



Inner and outer fuzzy approximations of confidence intervals

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Abstract

We extend the notion of confidence region to fuzzy data, by defining a pair of fuzzy inner and outer confidence regions. We show the connection with previous proposals, as well as with recent studies on hypothesis testing with low quality data.

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Keywords: Confidence region; Possibility measure; Hypothesis testing

1. Introduction

More often than not, the measurements of the attributes involved in a statistical problem do not coincide with their actual values. Sometimes, such difference is very small and it can be ignored. Some other times, it can be modeled by means of a probability distribution, and therefore, traditional additive random noise techniques can be applied. In other cases, we cannot model the expert knowledge about such difference by means of a unique probability distribution, but, even though, we can bound it or, at least, we can assess different confidence degrees to an ordered sequence of bounds. In such situations, we can assume that our data set comprises n instances, each one being an imprecise perception of a value of the attribute(s), which is either represented by means of

1. a set of mutually exclusive values, one of which is the attribute value of the object under concern or
2. a fuzzy subset of the real line, interpreted as a possibility distribution over the class of possible values for the attribute.

The development of useful statistical procedures to take decisions from such kind of imprecise observations of the attributes has gained recognition in recent years. In particular, the notions of p -value and test function have been already generalized (see [4,5,7,12,13], for instance). Let us briefly recall those generalizations. In the classical setting, the p -value represents the infimum significance value for which the null hypothesis can be rejected, based upon a specific sample. When the available data set comprises a tuple of n crisp sets (situation described in point 1), an interval of upper and lower bounds for the p -value can be computed. When both of them are on one side of the significance level α , the decision (reject or accept) is clear. But when such interval of bounds overlaps the threshold, we are undecided. Thus, the test function becomes a multi-valued mapping taking the values {1} (reject), {0} (accept) and {0, 1} (undecided). In a more general setting, when the data set comprises a tuple of n fuzzy instances, interpreted as possibility distributions over the class of possible values of the attribute (situation described in point 2), the p -value and the test function become

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into a fuzzy set and a fuzzy-valued mapping, respectively. The fuzzy memberships are naturally interpreted as degrees of possibility.

Following a similar approach, in this paper we generalize the concept of confidence region. We consider a specific $1 - \alpha$ confidence region for the parameter (a random subset of the real line containing the parameter with probability greater than or equal to $1 - \alpha$) and we express the available information about it. Such available information will be imprecise, due to the imprecision in the data set. When each one of the n inputs is a (crisp) set of values, the natural representation of such information will be a pair of “outer” and “inner” approximations of the actual confidence interval. When, in a more general setting, each input is a fuzzy subset on the class of possible values for the attribute, we will represent the imprecise information about the confidence interval by means of a pair of “inner” and “outer” fuzzy subsets of the real line. The close connection between these inner and outer approximations and the above-mentioned extensions of the notions of p -value and test will be laid bare in the paper.

We must mention that this is not the first generalization of the concept of confidence interval in the fuzzy literature. The closest precedent is the concept of “fuzzy confidence interval”, introduced by Kruse and Meyer in [15]. There, a convex fuzzy subset of \mathbb{R} is assigned to each fuzzy sample (each tuple of n fuzzy numbers). The calculation of such fuzzy confidence interval coincides, under some continuity restrictions, with the calculation of our fuzzy outer approximation, but their interpretations are completely different. There is one more difference between Kruse and Meyer’s approach and ours. The fuzzy inner approximation has no place in Kruse and Meyer’s approach. But, in our context, it expresses some relevant information that cannot be determined, in general, from the outer approximation. Thus, the (fuzzy) set-valued tests that can be derived from Kruse and Meyer construction [14] are, in general, less informative than the tests derived from our construction.

The paper is organized as follows: In Section 2, we will introduce the new notions of inner and outer approximations of confidence regions. Section 2.1 will be devoted to the case of crisp set-valued data and Section 2.2 will generalize the former ideas to the case of fuzzy data sets. In Section 2.3 we will compare this new concept with the notion of fuzzy confidence interval introduced by Kruse and Meyer [15]. In Section 3 we will provide a method to construct fuzzy test functions from pairs of (fuzzy) inner and outer approximations. In Section 4 we will propose a pair of crisp outer and inner approximations of the actual confidence region, even for the case of fuzzy data set. Their interpretation is based on the suggestions proposed by Kyburg and Walley about combining different sources of uncertainty into the same probability model in the decision-making stage [16,29]. We finish the paper with some concluding remarks.

2. Imprecise confidence regions associated to low quality data

Let $X^* : \Omega \rightarrow \mathbb{R}$ be a random variable with distribution function F^* and let $\mathbf{X}^* = (X_1^*, \dots, X_n^*) : \Omega^n \rightarrow \mathbb{R}^n$ be a simple random sample of size n from F^* (a collection of n independent and identically distributed random variables with common distribution F^*). They represent n independent observations of X^* . Let us now suppose that the distribution function F^* is known, except for the value of a certain parameter $\theta^* \in \mathbb{R}^k$, with $k \geq 1$. We will consider a specific set-valued mapping $\text{Reg} : \mathbb{R}^n \rightarrow \wp(\mathbb{R})$, associating a subset of the real line (usually, an interval, but not necessarily) to each possible sample realization $\mathbf{x} \in \mathbb{R}^n$. It is said to be a $(1 - \alpha)$ -level confidence region for a certain unknown parameter θ_{X^*} of the df F^* , when it satisfies the following restriction:

$$P_{\theta_{X^*}}(\{\mathbf{x} \in \mathbb{R}^n : \text{Reg}(\mathbf{x}) \ni \theta_{X^*}\}) \geq 1 - \alpha.$$

In the above formula, $P_{\theta_{X^*}}$ denotes the probability measure associated to the random vector \mathbf{X}^* . (Such a probability depends on F^* , which in turn, is written as a function of the parameter θ_{X^*} .)

Now assume that we have got imprecise information about a sample realization \mathbf{x}^* . We will divide our study into two phases. In Section 2.1 we will assume that such imprecise information is represented by a crisp subset of \mathbb{R}^n . In Section 2.2 we will generalize this approach to the (more general) case where such information is represented by means of a fuzzy subset of \mathbb{R}^n .

2.1. Imprecise confidence regions from (crisp) set-valued data

Along this subsection, we will describe our imprecise information about the sample realization $\mathbf{x}^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ by means of a set of mutually exclusive points of \mathbb{R}^n , $\gamma \subseteq \mathbb{R}^n$. In other words, all we know about \mathbf{x}^* is that it belongs

to γ . In this setting, we can provide a pair of inner and outer approximations of the $1 - \alpha$ -confidence region $\text{Reg}(\mathbf{x}^*)$:

$$\text{InnReg}(\gamma) = \bigcap_{\mathbf{x} \in \gamma} \text{Reg}(\mathbf{x})$$

and

$$\text{OutReg}(\gamma) = \bigcup_{\mathbf{x} \in \gamma} \text{Reg}(\mathbf{x}) \quad (1)$$

The above approximations reflect our knowledge about the actual confidence interval $\text{Reg}(\mathbf{x}^*)$: it is known with certainty that $\text{Reg}(\mathbf{x}^*)$ contains the inner approximation and that it is contained in the outer one, as $\text{Reg}(\mathbf{x}^*)$ coincides with $\text{Reg}(\mathbf{x})$ for some $\mathbf{x} \in \gamma$.

So, for an arbitrary point $y \in \mathbb{R}$ we can distinguish three possible situations:

- It belongs to $\text{InnReg}(\gamma)$. In such a case, we are sure that it belongs to the “true” confidence region, $\text{Reg}(\mathbf{x}^*)$.
- It does not belong to $\text{OutReg}(\gamma)$. In such a case, we are sure that it does not belong to the “true” confidence region, $\text{Reg}(\mathbf{x}^*)$.
- It belongs to $\text{OutReg}(\gamma)$, but it does not belong to $\text{InnReg}(\gamma)$. Then, we have no information about the membership of y to the actual confidence region $\text{Reg}(\mathbf{x}^*)$.

Example 2.1. We have a container of apples and we are asked about their expected weight. We use a scale, but we do not fully trust the obtained measurement. Let us denote the displayed quantity for each apple ω by $d = D(\omega)$. Suppose that all the measurements are within a 3g error margin. Let us denote by $x^* = X^*(\omega)$ the ill-known quantity describing the (actual) weight of an arbitrary apple $\omega \in \Omega$. For simplicity, we suppose that the random variable X^* is normally distributed, with known variance $\sigma^2 = 100g^2$ and unknown expectation, $E(X^*) = \theta$. We want to provide some confidence-interval information about θ , on the basis of a sample of 25 displayed quantities, (d_1, \dots, d_{25}) . We have imprecise information about the true tuple of weights $\mathbf{x}^* = (x_1^*, \dots, x_{25}^*)$ (based on the displayed quantities and our knowledge about the precision of the scale). We know that they belong to the Cartesian product $\gamma = [d_1 - 3, d_1 + 3] \times \dots \times [d_{25} - 3, d_{25} + 3]$. According to this information, we can provide a pair of inner and outer approximations of the 0.95-confidence interval $(\bar{x}^* - 3.92, \bar{x}^* + 3.92)$. We know with certainty that it is contained in $\text{OutReg}(\gamma) = (\bar{d} - 6.92, \bar{d} + 6.92)$ and, furthermore, that it contains the interval $\text{InnReg}(\gamma) = (\bar{d} - 0.92, \bar{d} + 0.92)$.

The main goal in the next subsection is to provide a suitable extension of the above definitions to the case where the expert’s information about the sample realization \mathbf{x}^* is given by means of a fuzzy subset.

2.2. Imprecise confidence regions from fuzzy-valued data

Let us now generalize the situation described in the last subsection, and suppose that the imprecise knowledge about \mathbf{x}^* is represented by a fuzzy set $\tilde{\mathbf{x}} \in \mathcal{F}(\mathbb{R}^n)$. First of all, we must clarify the meaning of such information under our approach. We will consider the possibilistic interpretation of fuzzy sets [11]. Thus, for each $\mathbf{x} \in \mathbb{R}^n$, the membership value $\tilde{\mathbf{x}}(\mathbf{x})$ represents the possibility degree that the “true” (imprecisely observed) realization \mathbf{x}^* coincides with \mathbf{x} . This possibility distribution reflects the limited confidence of the expert about his/her measurements. It may express a subjective judgment or it can be based on some empirical knowledge. We can find more detailed explanations about this interpretation and its relation with the theory of imprecise probabilities in [1–3,8]. Furthermore, the construction of specific membership functions based on empirical data, according to this approach is provided in [18–21,23–27].

In Section 2.1, we have assumed that our information about each sample realization \mathbf{x}^* was determined by a set $\gamma \subseteq \mathbb{R}^n$ of feasible “values” (points in \mathbb{R}^n). This information can be alternatively represented by means of a 0–1-valued possibility measure on \mathbb{R}^n . Let us now generalize the situation to the case where an arbitrary possibility measure on \mathbb{R}^n represents our imprecise knowledge about the sample realization $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$. We have the information provided by a nested family of sets $(\tilde{\mathbf{x}}_\delta)_{\delta \in (0,1)}$ each one associated to a confidence degree $1 - \delta$.

We will define two mappings OutReg and InnReg from $\mathcal{F}(\mathbb{R}^n)$ to $\mathcal{F}(\mathbb{R})$ as follows.

Definition 2.1. The outer fuzzy confidence region associated to $\tilde{\mathbf{x}}$ is the fuzzy set $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ defined as follows:

$$\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y) = \sup\{\tilde{\mathbf{x}}(\mathbf{x}) : \text{Reg}(\mathbf{x}) \ni y\}, \quad \forall y. \quad (2)$$

Taking into account that $\tilde{\mathbf{x}}$ is interpreted as a possibility distribution and, according to the supremum-preserving property of possibility measures, the membership value $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y)$ will represent the degree of possibility that \mathbf{x}^* belongs to the family of samples:

$$\{\mathbf{x} \in \mathbb{R}^n : \text{Reg}(\mathbf{x}) \ni y\}.$$

In other words, the quantity $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y)$ represents the possibility that the confidence region determined by \mathbf{x}^* contains y , when our imprecise perception about \mathbf{x}^* is represented by $\tilde{\mathbf{x}}$.

When, in particular, $\tilde{\mathbf{x}}$ is a crisp set $\gamma \subseteq \mathbb{R}^n$, then $\widetilde{\text{OutReg}}(\gamma)$ assigns possibility 1 to all the elements in the crisp set $\bigcup_{\mathbf{x} \in \gamma} \text{Reg}(\mathbf{x})$ and 0 otherwise. So the above definition extends the notion of “outer” approximation introduced in Section 2.1. As we pointed out there, the crisp set $\bigcup_{\mathbf{x} \in \gamma} \text{Reg}(\mathbf{x})$ is the most committed set that would contain $\text{Reg}(\mathbf{x}^*)$ with certainty.

Let us now extend the notion of inner approximation.

Definition 2.2. The inner fuzzy confidence region associated to $\tilde{\mathbf{x}}$ is the fuzzy set $\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})$ defined as follows:

$$\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y) = 1 - \sup\{\tilde{\mathbf{x}}(\mathbf{x}) : \text{Reg}(\mathbf{x}) \not\ni y\}.$$

According to the above formula, the supremum $\sup\{\tilde{\mathbf{x}}(\mathbf{x}) : \text{Reg}(\mathbf{x}) \not\ni y\}$ coincides with the membership of y to the complement of $\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})$. According to the properties of possibility measures, such supremum represents the possibility degree of the subset

$$\{\mathbf{x} : \text{Reg}(\mathbf{x}) \not\ni y\}. \quad (3)$$

Furthermore, due to the duality between possibility and necessity measures, the membership value $\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y)$ represents the degree of necessity (certainty) of the complementary of the set given in Eq. (3), i.e.,

$$\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y) = \text{Nec}(\{\mathbf{x} : \text{Reg}(\mathbf{x}) \ni y\}).$$

In other words, the membership value $\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y)$ expresses the degree of certainty that the number y belongs to the “true” confidence region, $\text{Reg}(\mathbf{x}^*)$. Notice that the membership function $\neg \widetilde{\text{InnReg}}(\tilde{\mathbf{x}}) : \mathbb{R} \rightarrow [0, 1]$ (the membership function associated to the complementary of the inner fuzzy region) can be interpreted as a possibility distribution. For an arbitrary $y \in \mathbb{R}$, the membership value $\neg \widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y)$ represents the possibility of the set of samples $\{\mathbf{x} : \text{Reg}(\mathbf{x}) \not\ni y\}$. In other words, it represents the degree of possibility that the true confidence region does not contain the value y .

When, in particular, $\tilde{\mathbf{x}}$ is a crisp set $\gamma \subseteq \mathbb{R}^n$, the membership function $\widetilde{\text{InnReg}}(\gamma)$ assigns the necessity value 1 to the elements in the crisp set $\bigcap_{\mathbf{x} \in \gamma} \text{Reg}(\mathbf{x})$ and 0 to the elements in the complementary of it. So it extends the notion of inner region introduced in Section 2.1 and it is interpreted as the largest set contained in $\text{Reg}(\mathbf{x}^*)$ with certainty.

Example 2.2. Consider again the situation described in Example 2.1. But now suppose that the specifications of the scale are more complex: the scales are “under control” 90% of the time, and in such situation the measurements are within a 3g error margin. In the remaining 10% of the time, the scales are “out of control” and we can only guarantee an error lower than 15g. According to Couso et al. [2], we can describe this information by means of the fuzzy set $\tilde{\mathbf{x}} \in \mathcal{F}(\mathbb{R}^n)$ whose membership is defined as follows for each $\mathbf{x} = (x_1, \dots, x_{25})$:

$$\tilde{\mathbf{x}}(\mathbf{x}) = \begin{cases} 1 & \text{if } x_i \in [d_i - 3, d_i + 3], \forall i = 1, \dots, 25 \\ 0.1 & \text{if } x_i \in [d_i - 15, d_i + 15], \forall i = 1, \dots, 25, \\ & \text{and } x_j \notin [d_j - 3, d_j + 3] \text{ for some } j \in \{1, \dots, 25\} \\ 0 & \text{otherwise.} \end{cases}$$

Now we will provide the imprecise information we have about the 0.95-confidence interval about $\theta_{X^*} = E(X^*)$, $\text{Reg}(\mathbf{x}^*) = (\bar{x} - 3.92, \bar{x} + 3.92)$, based on our imprecise information about the realization \mathbf{x}^* . According to Eq. (2), we will define the fuzzy set $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ as follows:

$$\begin{aligned} \widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y) &= \sup\{\tilde{\mathbf{x}}(\mathbf{x}):y \in \text{Reg}(\mathbf{x})\} \\ &= \sup\{\tilde{\mathbf{x}}(\mathbf{x}):y \in (\bar{x} - 3.92, \bar{x} + 3.92)\} \\ &= \begin{cases} 1 & \text{if } y \in (\bar{d} - 6.92, \bar{d} + 6.92), \\ 0.1 & \text{if } y \in (\bar{d} - 18.92, \bar{d} + 18.92) \\ & \text{but } y \notin (\bar{d} - 6.92, \bar{d} + 6.92), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As we pointed out at the beginning of this section, $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y)$ represents the degree of possibility that y belongs to $\text{Reg}(\mathbf{x}^*)$. We can provide an alternative expression for $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ in terms of the crisp outer approximations associated to the level cuts of $\tilde{\mathbf{x}}$.

Proposition 2.1. $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y) = \sup\{\alpha \in [0, 1]:y \in \text{OutReg}(\tilde{\mathbf{x}}_\alpha)\}.$

Proof. The proof is straightforward, if we take into account that the membership function $\tilde{\mathbf{x}}$ can be expressed as the supremum $\tilde{\mathbf{x}}(\mathbf{x}) = \sup\{\alpha \in [0, 1]:\mathbf{x} \in \mathbf{x}_\alpha\}$ and that, according to Eq. (1), $\text{OutReg}(\tilde{\mathbf{x}}_\alpha)$ is defined as the union $\text{OutReg}(\tilde{\mathbf{x}}_\alpha) = \bigcup_{\mathbf{x} \in \tilde{\mathbf{x}}_\alpha} \text{Reg}(\mathbf{x})$. \square

Similarly, we can define the fuzzy set $\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})$ as follows:

$$\begin{aligned} \widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y) &= 1 - \sup\{\tilde{\mathbf{x}}(\mathbf{x}):y \notin \text{Reg}(\mathbf{x})\} \\ &= 1 - \sup\{\tilde{\mathbf{x}}(\mathbf{x}):y \in (-\infty, \bar{x} - 3.92) \cup (\bar{x} + 3.92, \infty)\} \\ &= \begin{cases} 0 & \text{if } y \in (-\infty, \bar{d} - 0.92) \cup (\bar{d} + 0.92, \infty), \\ 0.9 & \text{if } y \in [\bar{d} - 0.92, \bar{d} + 0.92]. \end{cases} \end{aligned}$$

$\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y)$ represents the degree of certainty that y belongs to $\text{Reg}(\mathbf{x}^*)$.

2.3. Similarities and differences with respect to Kruse and Meyer fuzzy confidence intervals

As we pointed out in Section 1, our notions of inner and outer approximations are somehow related to the earlier concept of “fuzzy confidence region” introduced by Kruse and Meyer in [15]. In this subsection, we will highlight the differences between them.

First of all, we need to recall the notion of “fuzzy perception” of a parameter in Kruse and Meyer’s context. According to the nomenclature established at the beginning of Section 2, let us denote θ_{X^*} a certain parameter associated to the df F^* of the “original” random variable $X^* : \Omega \rightarrow \mathbb{R}$. Let the fuzzy-valued mapping $\tilde{X} : \Omega \rightarrow \mathcal{F}(\mathbb{R})$ represent the “fuzzy perception” of X^* . For each $\omega \in \Omega$ and each $x \in \mathbb{R}$, the membership value $\tilde{X}(\omega)(x)$ represents the possibility degree that “true” image of ω , $X^*(\omega)$ coincides with x . The fuzzy-valued mapping \tilde{X} determines a fuzzy set on the class of all random variables from Ω to \mathbb{R} which is called the “acceptability function” and is defined as follows:

$$\text{acc}_{\tilde{X}}(X) = \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)) \quad \text{for all random variable } X : \Omega \rightarrow \mathbb{R}.$$

Such fuzzy set represents a possibility distribution over the class of random variables that assigns, to each variable X , the possibility degree that it coincides with the “original” X^* . The *fuzzy perception of θ_{X^*}* is the fuzzy set $\tilde{\theta}_{\tilde{X}}$ defined as follows:

$$\tilde{\theta}_{\tilde{X}}(y) = \sup\{\text{acc}_{\tilde{X}}(X):\theta_X = y\}.$$

The membership value $\tilde{\theta}_{\tilde{X}}(y)$ represents the degree of possibility that the true value of the parameter θ_{X^*} coincides with y . This possibility degree is calculated on the basis of the vague perception of X^* .

Example 2.3. Consider the scale specifications described in Example 2.2: the scales are “under control” 90% of the time, and in such situation the measurements are within a 3g error margin. In the remaining 10% of the time, the scales are “out of control” and we can only guarantee an error lower than 15g. Let D denote the random variable that represents the values displayed by the scale. The fuzzy perception of the expected weight $\theta_{X^*} = E(X^*)$ is the fuzzy number:

$$\tilde{E}(\tilde{X})(x) = \begin{cases} 1 & \text{if } x \in [E(D) - 3, E(D) + 3], \\ 0.1 & \text{if } x \in [E(D) - 15, E(D) + 15] \setminus [E(D) - 3, E(D) + 3], \\ 0 & \text{otherwise.} \end{cases}$$

It coincides with the fuzzy expectation [22] of the fuzzy random variable $\tilde{X} : \Omega \rightarrow \mathcal{F}(\mathbb{R})$ defined from D as follows:

$$\tilde{X}(\omega)(x) = \begin{cases} 1 & \text{if } x \in [D(\omega) - 3, D(\omega) + 3], \\ 0.1 & \text{if } x \in [D(\omega) - 15, D(\omega) + 15] \setminus [D(\omega) - 3, D(\omega) + 3], \\ 0 & \text{otherwise, } \forall \omega \in \Omega. \end{cases}$$

The frv \tilde{X} represents the incomplete knowledge about X^* in the sense that $\tilde{X}(\omega)$ describes a possibility distribution over the class of possible values for $X^*(\omega)$. According to this possibilistic interpretation, $\tilde{E}(\tilde{X})(x)$ represents the possibility degree that the actual expectation of X^* coincides with x , for all $x \in \mathbb{R}$. In practice, we cannot calculate $E(D)$, because we have only information about a sample of displayed weights, so we cannot determine $\tilde{E}(\tilde{X})$.

Let now the fuzzy-valued mapping $\tilde{\mathbf{X}} : \Omega^n \rightarrow \mathcal{F}(\mathbb{R}^n)$ represent the fuzzy perception of a random sample from X^* , $\mathbf{X}^* = (X_1^*, \dots, X_n^*)$. For each $\tilde{\omega} \in \Omega^n$ and each $\mathbf{x} = (x_1, \dots, x_n)$, the membership value $\tilde{\mathbf{X}}(\tilde{\omega})(\mathbf{x})$ represents the degree of possibility that the true sample realization $\mathbf{X}^*(\tilde{\omega}) = (x_1^*, \dots, x_n^*)$ coincides with \mathbf{x} . Let us now recall the definition of a *fuzzy confidence interval* introduced in [15].

Definition 2.3 (Kruse and Meyer [15]). A convex fuzzy-valued mapping $\Pi : \mathcal{F}(\mathbb{R}^n) \rightarrow \mathcal{F}(\mathbb{R})$ is a *fuzzy confidence interval* when it satisfies the restrictions

$$P(\{\tilde{\omega} \in \Omega^n : (\tilde{\theta}_{\tilde{X}})_\delta \subseteq \Pi_\delta(\tilde{\mathbf{X}}(\tilde{\omega}))\}) \geq 1 - \alpha, \quad \forall \delta \in (0, 1). \tag{4}$$

Just from a formal point of view, there is a connection between our definition of fuzzy outer approximation and the above notion of fuzzy confidence interval.

Proposition 2.2. $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ satisfies Definition 2.3.

Proof. According to the properties of the (weak) level cuts of fuzzy sets, the following equalities hold:

$$[\tilde{\theta}_{\tilde{X}}]_\delta = \bigcap_{\beta < \delta} \{\theta_X : X(\omega) \in [\tilde{X}(\omega)]_\beta, \forall \omega \in \Omega\}$$

$$[\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})]_\delta = \bigcap_{\beta < \delta} \text{OutReg}(\tilde{x}_\beta).$$

Having the above equalities into account, the proof is straightforward. \square

Kruse and Meyer [15] propose a method to construct a specific fuzzy confidence interval from a standard confidence interval. Let $\text{Reg} = (T_1, T_2) : \mathbb{R}^n \rightarrow \wp(\mathbb{R})$ denote a $1 - \alpha$ -confidence interval for θ_{X^*} based on a specific realization \mathbf{x}^* and let $\tilde{\mathbf{x}}$ denote the fuzzy perception of \mathbf{x}^* . The authors consider the convex fuzzy set $\Pi(\tilde{\mathbf{x}})$ defined as follows:

$$\Pi(\tilde{\mathbf{x}})(y) = \sup\{\delta \in [0, 1] : y \in (\Pi_\delta^1(\tilde{\mathbf{x}}), \Pi_\delta^2(\tilde{\mathbf{x}}))\}, \tag{5}$$

where $\Pi_{\delta}^1(\tilde{\mathbf{x}})$ and $\Pi_{\delta}^2(\tilde{\mathbf{x}})$ are given by

$$\Pi_{\delta}^1(\tilde{\mathbf{x}}) = \inf\{T_1(\mathbf{x}) : \mathbf{x} \in \tilde{\mathbf{x}}_{\delta}\}$$

and

$$\Pi_{\delta}^2(\tilde{\mathbf{x}}) = \sup\{T_2(\mathbf{x}) : \mathbf{x} \in \tilde{\mathbf{x}}_{\delta}\}.$$

They check that the convex fuzzy set Π defined in Eq. (5) is a fuzzy confidence interval, according to Definition 2.3.

When the extremes of the confidence interval, T_1 and T_2 , are continuous functions from \mathbb{R}^n to \mathbb{R} , and the δ -cuts $\tilde{\mathbf{x}}_{\delta}$ are convex, the fuzzy outer approximation $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ is a convex fuzzy set and it coincides with the fuzzy interval defined in Eq. (5), in virtue of the Mean Value Theorem. In a more general setting it is easy to check that the fuzzy outer approximation is at least as committed as the fuzzy confidence interval, i.e.,

$$\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y) \leq \Pi(\tilde{x})(y), \quad \forall y \in \mathbb{R}, \quad \forall \mathbf{x} \in \mathbb{R}^n. \quad (6)$$

But, even if there is a very close formal connection between the fuzzy outer approximation of a confidence region and Kruse and Meyer fuzzy confidence intervals, their interpretation is not the same: while Kruse and Meyer aim to cover the imprecise (fuzzy) perception, $\tilde{\theta}_{\tilde{x}}$ of the parameter θ_{X^*} , we aim to describe the available imprecise information about the (crisp) confidence region $\text{Reg}(\mathbf{x}^*)$. There is a further difference between Kruse and Meyer approach and the one presented here: the inner fuzzy region $\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})$ has no place within their approach, while it is needed in our context. In Section 3, we will show that we need to take it into account, when we aim to construct fuzzy test functions (see [4,7]) from fuzzy regions.

3. Fuzzy tests and p -values associated to fuzzy confidence regions

In classical statistics, there is a strong connection between the construction of confidence regions, the definition of parametric test functions, and the calculation of critical values. In Section 3.1, we will recall these relationships. We will make use of these ideas along the rest of the paper, in order to connect the concepts of inner and outer approximations with the notions of fuzzy test [4,7] and fuzzy p -value introduced in the recent literature. (See [7,13] for the specific definitions for some particular tests, and [4] for a more general definition). In particular, in Section 3.2, we will give a procedure to define fuzzy test functions in the sense of [4,7] from the inner and outer fuzzy regions introduced in the last section. In Section 4, we will take into account the connection between confidence regions and critical values, to propose a meaningful “defuzzification” of the inner and outer fuzzy approximations. First, let us recall some notions in the classical context.

3.1. Classical connection between tests, critical values and confidence regions

Let $X^* : \Omega \rightarrow \mathbb{R}$ be a random variable and let us state the null hypothesis

$$H_0 : \theta_{X^*} = \theta_0 \quad \text{against the alternative} \quad H_1 : \theta_{X^*} \in \Theta_1, \quad (7)$$

where θ_{X^*} is a parameter that depends on the probability distribution F^* induced by X^* . (The last equation contains, as particular cases, all one-sided and two-sided parametrical tests.) Let $\varphi^{\theta_0} : \mathbb{R}^n \rightarrow \{0, 1\}$ be a non-randomized test that represents the decision rule that will lead to accept or reject the null hypothesis. The critical region associated to φ^{θ_0} is

$$C^{\theta_0} = \{\mathbf{x} \in \mathbb{R}^n : \varphi^{\theta_0}(\mathbf{x}) = 1\}.$$

The size of the test φ^{θ_0} is the quantity

$$E_{\theta_0}(\varphi^{\theta_0}) = P_{\theta_0}(C) \leq \alpha.$$

Now assume that we have an α -test (a test of size α), $\varphi_{\alpha}^{\theta_0}$, for each $\alpha \in (0, 1]$ with critical region $C_{\alpha}^{\theta_0}$. Let us suppose that those critical regions are one included in another, as usual, i.e.,

$$[\alpha_1 \leq \alpha_2 \Rightarrow C_{\alpha_1}^{\theta_0} \subseteq C_{\alpha_2}^{\theta_0}].$$

Table 1
Summary of the classical relationships.

$(\varphi_\alpha^\theta)_{\alpha \in [0,1]} : \mathbb{R}^n \rightarrow \{0, 1\}$ (family of α -tests)	$p^\theta(x) = \inf\{\alpha : \varphi_\alpha^\theta(\mathbf{x}) = 1\}$ (critical level)	$\text{Reg}_{\varphi_\alpha^\theta}(x) = \{\theta : \varphi_\alpha^\theta(\mathbf{x}) = 0\}$ (confidence region)
$\text{Reg} : \mathbb{R}^n \rightarrow \wp(\mathbb{R})$ ($1 - \alpha$ confidence region)	$\varphi_{\text{Reg}}^\theta(x) = \begin{cases} 1 & \text{if } \text{Reg}(\mathbf{x}) \not\ni \theta \\ 0 & \text{if } \text{Reg}(\mathbf{x}) \ni \theta \end{cases}$ (test with significance level α)	
$p^\theta : \mathbb{R}^n \rightarrow [0, 1]$ (p -value)	$\varphi_{(p^\theta, \alpha)} = \begin{cases} 1 & \text{if } p^\theta(\mathbf{x}) \leq \alpha \\ 0 & \text{if } p^\theta(\mathbf{x}) > \alpha \end{cases}$ (family of α -tests)	$\text{Reg}_{(p^\theta, \alpha)}(\mathbf{x}) = \{\theta : p^\theta(\mathbf{x}) > \alpha\}$ ($1 - \alpha$ confidence region)

The critical level (or “the p -value”) associated to this family of tests is the mapping $p^\theta : \mathbb{R}^n \rightarrow [0, 1]$ defined as

$$p^\theta(\mathbf{x}) = \inf\{\alpha \in [0, 1] : \mathbf{x} \in C_\alpha^{\theta_0}\}. \tag{8}$$

Conversely, let $p^\theta : \mathbb{R}^n \rightarrow [0, 1]$ be the p -value function associated to the family of tests $(\varphi_\alpha^\theta)_{\alpha \in [0,1]}$. For a specific α , consider the mapping: $\varphi_{(p^\theta, \alpha)} : \mathbb{R}^n \rightarrow [0, 1]$:

$$\varphi_{(p^\theta, \alpha)}(\mathbf{x}) = \begin{cases} 1 & \text{if } p^\theta(\mathbf{x}) \leq \alpha, \\ 0 & \text{if } p^\theta(\mathbf{x}) > \alpha. \end{cases} \tag{9}$$

It can be easily checked that $\varphi_{(p^\theta, \alpha)}$ coincides with the infimum $\inf_{\beta > \alpha} \varphi_\beta^{\theta_0}$ and so, it is an α -test. Furthermore, under some continuity conditions, $\varphi_{(p^\theta, \alpha)}$ and $\varphi_\alpha^{\theta_0}$ do coincide.

Now we will recall the connection between α -tests and $1 - \alpha$ -confidence regions. Let $\text{Reg} : \mathbb{R}^n \rightarrow \wp(\mathbb{R})$ denote a $1 - \alpha$ -confidence region, in the sense that

$$P_{\theta_{X^*}}(\{\mathbf{x} \in \mathbb{R}^n : \text{Reg}(\mathbf{x}) \ni \theta_{X^*}\}) \geq 1 - \alpha.$$

It is well known that the test $\varphi_{\text{Reg}}^{\theta_0}$ defined as follows:

$$\varphi_{\text{Reg}}^{\theta_0}(\mathbf{x}) = \begin{cases} 1 & \text{if } \text{Reg}(\mathbf{x}) \not\ni \theta_0, \\ 0 & \text{if } \text{Reg}(\mathbf{x}) \ni \theta_0 \end{cases} \tag{10}$$

is a test of size less than or equal to α for the hypotheses given in Eq. (7). Furthermore, if the probability $P_\theta(\{\mathbf{x} : \text{Reg}(\mathbf{x}) \ni \theta\})$ equals $1 - \alpha$, then the size of the test $\varphi_{\text{Reg}}^{\theta_0}$ is exactly α . Conversely, let us start from an α -test, $\varphi_\alpha^\theta : \mathbb{R}^n \rightarrow [0, 1]$ for $H_0 : \theta = \theta_0$ against $H_1 : \theta \in \Theta_1$. Then the multi-valued mapping $\text{Reg} : \mathbb{R}^n \rightarrow \wp(\mathbb{R}^n)$ defined as $\text{Reg}'_{\varphi_\alpha^\theta}(\mathbf{x}) = \{\theta \in \Theta : \varphi_\alpha^\theta(\mathbf{x}) = 0\}$ is a $1 - \alpha$ -confidence region for θ , and furthermore, the probability

$$P_\theta(\{\mathbf{x} : \text{Reg}'_{\varphi_\alpha^\theta}(\mathbf{x}) \ni \theta\})$$

is exactly equal to $1 - \alpha$. Table 1 summarizes the ideas recalled in this subsection. The first column represents the starting notion and the other two columns represent the notions that can be derived from it. The third column in the second line of the table is empty, because we should need a family of (nested) $1 - \alpha$ regions (one region for each $\alpha \in (0, 1)$) to derive the calculation of the critical value, and not a single confidence region for a specific α . Furthermore, they should be exact confidence intervals, in the sense that, for each α , the probability of coverage of the parameter should be exactly $1 - \alpha$.

3.2. (Fuzzy)-set-valued test functions derived from inner and outer confidence regions

In some recent papers [4,7,12] the notion of α -test has been extended, according to Zadeh's Extension Principle, to the case where the incomplete information about sample realization $\mathbf{x}^* = (x_1^*, \dots, x_n^*)$ is represented by means of a (crisp or fuzzy) subset of \mathbb{R}^n . The generalized test functions are interpreted as possibility distributions.

In this subsection we will show a procedure to construct such kind of (fuzzy)-set-valued test functions on the basis of a pair of inner and outer fuzzy approximations. Furthermore, we will compare it with another procedure proposed by Grzegorzewski in [14] to define fuzzy tests from Kruse and Meyer's fuzzy confidence regions.

We will first recall the construction of fuzzy tests proposed in [4,7]. Let us initially suppose that the sample of imprecise perceptions of the attribute is represented by means of a crisp set of mutually exclusive points in \mathbb{R}^n . Then, given an α -test $\varphi : \mathbb{R}^n \rightarrow \{0, 1\}$, it is extended, in a natural way to $\wp(\mathbb{R})$ as follows:

$$\begin{aligned} \varphi(\gamma) &= \{\varphi(\mathbf{x}) : \mathbf{x} \in \gamma\} \\ &= \begin{cases} \{1\} \text{ (reject)} & \text{if } \varphi(\mathbf{x}) \in C, \forall \mathbf{x} \in \gamma, \\ \{0\} \text{ (accept)} & \text{if } \varphi(\mathbf{x}) \in C^c, \forall \mathbf{x} \in \gamma, \\ \{0, 1\} \text{ (inconclusive)} & \text{otherwise.} \end{cases} \end{aligned} \quad (11)$$

The assignation $\varphi(\gamma) = \{0, 1\}$ means that our perception, γ , of \mathbf{x}^* is too imprecise and prevents us to take a clear decision (rejecting (1) or no rejecting (0)). So we would need further information to be able to take a decision.

Let us now suppose the more general case where the imprecise perception of \mathbf{x}^* is represented by a fuzzy set $\tilde{\mathbf{x}}$. Then, according to Zadeh's Extension Principle (that extends the construction proposed in Eq. (11)), φ is extended (see [4,7]) from \mathbb{R}^n to $\mathcal{F}(\mathbb{R}^n)$ as follows:

$$\tilde{\varphi}(\tilde{\mathbf{x}})(1) = \sup\{\tilde{\mathbf{x}}(\mathbf{x}) : \varphi(\mathbf{x}) = 1\}$$

and

$$\tilde{\varphi}(\tilde{\mathbf{x}})(0) = \sup\{\tilde{\mathbf{x}}(\mathbf{x}) : \varphi(\mathbf{x}) = 0\}. \quad (12)$$

Using the possibilistic interpretation of the fuzzy set $\tilde{\mathbf{x}}$, $\tilde{\varphi}(\tilde{\mathbf{x}})(1)$ represents the possibility degree of the family of samples

$$\{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}) = 1\}.$$

In other words, it represents the possibility that the null hypothesis would be rejected, had the sample realization \mathbf{x}^* been precisely observed. Similarly, $\tilde{\varphi}(\tilde{\mathbf{x}})(0)$ represents the possibility that H_0 would be accepted (no rejected), had the data been precisely observed. Equivalently, the quantity $1 - \tilde{\varphi}(\tilde{\mathbf{x}})(0)$ represents the degree of necessity (certainty) of rejection of the null hypothesis.

Now, consider a specific confidence region $\text{Reg} : \mathbb{R}^n \rightarrow \wp(\mathbb{R})$ and let $\varphi_{\text{Reg}}^{\theta_0}$ denote the test function derived from it according to Eq. (10). Let $\tilde{\varphi}_{\text{Reg}}$ denote its fuzzy extension according to Eq. (12). We observe that it can be alternatively written in terms of the pair of outer and inner fuzzy approximations of Reg as follows:

$$\begin{aligned} \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0) &= \widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(\theta_0), \\ \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(1) &= 1 - \widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(\theta_0). \end{aligned} \quad (13)$$

Note that the above formulae involve not only the outer approximation, but also the inner one. This is the main difference with respect to the procedure proposed by Grzegorzewski [14] to derive fuzzy tests from fuzzy confidence regions. Grzegorzewski method is based on the fuzzy confidence intervals proposed by Kruse and Meyer. We will next analyze the differences between both approaches. First, we need to recall Grzegorzewski formulation.

Consider a $1 - \alpha$ confidence interval $[T_1, T_2]$ for the parameter θ_{X^*} . Suppose that the fuzzy set $\tilde{\mathbf{x}} \in \mathcal{F}(\mathbb{R}^n)$ represents imprecise perception of the sample $\mathbf{x}^* \in \mathbb{R}^n$. Now, consider the convex fuzzy set defined in Eq. (5), $\Pi(\tilde{\mathbf{x}})$. Let $\neg\Pi(\tilde{\mathbf{x}})$ denote the (usual) complement of the fuzzy set $\Pi(\tilde{\mathbf{x}})$, i.e.,

$$\neg\Pi(\tilde{\mathbf{x}})(y) = 1 - \Pi(\tilde{\mathbf{x}})(y), \quad \forall y \in \mathbb{R}.$$

Let $\Pi(\tilde{\mathbf{x}})_\delta$ and $[\neg\Pi(\tilde{\mathbf{x}})](\delta)$ denote the respective (weak) δ -cuts of those fuzzy sets. Grzegorzewski [14] proposes the following level-wise definition of a fuzzy test function $\tilde{\varphi}_G$ to test the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$:

$$\tilde{\varphi}_G(\tilde{\mathbf{x}})_\alpha = \begin{cases} \{0\} & \text{if } \theta_0 \in \Pi(\tilde{\mathbf{x}})_\delta \setminus [\neg\Pi(\tilde{\mathbf{x}})]_\delta, \\ \{1\} & \text{if } \theta_0 \in [\neg\Pi(\tilde{\mathbf{x}})]_\delta \setminus \Pi(\tilde{\mathbf{x}})_\delta, \\ \{0, 1\} & \text{if } \theta_0 \in \Pi(\tilde{\mathbf{x}})_\delta \cap [\neg\Pi(\tilde{\mathbf{x}})]_\delta, \\ \emptyset & \text{otherwise.} \end{cases} \quad (14)$$

To compare this fuzzy test with the one given in Eq. (13), we will first replace this level-wise expression with the membership function expression. The following equalities are straightforwardly obtained:

$$\begin{aligned} \tilde{\varphi}_G(\tilde{\mathbf{x}})(0) &= \Pi(\tilde{\mathbf{x}})(\theta_0), \\ \tilde{\varphi}_G(\tilde{\mathbf{x}})(1) &= \neg\Pi(\tilde{\mathbf{x}})(\theta_0) = 1 - \varphi_G(\tilde{\mathbf{x}})(0). \end{aligned} \quad (15)$$

Now, let us compare the expressions considered in Eqs. (13) and (15). First, let us recall that the outer fuzzy region $\widetilde{\text{OutReg}}$ is always included in Π , as we point out in Section 2.3, Eq. (6). In order to compare both fuzzy tests, we will first restrict ourselves to the case where $\widetilde{\text{OutReg}}$ coincides with Π (We provide sufficient conditions in Section 2.3.). In that case, the membership of the value 0 (acceptance) coincides in both tests, i.e.,

$$\tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0) = \tilde{\varphi}_G(\tilde{\mathbf{x}})(0).$$

Thus, the membership value $\tilde{\varphi}_G(\tilde{\mathbf{x}})(0)$ can be also interpreted as the possibility degree of acceptance of the null hypothesis, according to the incomplete perception of the true sample realization. On the contrary, the quantity $\tilde{\varphi}_G(\tilde{\mathbf{x}})(1) = 1 - \varphi_G(\tilde{\mathbf{x}})(0)$ does not represent the possibility degree of rejection of the null hypothesis, but the necessity degree. In short:

$$\begin{aligned} \tilde{\varphi}_G(\tilde{\mathbf{x}})(0) &= \text{Poss}(\text{accept}) = \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0), \\ \tilde{\varphi}_G(\tilde{\mathbf{x}})(1) &= \text{Nec}(\text{reject}) \leq \text{Poss}(\text{reject}) = \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(1). \end{aligned}$$

So, both tests would only coincide when the inequality $\text{Nec}(\text{reject}) \leq \text{Poss}(\text{reject})$ becomes an equality. The fuzzy set $\tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})$ is normal by construction (it represents a possibility distribution on the binary set $\{0, 1\}$), and hence, such equality is only satisfied when both tests are crisp.

In order to clarify even more the differences between both tests, we will next analyze them in the particular case where the imprecise perception of the sample realization \mathbf{x}^* is represented by means of a crisp set $\gamma \in \mathbb{R}^p$. In such situation, each fuzzy test becomes into a set-valued function. On the one hand, our test derived from the inner–outer approximations is

$$\varphi_{\text{Reg}}(\gamma) = \begin{cases} \{0\} \text{ (reject)} & \text{if } \theta_0 \in \text{InnReg}(\gamma), \\ \{1\} \text{ (accept)} & \text{if } \theta_0 \notin \text{OutReg}(\gamma), \\ \{0, 1\} \text{ (inconclusive)} & \text{otherwise.} \end{cases}$$

On the other hand, Grzegorzewski test becomes into the following crisp test:

$$\varphi_G(\gamma) = \begin{cases} \{0\} \text{ (reject)} & \text{if } \theta_0 \in \text{OutReg}(\gamma), \\ \{1\} \text{ (accept)} & \text{if } \theta_0 \notin \text{OutReg}(\gamma). \end{cases}$$

The difference between both set-valued tests is clear: Gregorzewski’s one is never inconclusive, since it does not distinguish between the situations $\theta_0 \in \text{InnReg}(\gamma)$ and $\theta_0 \in \text{OutReg}(\gamma) \setminus \text{InnReg}(\gamma)$. On the contrary, the test φ_{Reg} only takes a decision (accept) in the first case, and it is inconclusive in the second case. Let us illustrate the above ideas with an example.

Example 3.1. Let us consider again the situation described in Example 2.1 and let us suppose that the mean of the displayed quantities has been $\bar{d} = 95g$. The imprecise perception of the sample realization is represented by the

Cartesian product $\gamma = [d_1 - 3, d_1 + 3] \times \cdots \times [d_{25} - 3, d_{25} + 3]$. The outer and the inner confidence regions are respectively the crisp sets:

$$\text{OutReg}(\gamma) = (88.08, 101.92)$$

and

$$\text{InnReg}(\gamma) = (94.08, 95.92).$$

According to this information, if we want to test the hypothesis $H_0 : E(X^*) = \theta_0$ against $H_1 : E(X^*) \neq \theta_0$ the set-valued test φ_{Reg} takes the following decisions for the respective values θ_0 :

- Reject, for all $\theta_0 \leq 88.08$ and $\theta_0 \geq 101.92$.
- Accept, for all $\theta_0 \in (94.08, 95.98)$.
- Undecided, if $\theta_0 \in (88.08, 94.08] \cup [95.92, 101.92)$.

On the other hand, Grzegorzewski set-valued test would assign the following decision values:

- Reject, for all $\theta_0 < 88.08$ and $\theta_0 > 101.92$.
- Accept, for all $\theta_0 \in [88.08, 101.92]$.

Now consider the situation described in Example 2.2, and suppose again that the sample mean of the displayed quantities is 95g ($\bar{d} = 95$).

So the outer and the inner fuzzy regions are

$$\begin{aligned} \widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(y) &= \sup\{\tilde{\mathbf{x}}(\mathbf{x}) : y \in \text{Reg}(\mathbf{x})\} \\ &= \begin{cases} 1 & \text{if } y \in (88.08, 101.92), \\ 0.1 & \text{if } y \in (76.08, 88.08) \cup (101.92, 113.92), \\ 0 & \text{if } y < 76.08 \text{ or } y > 113.92 \end{cases} \end{aligned}$$

and

$$\widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(y) = \begin{cases} 0.9 & \text{if } y \in (94.08, 95.92), \\ 0 & \text{otherwise.} \end{cases}$$

If, for instance, we want to test the null hypothesis $H_0 : E(X^*) = 95.5$ against $H_1 : E(X^*) \neq 95.5$, the fuzzy test associated to the above fuzzy confidence regions is the following fuzzy subset of $\{0, 1\}$:

$$\begin{aligned} \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0) &= \widetilde{\text{OutReg}}(\tilde{\mathbf{x}})(95.5) = 1, \\ \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(1) &= 1 - \widetilde{\text{InnReg}}(\tilde{\mathbf{x}})(95.5) = 0.1. \end{aligned}$$

According to this fuzzy test, the possibility of acceptance is 1 and the possibility of rejection is 0.1. In the next section we will provide a meaningful defuzzification of this kind of tests, and we will also derive a pair of upper and lower (crisp) approximations of the confidence region from the fuzzy ones. On the other hand, Grzegorzewski's test would take the values:

$$\begin{aligned} \tilde{\varphi}_{\text{G}}(\tilde{\mathbf{x}})(0) &= \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0) = 1, \\ \tilde{\varphi}_{\text{G}}(\tilde{\mathbf{x}})(1) &= 1 - \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0) = 0. \end{aligned}$$

The membership value $\tilde{\varphi}_{\text{G}}(\tilde{\mathbf{x}})(0) = 1$ represents again the possibility of acceptance. Its dual, $\tilde{\varphi}_{\text{G}}(\tilde{\mathbf{x}})(1) = 1 - \tilde{\varphi}_{\text{Reg}}(\tilde{\mathbf{x}})(0) = 0$ represents the degree of necessity or certainty of rejection.

Finally, we will briefly discuss the general case, where the fuzzy regions $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ and $\Pi(\tilde{\mathbf{x}})$ do not coincide. According to Eq. (6), the membership $\widetilde{\text{OutReg}}(\tilde{\mathbf{x}})$ is less than or equal to $\Pi(\tilde{\mathbf{x}})$. Thus, we easily observe that, in general,

the following inequalities are fulfilled:

$$\varphi_G(\tilde{\mathbf{x}})(0) \geq \varphi_{\text{Reg}}(\tilde{\mathbf{x}})(0)$$

and

$$\varphi_G(\tilde{\mathbf{x}})(1) = 1 - \varphi_{\text{Reg}}(\tilde{\mathbf{x}})(0) \leq 1 - \varphi_{\text{Reg}}(\tilde{\mathbf{x}})(0). \quad (16)$$

Thus, in the general case, the membership value $\varphi_G(\tilde{\mathbf{x}})(0)$ is just an upper bound for the possibility that the actual sample belongs to the acceptance region, while $\varphi_G(\tilde{\mathbf{x}})(1)$ is a lower bound for the necessity value.

4. Crisp inner and outer approximations from fuzzy data: how to “defuzzify”

In Section 3.1, we have recalled the link between confidence regions, tests and p -values in standard statistics. This formal connection also holds in the fuzzy case, as we have shown in Section 3.2. In fact, we have derived fuzzy tests from fuzzy inner and outer approximations of confidence regions, and we have shown that they can be interpreted as possibility distributions over the binary space $\{0, 1\}$: the membership values $\varphi_{\text{Reg}}(\tilde{\mathbf{x}})(0)$ and $\varphi_{\text{Reg}}(\tilde{\mathbf{x}})(1)$ respectively represent the possibility of acceptance and the possibility of rejection of the null hypothesis. But in practice, a crisp decision may be needed. In [4], we have proposed a method to convert the fuzzy p -value associated to a fuzzy sample realization into an interval in the real line, and given it a justification based on the Theory of Imprecise Probabilities. Taking into account the connection between p -values, tests and confidence regions, we can also convert a fuzzy-valued test function into a (crisp) set-valued one, and a pair of fuzzy inner and outer approximations into a pair of crisp ones. In Section 4.1, we will briefly recall the method proposed in [4] to convert fuzzy p -values into intervals. We will apply it to “defuzzify” inner and outer fuzzy approximations of confidence regions in Section 4.2.

4.1. Interval representation of fuzzy p -values

Let p^{θ_0} denote the p -value function associated to a nested family of α -test functions, $(\varphi_\alpha)_{\alpha \in (0,1)}$ for $H_0 : \theta = \theta_0$ against $H_1 : \theta \in \Theta_1$. According to Eq. (8), for a specific sample realization $\mathbf{x} \in \mathbb{R}$, the value $p^{\theta_0}(\mathbf{x})$ is defined as the infimum of the test sizes such that \mathbf{x} makes us reject the hypothesis. It can be also interpreted as the probability, under the null hypothesis, of getting a sample which is “less compatible” with H_0 than \mathbf{x} is. More formally, the p -value coincides with the success parameter of the Bernoulli random variable $D(\mathbf{x}) : \mathbb{R}^n \rightarrow \{0, 1\}$ defined by

$$D(\mathbf{x})(\mathbf{y}) = \begin{cases} 1 & \text{if } p^{\theta_0}(\mathbf{y}) < p^{\theta_0}(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

Let now $\tilde{\mathbf{x}}$ represent the imprecise perception of a sample realization $\mathbf{x}^* \in \mathbb{R}^n$. The fuzzy p -value associated to \mathbf{x}^* [4] is the fuzzy set:

$$\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})(p) = \sup\{\tilde{\mathbf{x}}(\mathbf{x}): p^{\theta_0}(\mathbf{x}) = p\}, \quad (17)$$

and it represents the possibility degree that the true critical value $p^{\theta_0}(\mathbf{x}^*)$ coincides with p . Therefore, according to the above interpretation of a classical p -value as the success parameter of a Bernoulli distribution, the membership value $\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})(p)$ can be interpreted as the possibility degree that the success parameter of $D(\mathbf{x}^*)$ coincides with p , for each particular $p \in [0, 1]$. In other words, it represents the possibility degree of the Bernoulli distribution $B(p)$, for each $p \in [0, 1]$. So, the fuzzy p -value univocally determines a second-order possibility measure (a possibility measure defined over the class of probability measures).

Second-order probabilities and possibilities concern statements of standard probabilities, while standard probabilities express degrees of belief about occurrence of events. Following Kyburg [16] and Walley [29] both of them can be combined into a joint probability space, even if they are conceptually different kinds of probabilities. According to [16], in any concrete application, the conceptual differences will disappear, because that application requires a “single-case” probability. Hence, any second-order probability can be re-interpreted as a first order one. We will illustrate Walley’s procedure with a very simple example. We will avoid any formalism, and we will only focus on the intuition behind. Let us consider the second-order probability measure \mathbb{P} that assigns strict positive probabilities only to the

Bernoulli distributions $B(0.2)$, $B(0.5)$ and $B(0.8)$, according to the following table:

Bernoulli success parameter	0.2	0.5	0.8
probability	0.1	0.2	0.7

The idea behind Walley's reduction is very simple: it is based on representing the following conditional statement:
If the true success parameter were p , then the probability of the event "success" (represented by the set $\{1\}$) would be p .

as a conditional probability as follows:

$$P(\{1\}|B(p)) = p.$$

Therefore, according to the Total Probability Theorem, the above second-order probability \mathbb{P} can be reduced into a standard one, P . The new standard probability measure would assign the following probability to the event "success":

$$P(\{1\}) = 0.1 \cdot 0.2 + 0.2 \cdot 0.5 + 0.7 \cdot 0.8 = 0.68.$$

The fuzzy p -value represents a second-order possibility measure, which is, in turn, equivalent to a family of second-order probabilities. (The family of second-order probabilities which are upper-bounded by it). According to Walley's reduction, each second-order probability can be turned into a standard one. In our particular situation, such a standard probability measure is Bernoulli, and therefore it is represented by means of a single number (the success parameter). Therefore, according to this procedure, we can transform the fuzzy p -value into a set of values, which can be proved to be an interval.

Furthermore, it is proved in [4] that such interval can be calculated as follows:

$$\left[\int_0^1 \inf[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_{\delta} d\delta, \int_0^1 \sup[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_{\delta} d\delta \right]. \quad (18)$$

The extreme points of such interval represent the most accurate bounds for the true p -value, $p_{\text{val}}(\mathbf{x}^*)$, based on our imprecise knowledge of \mathbf{x}^* . Note that this interval of bounds coincides with the *mean value* [9] of the fuzzy set $\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})$.

4.2. Crisp representation of the inner and outer fuzzy approximations

In the last subsection, we provided each fuzzy sample realization, $\tilde{\mathbf{x}}$, with an interval p -value, $[p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}}), \overline{p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}})}]$. The extreme points of such interval represent the most accurate bounds for the actual p -value, $p^{\theta_0}(\mathbf{x}^*)$, according to the incomplete information about \mathbf{x}^* provided by $\tilde{\mathbf{x}}$, when we combine second-order probabilities and (usual) first-order probabilities into the same probability model. On the other hand, we show in Table 1 (third column, third row) the way to derive confidence regions from critical values in classical statistics. Thus, if we assume that the actual critical value $p^{\theta_0}(\mathbf{x}^*)$ belongs to the interval $[p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}}), \overline{p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}})}]$, then the confidence region

$$\text{Reg}_{(p^{(\cdot)}(\mathbf{x}), \alpha)} = \{\theta: p^{\theta}(\mathbf{x}) > \alpha\}$$

belongs to the family of subsets of the real line $\{\text{Reg}_{(p, \alpha)}: p \in [p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}}), \overline{p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}})}]\}$. Such family is determined by the pair of inner and outer sets:

$$\text{InnReg}(\mathbf{x}) = \{\theta: p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}}) > \alpha\}$$

and

$$\text{OutReg}(\mathbf{x}) = \{\theta: \overline{p_{\text{val}}^{\theta_0}(\tilde{\mathbf{x}})} > \alpha\}. \quad (19)$$

This is the pair of crisp inner and outer approximations of the confidence region $\text{Reg}_{(p^{(\cdot)}(\mathbf{x}), \alpha)}$ calculated upon the incomplete information about \mathbf{x}^* given by $\tilde{\mathbf{x}}$.

Now, let us start from a family of nested exact $1 - \alpha$ confidence regions for θ , $(\text{Reg}_\alpha(\mathbf{x}))_{\alpha \in (0,1)}$. We can derive from it a family of α tests, and we can therefore calculate the associated p -value, p^θ . We can again derive a $1 - \alpha$ -confidence region, $\text{Reg}'_\alpha(\mathbf{x}) = \{\theta: p^\theta(\mathbf{x}) > \alpha\}$. We can easily check that this confidence region coincides with the initial one, $\text{Reg}_\alpha(\mathbf{x})$ if it satisfies the following continuity condition:

$$\text{Reg}_\alpha(\mathbf{x}) = \bigcup_{\beta > \alpha} \text{Reg}_\beta(\mathbf{x}).$$

In such a (very common) case, the crisp inner and outer regions considered in Eq. (19) are interpreted as inner and outer approximations of the initial confidence region $\text{Reg}(\mathbf{x}^*)$.

Example 4.1. Consider the container of apples of Examples 2.1 and 2.2. For simplicity, the weight was assumed to be a normal random variable with known standard deviation $\sigma = 10g$. For a particular sample realization of size 25, \mathbf{x}^* , and for each $\alpha \in [0, 1]$, the interval $\text{Reg}_\alpha(\mathbf{x}^*) = (\bar{\mathbf{x}}^* - 2 \cdot z_{1-\alpha/2}, \bar{\mathbf{x}}^* + 2 \cdot z_{1-\alpha/2})$ is an (exact) $1 - \alpha$ -confidence interval. (In the last formula, $z_{1-\alpha/2}$ denotes the $1 - \alpha/2$ quantile of the standard normal distribution.) We can derive a family of α -tests from the above family of intervals:

$$\varphi_{\text{Reg}_\alpha}(\mathbf{x}) = \begin{cases} 1 & \text{if } (\bar{\mathbf{x}} - 2 \cdot z_{1-\alpha/2}, \bar{\mathbf{x}} + 2 \cdot z_{1-\alpha/2}) \not\ni \theta, \\ 0 & \text{otherwise.} \end{cases}$$

The associated p -value is given by the formula:

$$p^\theta(\mathbf{x}^*) = \inf\{\alpha: (\bar{\mathbf{x}}^* - 2 \cdot z_{1-\alpha/2}, \bar{\mathbf{x}}^* + 2 \cdot z_{1-\alpha/2}) \not\ni \theta\} = 2 \left[1 - \phi \left(\frac{|\bar{\mathbf{x}}^* - \theta|}{2} \right) \right],$$

where ϕ denotes the distribution function of the standard normal distribution. Now assume that the fuzzy set $\tilde{\mathbf{x}}$ (based on the information provided by the displayed quantity and the specifications of the scale provided in Example 2.2) represents the imprecise perception of \mathbf{x}^* .

According to Eq. (17), the fuzzy p -value can be calculated as follows:

$$\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})(p) = \sup\{\tilde{\mathbf{x}}(\mathbf{x}): p(\mathbf{x}) = p\} = \sup \left\{ \tilde{\mathbf{x}}(\mathbf{x}): 2 \left[1 - \phi \left(\frac{|\bar{\mathbf{x}} - \theta|}{2} \right) \right] = p \right\}.$$

In order to calculate the mean value of the fuzzy set $\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})$, we will first determine each δ -cut. Let us notice that p^{θ_0} is, in this particular example, a continuous function from \mathbb{R}^n to \mathbb{R} . Therefore, according to [17], if we assume that the δ -cuts of $\tilde{\mathbf{x}}$ are compact sets, then their images by the function p^{θ_0} do coincide with the δ -cuts of the fuzzy set $\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})$. Thus, the δ -cut of the p -value is determined by the formula:

$$[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_\delta = \left\{ 2 \left[1 - \phi \left(\frac{|\bar{\mathbf{x}} - \theta_0|}{2} \right) \right] : \mathbf{x} \in \tilde{\mathbf{x}}_\alpha \right\}, \quad \forall \delta$$

and therefore, the maximum and the minimum of each δ -cut are given by

$$\min[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_\delta = \begin{cases} 2 \left[1 - \phi \left(\frac{|\bar{d} - \theta_0| + 3}{2} \right) \right] & \text{if } \delta \geq 0.1, \\ 2 \left[1 - \phi \left(\frac{|\bar{d} - \theta_0| + 15}{2} \right) \right] & \text{otherwise,} \end{cases}$$

$$\max[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_\delta = \begin{cases} 2 \left[1 - \phi \left(\max \left\{ \frac{|\bar{d} - \theta_0| - 3}{2}, 0 \right\} \right) \right] & \text{if } \delta \geq 0.1, \\ 2 \left[1 - \phi \left(\max \left\{ \frac{|\bar{d} - \theta_0| - 15}{2}, 0 \right\} \right) \right] & \text{otherwise.} \end{cases}$$

The maximum and the minimum of the mean value of the fuzzy p -value are, therefore, the Lebesgue integrals:

$$\begin{aligned}\bar{p}^{\theta_0}(\tilde{\mathbf{x}}) &= \int_0^1 \max[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_{\delta} d\delta \\ &= 0.9 \times 2 \left[1 - \phi \left(\max \left\{ \left(\frac{|\bar{d} - \theta_0| - 3}{2} \right), 0 \right\} \right) \right] + 0.1 \times 2 \left[1 - \phi \left(\max \left\{ \left(\frac{|\bar{d} - \theta_0| - 15}{2} \right), 0 \right\} \right) \right], \\ \underline{p}^{\theta_0}(\tilde{\mathbf{x}}) &= \int_0^1 \min[\tilde{p}^{\theta_0}(\tilde{\mathbf{x}})]_{\delta} d\delta \\ &= 0.9 \times 2 \left[1 - \phi \left(\frac{|\bar{d} - \theta_0| + 3}{2} \right) \right] + 0.1 \times 2 \left[1 - \phi \left(\frac{|\bar{d} - \theta_0| + 15}{2} \right) \right].\end{aligned}$$

According to the interpretation of $\underline{p}^{\theta_0}(\tilde{\mathbf{x}})$ and $\bar{p}^{\theta_0}(\tilde{\mathbf{x}})$ suggested in Section 4.1, they are the most committed bounds for the true p -value $p^{\theta_0}(\mathbf{x}^*)$, according to the incomplete information we have about the sample realization.

Now, we will use the formula given in the third column of the third row in Table 1 to determine a $1 - \alpha$ confidence region from critical values:

$$\text{Reg}'_{\alpha}(\mathbf{x}^*) = \{\theta: p^{\theta}(\mathbf{x}) > \alpha\}.$$

We observe that $\text{Reg}'_{\alpha}(\mathbf{x}^*) = \bigcup_{\beta > \alpha} \text{Reg}_{\beta}(\mathbf{x}^*) = \text{Reg}_{\alpha}(\mathbf{x}^*)$. Thus, according to the information provided by the fuzzy sample, and according to the interpretation of $\underline{p}^{\theta_0}(\tilde{\mathbf{x}})$ and $\bar{p}^{\theta_0}(\tilde{\mathbf{x}})$ as bounds for the true p -value, $p^{\theta_0}(\mathbf{x}^*)$, we can provide a pair of (crisp) outer and inner approximations of $\text{Reg}(\mathbf{x}^*)$ as follows:

$$\text{OutReg}(\tilde{\mathbf{x}}) = \{\theta: \bar{p}^{\theta}(\tilde{\mathbf{x}}) > \alpha\} \quad \text{and} \quad \text{InnReg}(\tilde{\mathbf{x}}) = \{\theta: \underline{p}^{\theta}(\tilde{\mathbf{x}}) > \alpha\}.$$

Let us explicitly calculate the (crisp) inner approximation:

$$\begin{aligned}\text{InnReg}(\tilde{\mathbf{x}}) &= \{\theta: \underline{p}^{\theta}(\tilde{\mathbf{x}}) > \alpha\} \\ &= \left\{ \theta: 0.9 \times 2 \left[1 - \phi \left(\frac{|\bar{d} - \theta_0| + 3}{2} \right) \right] + 0.1 \times 2 \left[1 - \phi \left(\frac{|\bar{d} - \theta_0| + 15}{2} \right) \right] > \alpha \right\} \\ &= \left\{ \theta: 2 \left[1 - \phi \left(\frac{0.9 \times (|\bar{d} - \theta_0| + 3) + 0.1 \times (|\bar{d} - \theta_0| + 15)}{2} \right) \right] > \alpha \right\} \\ &= \left\{ \theta: 0.9 \times \phi(|\bar{d} - \theta_0| + 3) + 0.1 \times \phi(|\bar{d} - \theta_0| + 15) > 1 - \frac{\alpha}{2} \right\} \\ &= \left\{ \theta: \phi^{-1} [0.9 \times \phi(|\bar{d} - \theta_0| + 3) + 0.1 \times \phi(|\bar{d} - \theta_0| + 15)] > z_{1-\alpha/2} \right\}.\end{aligned}\tag{20}$$

Concerning the interpretation of the above calculations, the interval $[\underline{p}^{\theta_0}(\tilde{\mathbf{x}}), \bar{p}^{\theta_0}(\tilde{\mathbf{x}})]$ represents the bounds of the actual p -value, when we combine (according to [16,29]) two different sources of uncertainty into the same model: (1) randomness, concerning the selection of different apples from the container and (2) ill-knowledge about the state of the scale (it is under control with probability 0.9 and out of control, with probability 0.1). The crisp inner and outer approximations represent the most committed “bounds” of $\text{Reg}(\mathbf{x}^*)$, under this assumption.

5. Concluding remarks and future work

We have extended the notion of confidence region of a parameter to the case where the sample is a collection of n crisp or fuzzy subsets of the real line. We suppose that each (fuzzy) subset represents some imprecisely observed quantity, interpreted as a possibility distribution (when the subset is crisp, such possibility distribution is 0–1 valued). We have defined a pair of inner and outer (fuzzy) confidence regions that represent our knowledge about the confidence

region determined by the actual sample. So, we complete the puzzle about the relationship between confidence regions, test functions and p -values within the fuzzy set environment.

We have also provided a pair of crisp approximations of the actual confidence region, and we have provided them with an interpretation within the theory of imprecise probabilities. They have been calculated according to the Kyburg [16] and Walley [29] suggestions of converting second-order probabilities into first order ones (by combining “probabilities about probabilities” and “probabilities about statements” into the same model).

Acknowledgements

This work was supported by the FEDER-MEC Grants TIN2008-06681-C06-04 and TIN2010-20900-C04-01. This financial support is gratefully acknowledged.

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