* [2, 3] is based on the probabilistic relation $Q(X,Y) = P(X > Y) + \frac{1}{2}P(X = Y)$ and it states that X is preferred to Y when $Q(X,Y) \geq 0.5$. Independently, a similar criterion has been proposed in [4, 5] in the framework of possibility theory. In this paper, we aim to extend the notion of statistical preference to the general theory of imprecise probabilities, relating it to the notions of *desirability* and *preference* between gambles. The problem of reconciling two different ways of treating preference relations will arise. In fact, a preference relation for pairs of variables (or *gambles*) can be understood in two different ways:

- The expert initial information is assessed by means of a preference criterion and, afterwards, a set of joint feasible *linear previsions*[3] is derived from it. This is the approach followed in the general theory of imprecise probabilities (see [1, 6]).
- A joint probability is assumed for any pair of gambles on the universe, and a preference relation is derived from it. This is the approach considered in [4, 5, 2, 3], for instance.

[3]The notion of linear prevision generalizes the notion of probability.

Taking into account the above duality, we will first start from an initial set of desirable gambles and we will say that a gamble X is signed-preferred to another gamble Y when the sign of their difference is a desirable gamble. Afterwards, we will show that signed-almost-preference becomes into statistical preference, when the initial set of desirable gambles induces a singleton as a credal set. In a second approach, we will state signed-preference and signed-desirability as primary concepts, appealing to a new idea of desirability: X will be said to be desirable when we have stronger beliefs about $X > 0$ than about $X < 0$. In words, we accept the gamble X, because we have stronger beliefs on making money than on loosing it –no matter how much money–. Based on this desirability definition, we can define a lower prevision as the supremum of the constants c satisfying that $X - c$ is desirable, according to the new definition. Such supremum makes sense as a threshold for buying prices: for any strictly lower price, you have stronger beliefs on earning money that on loosing it. Analogously, we will define an upper prevision as an infimum threshold for selling prices. Once introduced both approaches (the interpretation of sign-desirability as a secondary an as a primary concept), we will relate them, and we will derive an interesting conclusion: the pair of lower and upper previsions defined for the set of signed-desirable gambles generalizes the notion of median, providing it with a meaningful behavioral interpretation. As a consequence of that, we will be able to show that there exists a very strong connection between the relation of statistical preference of two random variables and the sign of the median of their difference. This result adds another piece to the puzzle about the relationships between different stochastic orderings proposed in the literature.

2 Sets of desirable gambles and partial preference orderings

Let Ω denote the set of outcomes of an experiment. A *gamble*, X, on Ω is a bounded mapping from Ω to $\mathbb{R}$ (the real line). If you were to accept gamble X and ω turned to be true, then you would gain $X(\omega)$. (This reward can be negative, and then it will represent a loss.) Let $\mathcal{L}$ denote the set of all gambles (bounded mappings from Ω to $\mathbb{R}$). A subset $\mathcal{D}$ of $\mathcal{L}$ is said to be a *coherent set of desirable gambles* [6] when it satisfies the following four axioms:

D1. If $X \leq 0$ then $X \notin \mathcal{D}$, *(Avoiding partial loss)*
D2. If $X \in \mathcal{L}$, $X \geq 0$ and $X \neq 0$, then $X \in \mathcal{D}$. *(Accepting partial gain)*
D3. If $X \in \mathcal{D}$ and $c \in \mathbb{R}^+$, then $cX \in \mathcal{D}$. *(Positive homogeneity)*
D4. If $X \in \mathcal{D}$ and $Y \in \mathcal{D}$ then $X + Y \in \mathcal{D}$. *(Addition)*

For a detailed justification of each of the above axioms concerning coherence in assessments of a subject, we refer the reader to (cf.[6], Section 2.2.4).

The *lower prevision induced by a set of desirable gambles* $\mathcal{D}$ is the set function $\underline{P} : \mathcal{L} \to \mathbb{R}$ defined as follows:

$$\underline{P}(X) = \sup\{c : X - c \in \mathcal{D}\}.$$

It is interpreted as your supremum acceptable buying price for X, so you are disposed to pay $\underline{P}(X) - \epsilon$, for the reward determined by the gamble X, for any $\epsilon > 0$. The *upper prevision induced by* $\mathcal{D}$ is the set function $\overline{P} : \mathcal{L} \to \mathbb{R}$ defined as follows:

$$\overline{P}(X) = \inf\{c : c - X \in \mathcal{D}\}.$$

It can be regarded as an infimum selling price for the gamble X.

The set of linear previsions[4] induced by a coherent set of gambles $\mathcal{D}$ is defined as:

$$\mathcal{P}_{\mathcal{D}} = \{P : P(X) \geq 0 \text{ for all } X \in \mathcal{D}\}.$$

$\mathcal{P}_{\mathcal{D}}$ is always a *credal set* (a closed and convex set of finitely additive probability measures). $\underline{P}$ and $\overline{P}$ are dual and they respectively coincide with the infimum and the supremum of $\mathcal{P}_{\mathcal{D}}$. There is not a one-to correspondence between sets of desirable gambles and credal sets, as there can be two different sets of desirable gambles $\mathcal{D} \neq \mathcal{D}'$ inducing the same class of linear previsions $\mathcal{P}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}'}$. On the other hand, a subset $\mathcal{D}^- \subset \mathcal{L}$ satisfying axioms D2–D4 and

D1'. If $\sup X < 0$ then $X \notin \mathcal{D}^-$. *(Avoiding sure loss)*
D5. If $X + \delta \in \mathcal{D}^-$, for all $\delta > 0$ then $X \in \mathcal{D}^-$. *(Closure)*

is called a coherent set of *almost desirable gambles.* (Let the reader notice that axiom D1' is weaker than D1.) A set of almost desirable gambles $\mathcal{D}^-$ determines a pair of lower and upper previsions, and a credal set, by means of expressions analogous to the case of desirable gambles. Conversely, a credal set univocally determines a coherent set of almost desirable gambles via the formula:

$$\mathcal{D}^-_{\mathcal{P}} = \{X \in \mathcal{L} : P(X) \geq 0,\ \forall\, P \in \mathcal{P}\}.$$

Finally, a set $\mathcal{D}^+ \subset \mathcal{L}$ is said to be a coherent set of *strict desirable gambles* if it is a coherent set of desirable gambles, and it satisfies, in addition, the following axiom:

D6. If $X \in \mathcal{D}^+$, then either $X \geq 0$ or $X - \delta \in \mathcal{D}^+$, for some $\delta > 0$. (openness)

A coherent set of strict desirable gambles can be derived from a credal set as follows:

$$\mathcal{D}^+_{\mathcal{P}} = \{X : X \geq 0 \text{ and } X \neq 0 \text{ or } P(X) > 0\ \forall\, P \in \mathcal{P}\}.$$

The notion of *desirability* of gambles is closely related to *partial preference ordering* between gambles. A gamble X is said to be *preferred* to another gamble Y ($X \succ Y$), Coherent preference orderings can be characterized through a set of axioms closely related to D1–D5.

[4] A *linear prevision* is a linear functional $P : \mathcal{L} \to \mathbb{R}$ satisfying the constraint $P(1) = 1$. So it generalizes the notions of expectation and (finitely additive) probability measure at the same time.

3 Generalized statistical preference

Probabilistic relations are usual representation of several relational preference models. A *probabilistic relation* (see [3]) Q on a set of alternatives A is a mapping from $A \times A$ to $[0,1]$ satisfying the equality $Q(a,b) + Q(b,a) = 1$ for any pair of alternatives $(a,b) \in A^2$. On the other hand, De Schuymer et al. [2, 3] introduced the notions of *strict preference,* $P(X,Y) = P(X > Y)$, and *indifference,* $I(X,Y) = P(X = Y)$, for comparing pairs of random variables. A probabilistic relation can be naturally derived from P and I as follows:

$$Q(X,Y) = P(X,Y) + \frac{1}{2} I(X,Y).$$

Based on it, a total preorder can be defined on the class of random variables defined on a probability space:

Definition 1. *[3] A random variable X is* statistically preferred *to another random variable Y if $Q(X,Y) \geq 0.5$. We will denote it by $X \geq_{SP} Y$. Furthermore, we will use the notation $X >_{SD} Y$ when $X \geq_{SP} Y$, but not $Y \geq_{SP} X$.*

The following result follows from the fact that the probabilistic relation $D(X,Y)$ is greater than 0.5 if and only if it is greater than $D(Y,X)$.

Proposition 1. *Consider two random variables defined on the same probability space. Then, $X \geq_{SP} Y$ if and only if $P(X > Y) \geq P(X < Y)$. Consequently $X >_{SP} Y$ iff $P(X > Y) > P(X < Y)$.*

According to the last straightforward result, a random variable (from now on, a *gamble*) is statistically preferred to another gamble Y if and only if they satisfy the inequality $P(X - Y > 0) \geq P(Y - X > 0)$. According to the behavioral interpretation of previsions in the general theory of imprecise probabilities, the above inequality is related to the following preference assessment: you are disposed to give up $1_{Y-X>0}$ in return for $1_{X-Y>0}$, where 1_A denotes the indicator of A. So, statistical preference of X over Y is connected to your acceptance of a reward of one unit of *probability currency* [6] if X takes an strictly higher value than Y in exchange to the reward or one unit if Y takes a strictly higher valued than X. (Because you have stronger belief on the occurrence of $X > Y$ than on the occurrence of $Y > X$.)

As we pointed out in the last section, there is a strong connection between the notions of *desirability* and *preference* of gambles, as a gamble X is preferred to another one Y when $X - Y$ is desirable, and, conversely, X is desirable when it is preferred to the null gamble. According to this connection, we will start by introducing the notion of *signed-desirable gamble* as a primary notion, and we will derive from it the concept of *signed-preference relation.* This last concept will be the generalization of the notion of statistical preference to the theory of imprecise probabilities.

Definition 2. *Consider a coherent set of desirable gambles $\mathcal{D}$ in $\mathcal{L}$. We will say that a gamble $X \in \mathcal{L}$ is* signed-desirable *if the gamble*

$$sgn(X) = 1_{X>0} - 1_{X<0}$$

belongs to $\mathcal{D}$. (In the above expression, sgn denotes the well know "sign function" and 1_B denotes the indicator of the subset B.)

In words, a gamble X is signed-desirable when you are disposed to give up the gamble $1_{X<0}$ (it means, paying one probability currency unit if X takes a negative value) in return for the gamble $1_{X>0}$ (receiving 1 unit if X takes a -strictly- positive value.)

Remark 1. Analogously to Definition 2, we can introduce the notions of *signed-almost desirable gamble*, as a gamble X satisfying the restriction $\text{sgn}(X) \in \mathcal{D}^-$ and *signed- strictly desirable* as a gamble satisfying the condition $\text{sgn}(X) \in \mathcal{D}^+$, where $\mathcal{D}^-$ and $\mathcal{D}^+$ respectively denote coherent families of almost/strict desirable gambles. We will use the respective notations $X \in \mathcal{D}_S^-$ and $X \in \mathcal{D}_S^+$.

Proposition 2. *Consider a coherent set of desirable gambles $\mathcal{D}$, and the associated sets of almost/strict desirable gambles, respectively denoted $\mathcal{D}^-$ and $\mathcal{D}^+$. Then:*

1. *The family of signed-desirable gambles $\mathcal{D}_S$ satisfies axioms D1 to D3.*
2. *The family of signed-almost desirable gambles $\mathcal{D}_S^-$ satisfies D1', D2, D3 and D5.*
3. *The family of signed-strict desirable gambles $\mathcal{D}_S^-$ satisfies D1 to D3, and D6.*

None of the above sets of gambles satisfies axiom D4 of additivity. It is a key axiom to identify coherent sets of (almost desirable) gambles with coherent lower previsions in the theory of imprecise probabilities. The notion of lower prevision extends the concept of expectation in (classical) probability theory. In the next section, we will associate sets of signed-desirable gambles with lower medians.

Based on the above definition of signed-desirability, we can derive the following three partial preference orderings.

Definition 3. *Consider a coherent set of desirable gambles $\mathcal{D}$ in $\mathcal{L}$. A gamble X is said to be* signed -preferred *to another gamble Y if $X - Y$ is signed-desirable, i.e., if $X - Y \in \mathcal{D}_S$.*

The notions of *signed-almost preference* and *signed-strict preference* can be introduced analogously, referring to the membership of the gamble $X - Y$ to the respective sets $\mathcal{D}_S^-$ and $\mathcal{D}_S^+$. In the next proposition, we will show that the above preference partial orderings generalize the notion of statistical preference.

Proposition 3. *Let P be a linear prevision and let us respectively denote by $\mathcal{D}^-$ and $\mathcal{D}^+$ the sets of gambles $\mathcal{D}^- = \{X : P(X) \geq 0\}$ and $\mathcal{D}^+ = \{X : P(X) > 0,$ or $[X \geq 0$ and $X \neq 0]\}$. Then, for a pair of gambles X and Y:*

- *$X \geq_{SP} Y$ if and only if $X - Y \in \mathcal{D}_S^-$.*
- *$X >_{SP} Y$ if and only if $X - Y \in \mathcal{D}_S^+$.*

The above result states that almost signed-preference generalizes statistical preference and signed-strict preference generalizes strict statistical preference. The notion of signed-preference is in between the two, and it has no counterpart within the classical theory of probability. The distinction between almost desirability and desirability becomes important within the theory of imprecise probabilities. For instance, different coherent sets of gambles inducing the same credal set propagate different information about conditioning, as it is illustrated in [1, 6], for instance. It will be a matter of future study whether the distinction between signed-almost desirability and signed-(plain) desirability is also of importance or not.

4 Behavioral interpretation of the median

According to the definitions introduced in the last section, a coherent set of desirable gambles determines a set of signed-desirable gambles. Now, let us start from signed-desirability as a primary notion and consider the lower prevision of X:

$$\underline{P}_{\mathcal{D}_S}(X) = \sup\{c : X - c \in \mathcal{D}_S\}$$

It is interpreted as a threshold for the desirability in the following sense: for any strictly lower quantity $c < \underline{P}_S(X)$, you are disposed to pay some fixed quantity (say 1 probability currency unit) if $X < c$ holds, in return for the same quantity if $X > c$ occurs, because you have stronger beliefs on the event $X > c$ than on $X > c$. For any strictly higher quantity, you are not. We can give a dual interpretation, as a threshold for the desirability of $c - X$ to the infimum:

$$\overline{P}_{\mathcal{D}_S}(X) = \inf\{c : c - X \in \mathcal{D}_S\}.$$

The next result connects the above definitions with the classical notion of median. It is parallel to the connection existing between pairs of lower and upper previsions of a gamble and the bounds of its expectations, when we range the probability measures in the credal set.

Theorem 1. *Let $\mathcal{P}$ be a credal set and let be $\mathcal{D}^+$ the coherent set of strict desirable gambles:*

$$\mathcal{D}^+ = \{X \in \mathcal{L} : P(X) > 0 \; \forall P \in \mathcal{P} \text{ or } [X \geq 0 \text{ and } X \neq 0]\}.$$

Given a linear prevision, and an arbitrary gamble X, let $\mathrm{Me}_P(X)$ denote the interval of the medians of X,

$$\mathrm{Me}_P(X) = \{x : P(1_{X \geq x}) \geq 0.5 \textit{ and } P(1_{X \leq x}) \geq 0.5\}.$$

Then the following equalities hold:

$$\sup\{c : \mathrm{sgn}(X-c) \in \mathcal{D}^+\} = \inf \cup_{P \in \mathcal{P}} \mathrm{Me}_P(X) \textit{ and}$$

$$\inf\{c : \mathrm{sgn}(c-X) \in \mathcal{D}^+\} = \sup \cup_{P \in \mathcal{P}} \mathrm{Me}_P(X).$$

According to the last theorem, we can introduce the notions of lower and upper median as follows:

Definition 4. *Let $\mathcal{D}^+ \subset \mathcal{L}$ be a coherent set of strict desirable gambles. The* lower median *of an arbitrary gamble $X \in \mathcal{L}$ is defined as the quantity*

$$\underline{\mathrm{Me}}(X) = \sup\{c : \mathrm{sgn}(X-c) \in \mathcal{D}^+\}.$$

Analogously, the upper median *of X is defined as the quantity*

$$\overline{\mathrm{Me}}(X) = \inf\{c : \mathrm{sgn}(c-X) \in \mathcal{D}^+\}.$$

In the general theory of imprecise probabilities, there is a well know connection between the value of the lower prevision of a gamble and its desirability: a gamble is almost-desirable if and only if its lower prevision is non negative. Furthermore, if the lower prevision is strictly positive, then it is strictly desirable. In the next result we will show a parallel connection between the value of the lower median and the sign-desirability of a gamble:

Proposition 4. *Consider a coherent set of desirable gambles $\mathcal{D}$ and let $\mathcal{P}_{\mathcal{D}}$ the associated credal set. Let $\mathcal{D}^-$ (resp. $\mathcal{D}^+$) denote the coherent sets of almost- (resp. strict-)desirable gambles derived from it. The following implications hold:*

$$\underline{\mathrm{Me}}(X) > 0 \Rightarrow \mathrm{sgn}(X) \in \mathcal{D}^+ \Rightarrow \mathrm{sgn}(X) \in \mathcal{D} \Rightarrow \mathrm{sgn}(X) \in \mathcal{D}^- \Rightarrow \underline{\mathrm{Me}}(X) \geq 0.$$

As a consequence of the above result, when we restrict to a single probability, the statistical preference of a random variable X over another one Y is very closely related to the sign of the median of their difference:

Corollary 1. *Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space and let (X, Y) be a random vector defined on it. Let $Me_P(X-Y)$ denote the set of medians of $X-Y$, i.e.,*

$$Me_P(X-Y) = \{x : P(X-Y \geq x) \geq 0.5 \textit{ and } P(X-Y \leq x) \geq 0.5\}.$$

Then, the following implications hold:

$$\inf Me_P(X-Y) > 0 \Rightarrow X >_{SP} Y \Rightarrow X \geq_{SP} Y \Rightarrow \inf Me_P(X-Y) \geq 0$$

Corollary 2. *Let (X,Y) be random vector with absolutely continuous distribution. Then:*

- $X \geq_{SP} Y$ *if and only if* $\inf Me_P(X-Y) \geq 0$
- $X >_{SP} Y$ *if and only if* $\inf Me_P(X-Y) > 0$

We easily derive from the above result that X is statistically preferred to Y if and only if the expectation of X is greater than the expectation of Y, when the difference $X-Y$ is absolutely continuous and it has a symmetric distribution.

5 Concluding remarks

We have extended the concept of median to Imprecise Probabilities, and provided it with a behavioral meaning. We have also introduced the notion of (almost)-signed preference as a generalization of the so-called statistical preference. X is said to be signed-preferred to Y when the gamble $\mathrm{sgn}(X-Y)$ is desirable, and therefore $\underline{P}(1_{X-Y>0} - 1_{Y-X>0}) \geq 0$. The last condition is weaker than the condition $\underline{P}(1_{X-Y>0}) \geq \overline{P}(1_{Y-X>0})$, which simultaneously extends statistical preference, and the preference relation considered in [4, 5]. In the future, we will investigate further connections between both extensions.

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References

1. Couso I, Moral S, Sets of Desirable Gambles and Credal Sets, 6th International Symposium on Imprecise Probability: Theories and Applications, Durham, United Kingdom, 2009.
2. De Schuymer B, De Meyer H, De Baets B, Cycle-transitive comparison of independent random variables, Journal of Multivariate Analysis 96 (2005) 352-373.
3. De Schuymer B, De Meyer H, De Baets B, Jenei S, On the cycle-transitivity of the dice model, Theory and Decision 54 (2003) 261-285.
4. Sánchez L, Couso I, Casillas J, Modeling Vague Data with Genetic Fuzzy Systems under a Combination of Crisp and Imprecise Criteria, Proceedings of the 2007 IEEE Symposium on Computational Intelligence in Multicriteria Decision Making (MCDM 2007).
5. Sánchez L, Couso I, Casillas J, Genetic Learning of Fuzzy Rules based on Low Quality Data, Fuzzy Sets and Systems, 160 (2009) 2524-2552.
6. Walley P, Statistical Reasoning with Imprecise Probabilities, Chapman and Hall, 1991.
7. Walley P, Towards a unified theory of imprecise probability, International Journal of Approximate Reasoning, 24, 2000, 125 –148.